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GROUPS WITH DECOMPOSABLE SET OF QUASINORMAL SUBGROUPS

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Abstract. A subgroup H of a group G is said to be *quasinormal* if $HX = XH$ for all subgroups X of G. In this article groups are characterized for which the partially ordered set of quasinormal subgroups is decomposable.

1. Introduction. A partially ordered set $\mathfrak{L}$ is said to be *decomposable* if it is isomorphic to the direct product of two non-trivial partially ordered sets. The structure of groups for which a certain relevant system $\mathfrak{X}$ of subgroups is decomposable has been investigated for several choices of the system $\mathfrak{X}$. In particular, about fifty years ago Suzuki [7] proved that the lattice $\mathfrak{L}(G)$ of all subgroups of a group G is decomposable if and only if G is the direct product of two periodic coprime non-trivial subgroups. Groups with decomposable lattice of normal subgroups have been later described by Curzio [2], and more recently decomposition problems for the ordered sets of subnormal, ascendant or other generalized normal subgroups have been studied (see [3], [4], [5]).

A subgroup H of a group G is said to be *quasinormal* (or *permutable*) if $HX = XH$ for every subgroup X of G. Clearly the join of any set of quasinormal subgroups is likewise quasinormal, but the intersection of two quasinormal

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subgroups of a group need not be quasinormal. The aim of this article is to characterize groups G for which the partially ordered set $qn(G)$ of all quasinormal subgroups is decomposable.

Let G be a group, and let H be a subgroup of G. The H-*norm* of G is the subgroup $ker(G:H)$ consisting of all elements g of G such that $X^g = X$ for each subgroup X of G containing H, i.e.

$$ker(G:H) = \bigcap_{H \leq X} N_G(X).$$

Clearly, $H \leq ker(G:H) \leq N_G(H)$, and $ker(G:H) \leq ker(G:K)$ if $H \leq K \leq G$. Moreover, the $\{1\}$-norm $ker(G:\{1\})$ is the *norm* $N(G)$ of G, introduced by Baer [1]; it is well-known that the subgroup $N(G)$ is contained in the second term of the upper central series of G.

Theorem. *Let G be a group. The partially ordered set $qn(G)$ of all quasinormal subgroups of G is decomposable if and only if $G = G_1 \times G_2$, where G_1 and G_2 are non-trivial subgroups of G satisfying the following conditions:*

(a) *every quasinormal subgroup of G_i is quasinormal in G (for $i = 1, 2$);*

(b) *if H_1 and H_2 are quasinormal subgroups of G_1 and G_2, respectively, the factor groups $ker(G_1 : H_1)/H_1$ and $ker(G_2 : H_2)/H_2$ have no elements of the same prime order.*

It should be observed that condition (a) in the above statement is not trivial, since quasinormal subgroups of direct factors need not be quasinormal in the whole group. To show this, let H be a finite p-group (p prime) containing a quasinormal non-normal subgroup X, and let x and h be elements of X and H, respectively, such that $h^{-1}xh \notin X$; if h has order p^n and $\langle z \rangle$ is a cyclic group of order p^n, then X is not quasinormal into the direct product $G = H \times \langle z \rangle$, since $X\langle hz \rangle \neq \langle hz \rangle X$.

Most of our notation is standard; in particular, if H and K are subgroups of a group G with $H \leq K$, we shall denote by $[K/H]$ the set of all subgroups X of G such that $H \leq X \leq K$. We will use the monograph [6] as a general reference on quasinormal subgroups and lattice properties of groups.

2. Proof of the Theorem.

Lemma 1. *Let G be a group, and let H be a quasinormal subgroup of G. Then every subgroup of $ker(G:H)$ containing H is quasinormal in G.*

Proof. Let X be a subgroup of G such that

$$H \leq X \leq ker(G:H),$$

and let Y be any subgroup of G. Since X is contained in the normalizer of HY, it follows that the product $XY = X(HY)$ is a subgroup of G. Therefore $XY = YX$, and X is a quasinormal subgroup of G. $\square$

Lemma 2. *Let the group $G = G_1 \times G_2$ be the direct product of two subgroups G_1 and G_2, and let H be a subgroup of G_1 which is quasinormal in G. If g is an element of $ker(G_1 : H)$ such that the subgroup $\langle g, H\rangle$ is quasinormal in G and the index $|\langle g, H\rangle : H|$ is a prime number, then g belongs to $ker(G : H)$.*

Proof. Let X be any subgroup of G containing H, and assume by contradiction that $X^g \neq X$. Put $K = \langle g, H\rangle$ and $|\langle g, H\rangle : H| = p$, so that g^p lies in X, and there exists an element x of X such that the conjugate x^g does not belong to $\langle x\rangle H$. Write $x = x_1x_2$, with $x_1 \in G_1$ and $x_2 \in G_2$. Obviously

$$\langle x\rangle^{\langle x_1\rangle H} = \langle x\rangle^H \leq \langle x\rangle H,$$

and hence $g \notin \langle x_1\rangle H$. It follows that

$$H \leq K \cap \langle x_1\rangle H < K,$$

and so $K \cap \langle x_1\rangle H = H$. Since the subgroup K is quasinormal in G, we have $xg = g^m h x^n$, where h is an element of H and m, n are integers. Thus

$$(x_1 g)x_2 = xg = g^m h x^n = (g^m h x_1^n)x_2^n,$$

and hence $x_1 g = g^m h x_1^n$ and $x_2 = x_2^n$. On the other hand, g belongs to the subgroup $ker(G_1 : H)$, and so $x_1^g = x_1^r h_1$, where r is an integer and h_1 is an element of H. It follows that $g^m h x_1^n = g x_1^r h_1$, and $u = g^{m-1}h = x_1^r h_1 x_1^{-n}$ belongs to $K \cap \langle x_1\rangle H = H$. Therefore

$$x^g = x_1^g x_2 = x_1^r h_1 x_2 = u x_1^n x_2 = u x^n$$

is an element of $\langle x\rangle H$, and this contradiction proves the lemma. $\square$

Let G be a group, and let X be any subgroup of G. Since every join of quasinormal subgroups is likewise quasinormal, we can define the *quasinormal core* of X (in G) as the largest quasinormal subgroup of G which is contained in X. In particular, if H and K are quasinormal subgroups of a group G, the quasinormal core of $H \cap K$ in G will be denoted by $H \wedge K$; then $H \cap K$ is quasinormal in G if and only if $H \cap K = H \wedge K$. Note also that the partially ordered set $qn(G)$ is a lattice, but it is not in general a sublattice of the subgroup lattice $\mathfrak{L}(G)$ of G.

Proof of the Theorem. Suppose first that the ordered set $qn(G)$ is decomposable, and let

$$\varphi : qn(G) \longrightarrow \mathfrak{L}_1 \times \mathfrak{L}_2$$

be an order isomorphism, where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are non-trivial partially ordered sets. Clearly $\mathfrak{L}_i$ has smallest element O_i and largest element I_i $(i = 1, 2)$, and the preimages $G_1 = \varphi^{-1}(I_1, O_2)$ and $G_2 = \varphi^{-1}(O_1, I_2)$ are quasinormal subgroups of G such that $G_1G_2 = G$ and $G_1 \wedge G_2 = \{1\}$. If a_1 and a_2 are elements of $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively, we have

$$(a_1, a_2) = \sup\{(a_1, O_2), (O_1, a_2)\}$$

and

$$(a_1, O_2) = \inf\{(a_1, a_2), (I_1, O_2)\}, \quad (O_1, a_2) = \inf\{(a_1, a_2), (O_1, I_2)\}.$$

It follows that, if H is any quasinormal subgroup of G, then

$$H = (H \wedge G_1)(H \wedge G_2),$$

and so also

$$H = \langle H \cap G_1, H \cap G_2 \rangle.$$

Let g be any element of G_1; then G_2^g is a quasinormal subgroup of G, and hence

$$G_2^g = (G_2^g \wedge G_1)(G_2^g \wedge G_2) = (G_2 \wedge G_1)^g(G_2^g \wedge G_2) = G_2^g \wedge G_2,$$

so that $G_2^g \leq G_2$ and G_2 is normal in $G = G_1G_2$. A similar argument shows that also G_1 is a normal subgroup of G; then $G_1 \cap G_2$ is normal in G, and so

$$G_1 \cap G_2 = G_1 \wedge G_2 = \{1\}.$$

Therefore $G = G_1 \times G_2$ is the direct product of its non-trivial subgroups G_1 and G_2. Let X be any quasinormal subgroup of G_1. Then the subgroup XG_2 is quasinormal in G, and hence

$$XG_2 = (XG_2 \wedge G_1)(XG_2 \wedge G_2) = (XG_2 \wedge G_1)G_2.$$

As $XG_2 \wedge G_1$ is contained in X, it follows that $X = XG_2 \wedge G_1$ is a quasinormal subgroup of G. It can be proved similarly that every quasinormal subgroup of G_2 is also quasinormal in G. Let H_1 and H_2 be quasinormal subgroups of G_1 and G_2, respectively, and assume by contradiction that the factor groups

$ker(G_1 : H_1)/H_1$ and $ker(G_2 : H_2)/H_2$ contain cyclic subgroups $K_1/H_1 = \langle g_1 H_1\rangle$ and $K_2/H_2 = \langle g_2 H_2\rangle$, respectively, with the same prime order p. It follows from Lemma 1 that K_1 and K_2 are quasinormal in G_1 and G_2, respectively. Then H_1, H_2, K_1, K_2 are quasinormal subgroups of G, so that also $H = H_1 H_2$ and $K = K_1 K_2$ are quasinormal in G. Moreover, H is normal in K and

$$K/H = \langle g_1 H\rangle \times \langle g_2 H\rangle$$

is an elementary abelian group of order p^2. Let L be any subgroup of G such that $H < L < K$, and put $L/H = \langle g_1^m g_2^n H\rangle$. Application of Lemma 2 yields that $g_1 \in ker(G : H_1)$ and $g_2 \in ker(G : H_2)$, so that the subgroup $\langle g_1, g_2\rangle$ is contained in $ker(G : H)$, and $L = \langle g_1^m g_2^n, H\rangle$ is quasinormal in G by Lemma 1. Therefore the interval $[K/H]$ of the lattice $\mathfrak{L}(G)$ is contained in $qn(G)$, and it follows immediately that the lattice $\mathfrak{L}(K/H) \simeq [K/H]$ is decomposable. This contradiction proves that $ker(G_1 : H_1)/H_1$ and $ker(G_2 : H_2)/H_2$ have no elements of the same prime order.

Conversely, suppose that the group $G = G_1 \times G_2$ satisfies the conditions of the statement, and let H be any quasinormal subgroup of G. Put $H_1 = H \cap G_1$ and $H_2 = H \cap G_2$, and assume that the subgroup $H_1 H_2$ is properly contained in H. Consider an element h of $H \setminus H_1 H_2$, and write $h = x_1 x_2$ with $x_1 \in G_1$ and $x_2 \in G_2$. Thus $x_1 \in K_1 \setminus H_1$ and $x_2 \in K_2 \setminus H_2$, where $K_1 = HG_2 \cap G_1$ and $K_2 = HG_1 \cap G_2$. Let X be any subgroup of G_1 containing H_1; then

$$X^{K_1} \leq X^{HG_2} = X^H \leq XH,$$

and hence

$$X^{K_1} \leq XH \cap G_1 = XH_1 = X.$$

Therefore K_1 is contained in $ker(G_1 : H_1)$. A similar argument shows that K_2 is a subgroup of $ker(G_2 : H_2)$. For $i = 1, 2$ the subgroup H_i is quasinormal in G_i, and Lemma 1 yields that every subgroup of K_i containing H_i is also quasinormal in G_i. It follows now from the hypotheses that the factor groups K_1/H_1 and K_2/H_2 are periodic with

$$\pi(K_1/H_1) \cap \pi(K_2/H_2) = \emptyset.$$

Since the non-trivial cosets $x_1 H_1$ and $x_2 H_2$ have the same order, we get a contradiction. Therefore

$$H = (H \cap G_1)(H \cap G_2)$$

for every quasinormal subgroup H of G. As the sets $qn(G_1)$ and $qn(G_2)$ are contained in $qn(G)$, the position

$$H^\varphi = (H \cap G_1, H \cap G_2)$$

defines an order isomorphism

$$\varphi : qn(G) \longrightarrow qn(G_1) \times qn(G_2),$$

and hence the partially ordered set $qn(G)$ is decomposable. $\square$

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