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ON THE 3-COLOURING VERTEX FOLKMAN NUMBER $F(2,2,4)$

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Abstract. In this note we prove that $F(2,2,4) = 13$.

We consider only finite, non-oriented graphs, without loops and multiple edges. $V(G)$ and $E(G)$ denote the set of the vertices and the set of the edges of the graph G, respectively. We say that G is an n-vertex graph when $|V(G)| = n$. For $v \in V(G)$ we denote by $\mathrm{Ad}(v)$ the set of all vertices, adjacent to v. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p, such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$. A set of vertices in a graph G is said to be independent if no two of them are adjacent. The cardinality of any largest independent set of vertices in G is denoted by $\alpha(G)$.

If $W \subseteq V(G)$, then $G - W$ denotes the subgraph of the graph G, which is obtained from G by the removal of the vertices belonging to W. The simple cycle of length n is denoted by C_n. By $\overline{G}$ we denote the complementary graph of G.

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The Ramsey number $R(p,q)$ is the smallest natural number n, such that for arbitrary n-vertex graph G, either $\mathrm{cl}(G) \geq p$ or $\alpha(G) \geq q$. We need the identities $R(3,4) = R(4,3) = 9$, [3].

Definition. *Let G be a graph and $a_1, \dots, a_r$, $r \geq 2$, be positive integers. The symbol $G \to (a_1, \dots, a_r)$ means that for every r-colouring of the vertices of G*

$$V(G) = V_1 \cup \dots \cup V_r, \quad V_i \cap V_j = \emptyset,\ i \neq j,$$

there exists $i \in \{1, 2, \dots, r\}$, such that the graph G contains a monochromatic a_i-clique K of colour i, i.e. $K \subseteq V_i$.

We put

$$H(a_1, \dots, a_r) = \{G : G \to (a_1, \dots, a_r) \text{ and } \mathrm{cl}(G) = \max(a_1, \dots, a_r)\}$$

$$F(a_1, \dots, a_r) = \min\{|V(G)| : G \in H(a_1, \dots, a_r)\}.$$

Folkman proved in [2] that $H(a_1, \dots, a_r) \neq \emptyset$. $F(a_1, \dots, a_r)$ are called r-colouring vertex Folkman numbers. It is clear that

$$G \to (a_1, \dots, a_r) \iff G \to (a_{\varphi(1)}, \dots, a_{\varphi(r)})$$

for any permutation φ of the symmetric group S_r. Hence $F(a_1, \dots, a_r)$ is a symmetric function and thus we may assume that $a_1 \leq a_2 \leq \dots \leq a_r$. Note that if $a_1 = 1$, then $F(a_1, \dots, a_r) = F(a_2, \dots, a_r)$. Hence we may assume also that $a_i \geq 2$, $i = 1, \dots, r$.

For the 2-colouring vertex Folkman numbers $F(p,q)$ the following facts are known:

Theorem A ([5]). *For any $p \geq 2$, we have $F(2,p) = 2p + 1$.*

Theorem B ([6]). *Let $G \in H(2,p)$, $p \geq 2$, and $|V(G)| = 2p + 1$. Then $G = \overline{C}_{2p+1}$.*

Theorem C ([10]). *For any $p \geq 3$, the Folkman numbers $F(p,p)$ satisfy inequality $F(p,p) < \lfloor p!e \rfloor - 1$.*

Theorem D ([6]). *Let p, q be any integers such that $2 \leq p \leq q$. Then*

$$F(p,q) \leq 2\sum_{i=0}^{p-1} \frac{q!}{(q-i)!} - 1.$$

We constructed in [9] a 14-vertex graph $G \in H(3,3)$, showing that $F(3,3) \leq 14$. In a joint paper with E. Nedialkov [8], we proved that $F(3,3) \geq 12$. The work [13] provides a computer proof of the inequality $F(3,3) \geq 14$ and thus $F(3,3) = 14$. According to Theorem D, we have $F(3,4) \leq 33$. In [11], it is proved that $F(3,4) = 13$.

The numbers $F(2,2,2) = 11$ and $F(2,2,2,2) = 22$ are the only known vertex Folkman numbers for more that two colours. Mycielski [7], presented an 11-vertex graph $G \in H(2,2,2)$, proving that $F(2,2,2) \leq 11$. Chvatal [1], showed that the Mycielski graph is the smallest possible graph in the class $H(2,2,2)$ and hence $F(2,2,2) = 11$. The equality $F(2,2,2,2) = 22$ is proved by Jensen and Royle in [4]. The inequality $F(3,3) \leq 14$ obviously implies $F(2,2,3) \leq 14$, but the exact value of $F(2,2,3)$ is unknown.

In this note we prove the following:

Theorem. $F(2,2,4) = 13$.

In the proof of this theorem, we shall use the following:

Lemma. *Let G be a 12-vertex graph with* $\mathrm{cl}(G) = 4$ *and* $\alpha(G) = 2$. *Then $G \notin H(2,2,4)$.*

P r o o f. Assume the opposite, i.e. $G \to (2,2,4)$. It is proved in [12] that the graph G is a subgraph of the graph P (the complementary graph $\overline{P}$ is given in Fig. 1). Hence $P \to (2,2,4)$. Since in 3-colouring $V(P) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{v_1, v_2\}$, $V_2 = \{v_5, v_6\}$, the sets V_1 and V_2 are independent and V_3 contains no 4-cliques, this is a contradiction. □

P r o o f o f t h e T h e o r e m.

1. Proof of the inequality $F(2,2,4) \leq 13$. We consider the graph Q, which complementary graph $\overline{Q}$ is given in Fig. 2. This graph is a well-known construction of Greenwood and Gleason [3], which shows that $R(3,5) \geq 14$. We prove the inequality $F(2,2,4) \leq 13$ by showing that $Q \in H(2,2,4)$. Obviously

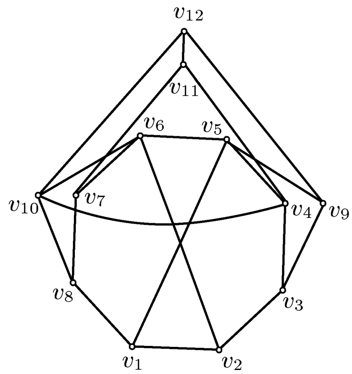

Fig. 1. Graph $\overline{P}$

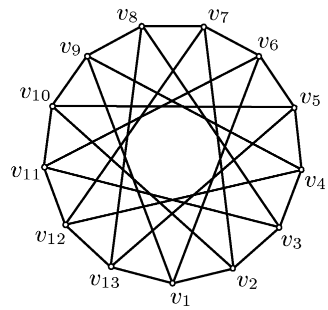

Fig. 2. Graph $\overline{Q}$

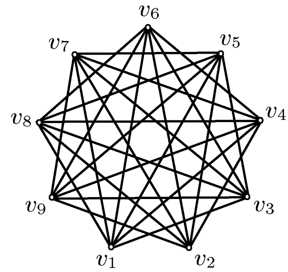

Fig. 3. Graph $\overline{C}_9$

$\alpha(Q) = 2$ and it is true that $\mathrm{cl}(Q) = 4$, [3]. Let $V_1 \cup V_2 \cup V_3$ be a 3-colouring of the vertices of the graph Q and suppose that V_1 and V_2 are independent sets of vertices in Q. From $\alpha(Q) = 2$ it follows that $|V_i| \leq 2$, $i = 1, 2$. Hence $|V_3| \geq 9$. From $\alpha(Q) = 2$ and $R(4, 3) = 9$ it follows that V_3 contains a 4-clique. So, $Q \in H(2, 2, 4)$. Since $|V(Q)| = 13$ it follows that $F(2, 2, 4) \leq 13$.

2. Proof of the inequality $F(2, 2, 4) \geq 13$. Assume the opposite. Let $G \in H(2, 2, 4)$ and $|V(G)| \leq 12$. By adding some isolated vertices, we may assume that $|V(G)| = 12$. Let A by an independent set of vertices of the graph G, $|A| = \alpha(G)$ and $G_1 = G - A$. From $G \in H(2, 2, 4)$ it follows that $G_1 \in H(2, 4)$. According to Theorem A, $|V(G_1)| \geq 9$. Hence $\alpha(G) = |A| \leq 3$. Since $\mathrm{cl}(G) = 4$, we have $\alpha(G) \geq 2$. The Lemma yields $|A| = 3$ and $|V(G_1)| = 9$. According to Theorem B, $G_1 = \overline{C}_9$ (see Fig. 3). We consider the set $M_1 = \{v_1, v_3, v_4, v_7, v_8\}$ of vertices of the graph $G_1 = \overline{C}_9$. We will prove that there is a vertex $u \in A$ such that $M_1 \subseteq \mathrm{Ad}(u)$. Assume the opposite. Then if $u \in A$ and $v_1, v_3, v_8 \in \mathrm{Ad}(u)$ it follows that $v_4 \notin \mathrm{Ad}(u)$ or $v_7 \notin \mathrm{Ad}(u)$. From $\mathrm{cl}(G) = 4$ it follows also that if $u \in A$ and $v_1, v_3, v_8 \in \mathrm{Ad}(u)$, then $v_5, v_6 \notin \mathrm{Ad}(u)$. We denote by W_1 the set of those of the vertices $u \in A$ for which $v_1, v_3, v_8 \in \mathrm{Ad}(u)$ and $v_4 \notin \mathrm{Ad}(u)$. By W_2 we denote the set of those $u \in A$ for which $v_1, v_3, v_4, v_8 \in \mathrm{Ad}(u)$ (and hence $v_7 \notin \mathrm{Ad}(u)$). Let $W_3 = A \setminus (W_1 \cup W_2)$. We consider the 3-colouring $V_1' \cup V_2' \cup V_3'$ of the $V(\overline{C}_9)$, where $V_1' = \{v_4, v_5\}$, $V_2' = \{v_6, v_7\}$. Let $V_i = V_i' \cup W_i$, $i = 1, 2, 3$. It is clear that $V_1 \cup V_2 \cup V_3$ is a 3-colouring of $V(G)$. Obviously, V_1 and V_2 are independent sets in G. Since V_3' have the unique 3-clique $\{v_1, v_3, v_8\}$, the set V_3 contains no 4-cliques, which is a contradiction.

So, there is a vertex $u \in A$ such that $M_1 \subseteq \mathrm{Ad}(u)$. The map σ defined by

$\sigma(v_i) = v_{i+1}$, $i = 1,\dots,8$, and $\sigma(v_9) = v_1$ is obviously an automorphism of the graph $G_1 = \overline{C}_9$. Hence for each $M_i = \sigma^{i-1}(M_1)$, $i = 1,\dots,9$, there is a vertex $u \in A$ such that $M_i \subseteq \mathrm{Ad}(u)$. From $|A| = 3$ it follows that for some of the vertices $u \in A$, there exist $i \neq j$, such that $M_i \cup M_j \subseteq \mathrm{Ad}(u)$. The set $M_i \cup M_j$, $i \neq j$, contains a 4-clique of the graph $\overline{C}_9$ (for example $M_1 \cup M_k$, $k \neq 1$ contains the 4-clique $\{v_1, v_3, v_5, v_8\}$ or the 4-clique $\{v_1, v_3, v_6, v_8\}$). Hence $\mathrm{cl}(G) \geq 5$, which is a contradiction. This ends the proof of the Theorem. □

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