Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **27** (2001), 107-114

Serdica
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

EQUIVARIANT EMBEDDINGS OF DIFFERENTIABLE SPACES

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Communicated by O. Muškarov

Abstract. Given a differentiable action of a compact Lie group G on a compact smooth manifold V, there exists [3] a closed embedding of V into a finite-dimensional real vector space E so that the action of G on V may be extended to a differentiable linear action (a linear representation) of G on E. We prove an analogous equivariant embedding theorem for compact differentiable spaces (∞-standard in the sense of [6, 7, 8]).

1. Preliminaries.

Differentiable algebras [2, 4, 5, 9]. $\mathcal{C}^\infty(\mathbb{R}^n)$ will denote the algebra of all smooth real-valued functions on $\mathbb{R}^n$, endowed with the usual Fréchet topology, so that polynomial functions are dense in $\mathcal{C}^\infty(\mathbb{R}^n)$. *Differentiable algebras* are defined to be quotients of $\mathcal{C}^\infty(\mathbb{R}^n)$ by closed ideals:

$$A \simeq \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}, \qquad \overline{\mathfrak{a}} = \mathfrak{a}$$

2000 *Mathematics Subject Classification:* 58A40, 22C05.
Key words: Affine differentiable spaces, actions of compact Lie groups, differentiable algebras.

and in such case we say that

$$(\mathfrak{a})_0 := \{x \in \mathbb{R}^n \ : \ f(x) = 0 \ \text{ for any } \ f \in \mathfrak{a}\}$$

is the real *spectrum* of A. If $x \in (\mathfrak{a})_0$, then $\mathfrak{m}_x := \{f \in A : f(x) = 0\}$ is a maximal ideal of A. If $f \in A$, then the *differential* $d_x f$ of f at x is defined to be the residue class of the increment $f - f(x) \in \mathfrak{m}_x$ in the *cotangent space* $\mathfrak{m}_x/\mathfrak{m}_x^2$ at the point x. We say that $f_1, \ldots, f_r \in A$ separate infinitely near points to x when $d_x f_1, \ldots, d_x f_r$ span the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Differentiable algebras have a $\mathcal{C}^\infty$-calculus: If $f_1, \ldots, f_r \in A$, then there exists a unique morphism of $\mathbb{R}$-algebras $\psi : \mathcal{C}^\infty(\mathbb{R}^r) \to A$ such that $\psi(x_i) = f_i$, $1 \leq i \leq r$. Moreover, ψ is surjective if and only if $f_1, \ldots, f_r$ separate infinitely near points and the map $(f_1, \ldots, f_r) : (\mathfrak{a})_0 \to \mathbb{R}^r$ defines a homeomorphism of $(\mathfrak{a})_0$ onto a closed subset of $\mathbb{R}^r$. In particular, when A has compact spectrum, ψ is surjective if and only if $f_1, \ldots, f_r$ separate points and infinitely near points.

Affine differentiable spaces [4, 6, 7, 8]. The category of *affine differentiable spaces* (local models of Spallek's standard ∞-differentiable spaces) is dual to the category of differentiable algebras. If X is an affine differentiable space, $\mathcal{C}^\infty(X)$ will denote the corresponding differentiable algebra, and the real spectrum of $\mathcal{C}^\infty(X)$ is denoted by X. Elements of $\mathcal{C}^\infty(X)$ are said to be *differentiable* functions on X. Morphisms of affine differentiable spaces $\phi : X \to Y$ are just morphisms of $\mathbb{R}$-algebras $\phi^* : \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$.

Any smooth manifold V (Hausdorff, separable and of finite dimension) defines an affine differentiable space since, according to Whitney's embedding theorem, $\mathcal{C}^\infty(V)$ is a differentiable algebra. Moreover, morphisms of differentiable spaces between smooth manifolds V, W are just differentiable maps, since it is well-known that every morphism of $\mathbb{R}$-algebras $\mathcal{C}^\infty(V) \to \mathcal{C}^\infty(W)$ is defined by a unique differentiable map $W \to V$.

The category of affine differentiable spaces has finite direct products.

A morphism of affine differentiable spaces $j : X \hookrightarrow X'$ is said to be a *closed embedding* if the corresponding morphism $j^* : \mathcal{C}^\infty(X') \to \mathcal{C}^\infty(X)$ is surjective. In such case the *restriction* to X of a differentiable function $f \in \mathcal{C}^\infty(X')$ is defined to be $j^* f$. When $j : X \hookrightarrow X'$ is a closed embedding, then so is $j \times (id) : X \times Y \hookrightarrow X' \times Y$ for any affine differentiable space Y.

Let X be an affine differentiable space and let V be a smooth manifold. A differentiable function f on $V \times X$ vanishes if and only if so does its restriction to $v \times X$ for any point $v \in V$. Hence, if Y is an affine differentiable space, two morphisms $\phi, \varphi : V \times X \to Y$ coincide if and only if they coincide on $v \times X$ for any point $v \in V$.

Peter-Weyl's theorem. Let G be a compact Lie group. A *continuous action* of G onto a topological space X is defined to be any continuous map $\theta : G \times X \to X$ such that $1 \cdot x = x$ and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for any $g_1, g_2 \in G$, $x \in X$; where $g \cdot x := \theta(g, x)$. Given two topological spaces X, Y, endowed with continuous actions of G, a continuous map $f\colon X \to Y$ is said to be *G-equivariant* when $f(g \cdot x) = g \cdot f(x)$ for any $x \in X$, $g \in G$.

A continuous action $\theta\colon G \times E \to E$ of G on a topological vector space E is said to be *linear* (or that G acts on E by linear automorphisms) when the map $\theta_g\colon E \to E$, $\theta_g(e) = g \cdot e$, is linear for every $g \in G$. A vector $e \in E$ is said to be of *representation* when its orbit $Ge := \theta(G \times e)$ spans a finite-dimensional vector subspace of E. If A is a topological algebra and $\theta : G \times A \to A$ is a continuous action, we say that G acts on A by automorphisms of algebras when $\theta_g\colon A \to A$ is an automorphism of algebras for any $g \in G$.

If $\theta : G \times E \to E$ is a linear continuous action of a compact Lie group G on a Fréchet space E, then Peter-Weyl's theorem ([1]) states that representation vectors are dense in E.

2. Continuous actions on differentiable algebras. Let E be a finite-dimensional real vector space. Continuous linear actions of G on E correspond with continuous linear representations (continuous morphism of groups) $\rho\colon G \to Gl(E)$, where $\rho(g)(e) = g \cdot e$. In such case, we have a continuous linear action of G on the dual space F of E, where $g \in G$ acts by: $(g \cdot f)(e) := f(g^{-1} \cdot e)$, $g \in G$, $f \in F$, $e \in E$. Since $F \subset \mathcal{C}^\infty(E)$, this linear action of G on the dual space F may be extended so as to obtain a continuous action of G on $\mathcal{C}^\infty(E)$ by automorphisms of algebras:

$$(g \cdot f)(e) := f(g^{-1} \cdot e), \qquad g \in G,\ f \in \mathcal{C}^\infty(E),\ e \in E$$

Lemma 2.1. *Let F be a finite-dimensional vector subspace of a differentiable algebra A and let E be the dual space of F. There exists a unique morphisms of $\mathbb{R}$-algebras $\mathcal{C}^\infty(E) \to A$ which is the identity on F. If we have a continuous action of G on A by automorphisms of algebras and F is a G-invariant subspace, then this morphism is G-equivariant.*

P r o o f. In order to show the existence of ψ, we may assume that $A = \mathcal{C}^\infty(\mathbb{R}^n)$. In such case, any point of $\mathbb{R}^n$ defines a linear map $\phi_x : F \to \mathbb{R}$,

$\phi_x(f) = f(x)$, and so we obtain a differentiable map $\phi : \mathbb{R}^n \to E$, $\phi(x) = \phi_x$. The morphism $\phi^* : \mathcal{C}^\infty(E) \to A$ is the identity on F.

Since the subalgebra of $\mathcal{C}^\infty(E)$ generated by F is dense, it follows that such morphism is unique and that it is G-equivariant whenever F is a G-invariant subspace. □

Theorem 2.2. *Let G be a compact Lie group and let A be a differentiable algebra with compact spectrum. If we have a continuous action of G on A by automorphisms of algebras, then there exists a continuous linear representation $G \to Gl(E)$ and a G-equivariant epimorphism $\mathcal{C}^\infty(E) \to A$.*

Proof. By definition, we have $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ for some closed ideal $\mathfrak{a}$, so that A is a Fréchet space and, according to Peter-Weyl's theorem, representation functions are dense in A. Hence, there are representation functions $f_1, \ldots, f_n \in A$ so close to the cartesian coordinates $x_1, \ldots, x_n$ that $d_x f_1, \ldots, d_x f_n$ span the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ at any point x of $K = (\mathfrak{a})_0$ (hence $f_1, \ldots, f_n$ separate infinitely near points). Moreover, we may assume that $f_1, \ldots, f_n$ separate points in a neighborhood of each point of K (although it may be that $f_1, \ldots, f_n$ do not separate points of K), and a finite number $U_1, \ldots, U_m$ of these neighborhoods cover K since it is assumed to be compact. Let ε be a positive real number such that, whenever the distance between two points $x, y \in K$ is $d(x, y) < \varepsilon$, then $x, y \in U_r$ for some $r = 1, \ldots, m$.

Let $h_1, \ldots, h_n \in A$ be representation functions so close to $x_1, \ldots, x_n$ that we have $d(x, y) < \varepsilon$ whenever $h_i(x) = h_i(y)$, $i = 1, \ldots, n$; so that $f_j(x) \neq f_j(y)$ for some index $j = 1, \ldots, n$. Since $f_1, \ldots, f_n, h_1, \ldots, h_n \in A$ are representation functions, their orbits $Gf_1, \ldots, Gf_n, Gh_1, \ldots, Gh_n$ span a finite-dimensional G-invariant vector subspace $F \subset A$.

Let E be the dual space of F. The morphism $\mathcal{C}^\infty(E) \to A$ provided by 2.1 is surjective, because $f_1, \ldots, f_n, h_1, \ldots, h_n$ separate points of K and infinitely near points, and it is G-equivariant according to 2.1. □

Definition. *A* differentiable action *of a Lie group G on an affine differentiable space X is defined to be a morphism of differentiable spaces $\theta\colon G \times X \to X$ such that the following diagrams are commutative:*

$$\begin{array}{ccc} X & \xrightarrow{1\times Id} & G \times X \\ & {\scriptstyle Id}\searrow \quad \swarrow{\scriptstyle \theta} & \\ & X & \end{array} \qquad\qquad \begin{array}{ccc} G \times G \times X & \xrightarrow{(id)\times\theta} & G \times X \\ \downarrow{\scriptstyle \mu\times(id)} & & \downarrow{\scriptstyle \theta} \\ G \times X & \xrightarrow{\theta} & X \end{array}$$

where $\mu : G \times G \to G$, $\mu(g', g) = g'g$, *stands for the operation of* G.

Let $\theta_X : G \times X \to X$ and $\theta_Y : G \times Y \to Y$ be differentiable actions of G on two affine differentiable spaces X, Y. A morphism of differentiable spaces $\phi : X \to Y$ is said to be G-equivariant when the following diagram is commutative:

$$\begin{array}{ccc} G\times X & \xrightarrow{\theta_X} & X \\ \big\downarrow{\scriptstyle (id)\times\phi} & & \big\downarrow{\scriptstyle \phi} \\ G\times Y & \xrightarrow{\theta_Y} & Y \end{array}$$

Example. Any continuous linear representation $\rho : G \to Gl(E)$ is a differentiable map, so that the corresponding linear action $\theta : G \times E \to E$, $\theta(g, e) = \rho(g)(e)$, is differentiable. In fact, $\Gamma_\rho := \{(g, \rho(g))\colon g \in G\}$ is a closed subgroup of $G \times Gl(E)$, and it is homeomorphic to G; therefore ([10]) Γ_ρ is a smooth submanifold of the same dimension as G. The first projection, $\pi_1\colon \Gamma_\rho \to G$ is a local diffeomorphism at some point by Sard's theorem, hence it is a diffeomorphism since it is an isomorphism of groups. It follows that the linear representation $\rho = \pi_2\pi_1^{-1}$ is differentiable, hence so is the corresponding linear action $\theta(g, e) = \rho(g)(e)$.

Let $\theta : G \times X \to X$ be a differentiable action of G on an affine differentiable space X. If $g \in G$, then the composition

$$X \simeq g \times X \hookrightarrow G \times X \xrightarrow{\theta} X$$

is an isomorphism $g\colon X \simeq X$, so that g induces an isomorphism of algebras $g^* : \mathcal{C}^\infty(X) \simeq \mathcal{C}^\infty(X)$. We obtain an action of G on $\mathcal{C}^\infty(X)$ by automorphisms of algebras:

$$g \cdot f = (g^{-1})^* f, \qquad g \in G,\ f \in \mathcal{C}^\infty(X)$$

and, by definition, $g \cdot f$ is just the restriction of $\theta^* f$ to $g^{-1} \times X \simeq X$.

Lemma 2.3. $\theta : G \times X \to X$ *be a differentiable action of* G *on an affine differentiable space* X. *The induced action of* G *on* $\mathcal{C}^\infty(X)$ *is continuous.*

Proof. If Y is any affine differentiable space, then we consider the map $\delta_Y\colon G \times \mathcal{C}^\infty(G \times Y) \to \mathcal{C}^\infty(Y)$, where $\delta_Y(g, f)$ is the restriction of f to $g \times Y \simeq Y$. It is easy to check that $\delta_{\mathbb{R}^n}$ is continuous. If $\mathcal{C}^\infty(X) = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$, then we have a commutative square

$$\begin{array}{ccc} G\times\mathcal{C}^\infty(G\times\mathbb{R}^n) & \xrightarrow{\delta_{\mathbb{R}^n}} & \mathcal{C}^\infty(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ G\times\mathcal{C}^\infty(G\times X) & \xrightarrow{\delta_X} & \mathcal{C}^\infty(X) \end{array}$$

where the vertical arrows are open surjective continuous maps. Since $\delta_{\mathbb{R}^n}$ is continuous, it follows that so is δ_X. Finally, the action of G on $\mathcal{C}^\infty(X)$ is continuous because it is the composition of the following continuous maps:

$$G\times\mathcal{C}^\infty(X) \xrightarrow{(id)\times\theta^*} G\times\mathcal{C}^\infty(G\times X) \xrightarrow{(inv)\times(id)} G\times\mathcal{C}^\infty(G\times X) \xrightarrow{\delta_X} \mathcal{C}^\infty(X)$$

3. Equivariant embedding theorem.

Theorem 3.1. *Let G be a compact Lie group and let X be a compact affine differentiable space. Any continuous action of G on $\mathcal{C}^\infty(X)$ by automorphisms of algebras is differentiable (it is induced by a differentiable action $G\times X\to X$).*

P r o o f. According to Theorem 2.2, there exists a continuous (hence differentiable) linear representation $G\to Gl(E)$ and a G-equivariant epimorphism $\mathcal{C}^\infty(E)\to\mathcal{C}^\infty(X)$; hence its kernel $\mathfrak{a}$ is a G-invariant closed ideal. This epimorphism defines a closed embedding $j: X\hookrightarrow E$ and, $\mathfrak{a}$ being G-invariant, $\theta_E^*(\mathfrak{a})$ vanishes on $g\times X$ for any $g\in G$. Therefore $\theta_E^*(\mathfrak{a})$ vanishes on $G\times X$ and the differentiable action $\theta_E: G\times E\to E$ induces a morphism of affine differentiable spaces $\theta_X: G\times X\to X$, so that the following square is commutative:

$$\begin{array}{ccc} G\times X & \xrightarrow{\theta_X} & X \\ \big\downarrow{\scriptstyle (id)\times j} & & \big\downarrow{\scriptstyle j} \\ G\times E & \xrightarrow{\theta_E} & E \end{array}$$

It follows that θ_X is a differentiable action, since so is θ_E and j is a closed embedding. Finally, it is easy to check that the action of G on $A=\mathcal{C}^\infty(X)$ defined by θ_X is just the initial one.

Theorem 3.2. *Let $\theta_X: G\times X\to X$ be a differentiable action of a compact Lie group G on a compact affine differentiable space X. There exists a differentiable linear representation $\rho: G\to Gl(E)$ and a G-equivariant closed embedding $j: X\hookrightarrow E$.*

Proof. By Lemma 2.3, the action of G on $\mathcal{C}^\infty(X)$ is continuous. By Theorem 2.2, there exists a closed embedding $j : X \hookrightarrow E$ such that the corresponding epimorphism $j^* : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(X)$ is G-equivariant. That is to say, if $g \in G$, then the following square is commutative:

$$\begin{array}{ccc} g \times X & \xrightarrow{\theta_X} & X \\ \Big\downarrow{\scriptstyle (id)\times j} & & \Big\downarrow{\scriptstyle j} \\ g \times E & \xrightarrow{\theta_E} & E \end{array}$$

Therefore, the following square

$$\begin{array}{ccc} G \times X & \xrightarrow{\theta_X} & X \\ \Big\downarrow{\scriptstyle (id)\times j} & & \Big\downarrow{\scriptstyle j} \\ G \times E & \xrightarrow{\theta_E} & E \end{array}$$

is commutative: the closed embedding j is G-equivariant.

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Received July 21, 2000