Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **27** (2001), 143-158

Serdica
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

EXAMPLES ILLUSTRATING SOME ASPECTS OF THE WEAK DELIGNE-SIMPSON PROBLEM

Vladimir Petrov Kostov

Communicated by J.-P. Françoise

To the memory of my mother

Abstract. We consider the variety of $(p+1)$-tuples of matrices A_j (resp. M_j) from given conjugacy classes $c_j \subset gl(n, \mathbf{C})$ (resp. $C_j \subset GL(n, \mathbf{C})$) such that $A_1 + \ldots + A_{p+1} = 0$ (resp. $M_1 \ldots M_{p+1} = I$). This variety is connected with the weak *Deligne-Simpson problem: give necessary and sufficient conditions on the choice of the conjugacy classes $c_j \subset gl(n, \mathbf{C})$ (resp. $C_j \subset GL(n, \mathbf{C})$) so that there exist $(p+1)$-tuples with trivial centralizers of matrices $A_j \in c_j$ (resp. $M_j \in C_j$) whose sum equals 0 (resp. whose product equals I).* The matrices A_j (resp. M_j) are interpreted as matrices-residua of Fuchsian linear systems (resp. as monodromy operators of regular linear systems) on Riemann's sphere. We consider examples of such varieties of dimension higher than the expected one due to the presence of $(p+1)$-tuples with non-trivial centralizers; in one of the examples the difference between the two dimensions is $O(n)$.

1. Introduction.

1.1. Formulation of the (weak) Deligne-Simpson problem. In the present article we consider examples related to the *Deligne-Simpson problem (DSP)*. The problem stems from the analytic theory of linear systems of ordinary differential equations but its formulation is purely algebraic:

2000 *Mathematics Subject Classification:* 15A30, 20G05.
Key words: regular linear system, Fuchsian system, monodromy group.
Research partially supported by INTAS grant 97-1644

Give necessary and sufficient conditions on the choice of the $p+1$ conjugacy classes $c_j \subset gl(n, \mathbf{C})$, resp. $C_j \subset GL(n, \mathbf{C})$, so that there exist irreducible $(p+1)$-tuples of matrices $A_j \in c_j$, $A_1 + \cdots + A_{p+1} = 0$, resp. of matrices $M_j \in C_j$, $M_1 \ldots M_{p+1} = I$.

Here I stands for the identity matrix and "irreducible" means "with no non-trivial common invariant subspace". The version with matrices A_j (resp. M_j) is called the *additive* (resp. the *multiplicative*) one. The matrices A_j are interpreted as *matrices-residua* of *Fuchsian* systems on Riemann's sphere (i.e. linear systems of ordinary differential equations with logarithmic poles). The sum of all matrices-residua of a Fuchsian system equals 0.

The matrices M_j are interpreted as *monodromy operators* of meromorphic linear *regular* systems on Riemann's sphere (i.e. linear systems of ordinary differential equations with moderate growth rate of the solutions at the poles). (Fuchsian systems are always regular.) A *monodromy operator* of a regular system is a linear operator acting on its solution space which maps the solution with a given initial value at a given *base point* a_0 onto the value at a_0 of its analytic continuation along some closed contour.

The monodromy operators generate the *monodromy group*. One usually chooses as generators of the monodromy group operators defined by contours which are freely homotopic to small loops each circumventing counterclockwise one of the poles of the system. For a suitable indexation of the poles the product of these generators equals I (and this is the only relation which they a priori satisfy).

Remark 1. In the multiplicative version the classes C_j are interpreted as *local monodromies* around the poles and the DSP admits the following interpretation:

For what $(p+1)$-tuples of local monodromies do there exist irreducible monodromy groups with such local monodromies.

The monodromy group of a regular system is the only invariant of a regular system under the linear changes of the dependent variables meromorphically depending on the time. Therefore the multiplicative version is more important than the additive one; nevertheless, the additive one is easier to deal with when computations are to be performed and one can easily deduce corollaries concerning the multiplicative version as well due to Remark 2.

Remark 2. If A denotes a matrix-residuum at a given pole of a Fuchsian system and if M denotes the corresponding operator of local monodromy, then in the absence of non-zero integer differences between the eigenvalues of A the operator M is conjugate to $\exp(2\pi i A)$.

By definition, the *weak DSP* is the DSP in which instead of irreducibility of the $(p+1)$-tuple of matrices one requires only its centralizer to be trivial. We say that the DSP (resp. the weak DSP) is *solvable* for a given $(p+1)$-tuple of conjugacy classes c_j or C_j if there exist matrices $A_j \in c_j$ whose sum is 0 or matrices $M_j \in C_j$ whose product is I such that their $(p+1)$-tuple is irreducible (resp. with trivial centralizer). By definition, the (weak) DSP is solvable for $n = 1$.

For given conjugacy classes c_j satisfying the condition $\sum \mathrm{Tr}(c_j) = 0$ and also the ones of Theorem 8 below consider the variety

$$\mathcal{V}(c_1, \ldots, c_{p+1}) = \{(A_1, \ldots, A_{p+1}) \mid A_j \in c_j \ , \ A_1 + \cdots + A_{p+1} = 0\}.$$

(One can define such a variety in a similar way in the case of matrices M_j as well.)

If the eigenvalues are generic (see the precise definition in Subsection 1.3), then the variety $\mathcal{V}(c_1, \ldots, c_{p+1})$ (or just $\mathcal{V}$ for short) is smooth, see [4]. If not, then it can have a complicated stratified structure defined by the invariants of the $(p+1)$-tuple of matrices; the centralizers of the $(p+1)$-tuples might be trivial on some strata and non-trivial on others; finally, a stratum on which the centralizer is non-trivial can be of greater dimension than the one of a stratum on which it is trivial; see [4] for some examples.

The aim of the present paper is to give further examples of varieties $\mathcal{V}$ and to discuss their stratified structure and dimension of the strata. This will be explained in some more detail in Subsection 1.4 after some necessary notions will be introduced in the next two subsections.

Remark 3. In what follows the sum of the matrices A_j is always presumed to be 0 and the product of the matrices M_j is always presumed to be I.

Notation 4. *Double subscripts indicate matrix entries. We denote by $E_{i,j}$ the matrix having zeros everywhere except in position (i,j) where it has a unit.*

1.2. Necessary conditions for the solvability of the (weak) DSP. The known results concerning the (weak) DSP are exposed in [2]. We recall in this and in the next subsection only the most necessary ones.

Definition 5. *A* Jordan normal form (JNF) of size n *is a collection of positive integers $\{b_{i,l}\}$ whose sum is n where $b_{i,l}$ is the size of the i-th Jordan block with the l-th eigenvalue; the eigenvalues are presumed distinct and for l fixed the numbers $b_{i,l}$ form a non-increasing sequence. Denote by $J(C)$ (resp. $J(A)$) the JNF defined by the conjugacy class C (resp. by the matrix A).*

Definition 6. *For a conjugacy class C in $GL(n, \mathbf{C})$ or $gl(n, \mathbf{C})$ denote by $d(C)$ its dimension; recall that it is always even. For a matrix $Y \in C$ set*

$r(C) := \min_{\lambda\in\mathbf{C}} \mathrm{rank}(Y-\lambda I)$. *The integer* $n-r(C)$ *is the maximal number of Jordan blocks of* $J(Y)$ *with one and the same eigenvalue. Set* $d_j := d(C_j)$ *(resp.* $d(c_j)$*),* $r_j := r(C_j)$ *(resp.* $r(c_j)$*). The quantities* $r(C)$ *and* $d(C)$ *depend only on the JNF* $J(Y) = J^n$*, not on the eigenvalues, so we write sometimes* $r(J^n)$ *and* $d(J^n)$*.*

The following two inequalities are necessary conditions for the existence of irreducible $(p+1)$-tuples of matrices A_j or M_j (their necessity in the multiplicative version was proved by C. Simpson, see [8], and in the additive one by the author, see [3]):

$$(\alpha_n) \qquad d_1 + \cdots + d_{p+1} \geq 2n^2 - 2,$$

$$(\beta_n) \qquad \text{for all } j,\ r_1 + \cdots + \hat{r}_j + \ldots + r_{p+1} \geq n.$$

The inequality

$$(\omega_n) \qquad r_1 + \cdots + r_{p+1} \geq 2n$$

is not a necessary condition (note that it implies (β_n)) but it is "almost sufficient", i.e. sufficient in most part of the cases, see the details in [2].

We formulate below a necessary condition for the solvability of the (weak) DSP which is a condition upon the $p+1$ JNFs $J_j^n = J(c_j)$ or $J(C_j)$ $(j = 1, \ldots, p+1$, the upper index indicates the size of the matrices) but not upon the classes c_j or C_j themselves.

Definition 7. *For a given* $(p+1)$*-tuple of JNFs* J_j^n *with* $n > 1$*, which satisfies condition* (β_n) *and doesn't satisfy condition* (ω_n) *set* $n_1 = r_1 + \cdots + r_{p+1} - n$*. Hence,* $n_1 < n$ *and* $n - n_1 \leq n - r_j$*. Define the* $(p+1)$*-tuple of JNFs* $J_j^{n_1}$ *as follows: to obtain the JNF* $J_j^{n_1}$ *from* J_j^n *one chooses one of the eigenvalues of* J_j^n *with greatest number* $n - r_j$ *of Jordan blocks, then decreases by 1 the sizes of the* $n - n_1$ smallest *Jordan blocks with this eigenvalue and deletes the Jordan blocks of size 0. Denote this construction by* $\Psi : (J_1^n, \ldots, J_{p+1}^n) \mapsto (J_1^{n_1}, \ldots, J_{p+1}^{n_1})$ *or just by* Ψ *for short.*

Theorem 8. *If the (weak) DSP is solvable for a given* $(p+1)$*-tuple of conjugacy classes* c_j *or* C_j *defining the JNFs* J_j^n*, satisfying condition* (β_n) *and not satisfying condition* (ω_n)*, then the map* Ψ *iterated as long as defined stops at a* $(p+1)$*-tuple of JNFs* $J_j^{n'}$ *either satisfying condition* $(\omega_{n'})$ *or with* $n' = 1$*.*

The theorem can be deduced from [2], see Theorem 8 there.

Remark 9. One can show that the results formulated by means of the map Ψ do not depend on the choice of an eigenvalue with maximal number of Jordan blocks belonging to it whenever such a choice is possible.

1.3. Generic eigenvalues and (poly)multiplicity vectors. We presume in the case of matrices M_j the necessary condition $\prod \det(C_j) = 1$ to hold. In the case of matrices A_j this is the condition $\sum \mathrm{Tr}(c_j) = 0$. In terms of the eigenvalues $\sigma_{k,j}$ (resp. $\lambda_{k,j}$) of the matrices from C_j (resp. c_j) repeated with their multiplicities, this condition reads $\prod_{k=1}^{n}\prod_{j=1}^{p+1}\sigma_{k,j} = 1$ (resp. $\sum_{k=1}^{n}\sum_{j=1}^{p+1}\lambda_{k,j} = 0$).

Definition 10. *An equality of the form* $\prod_{j=1}^{p+1}\prod_{k\in\Phi_j}\sigma_{k,j} = 1$, *resp.* $\sum_{j=1}^{p+1}\sum_{k\in\Phi_j}\lambda_{k,j} = 0$, *is called a* non-genericity relation*; the sets Φ_j contain one and the same number $< n$ of indices for all j. Eigenvalues satisfying none of these relations are called* generic. *If one replaces for all j the sets Φ_j by their complements in $\{1,\ldots,n\}$, then one obtains another non-genericity relation which we identify with the initial one.*

Remarks 11. 1) Reducible $(p+1)$-tuples exist only for non-generic eigenvalues. Indeed, if the $(p+1)$-tuple is block upper-triangular, then the eigenvalues of each diagonal block satisfy some non-genericity relation.

2) For generic eigenvalues the conditions of Theorem 8 are sufficient as well, see [2] (Theorem 8), [3] and [7].

Remark 12. Condition (β_n) admits the following generalizations which in certain cases of non-generic eigenvalues are stronger than (β_n) itself – these are the inequalities

$$(\delta_n) \qquad \min_{\substack{b_j\in\mathbf{C}\\ b_1+\ldots+b_{p+1}=0}} \sum_{j=1}^{p+1}\mathrm{rk}(A_j - b_jI) \geq 2n, \quad \min_{\substack{b_j\in\mathbf{C}^*\\ b_1\ldots b_{p+1}=1}} \sum_{j=1}^{p+1}\mathrm{rk}(b_jM_j - I) \geq 2n$$

which are necessary conditions for the existence of irreducible $(p+1)$-tuples of matrices A_j or M_j (see [2], Lemma 10 and the line after it).

Definition 13. *A* multiplicity vector (MV) *is a vector with positive integer components whose sum is n and which are the multiplicities of the eigenvalues of an $n\times n$-matrix. In the case of diagonalizable matrices the MV defines completely the JNF. A* polymultiplicity vector (PMV) *is a $(p+1)$-tuple of multiplicity vectors, the ones of the eigenvalues of the matrices A_j or M_j.*

Remark 14. For a diagonal JNF J^n defined by the MV $(m_1,\ldots,m_s)$, $m_1\geq\cdots\geq m_s$, one has $r(J^n) = m_2+\cdots+m_s$ and $d(J^n) = n^2 - m_1^2 - \cdots - m_s^2$. If the JNF $J(c_j)$ or $J(C_j)$ is diagonal, then the construction Ψ (see the previous subsection) results in decreasing the greatest component of the j-th MV by $n-n_1$.

1.4. The index of rigidity and the expected dimension of the variety $\mathcal{V}$.

Definition 15. *Call* index of rigidity *of the $(p+1)$-tuple of conjugacy classes c_j or C_j (or of the $(p+1)$-tuple of JNFs defined by them) the quantity $\kappa = 2n^2 - d_1 - \cdots - d_{p+1}$. This notion was introduced in [1].*

Remarks 16. 1) If condition (α_n) holds, then κ can take the values 2, 0, -2, -4, ...

2) If $\kappa = 2$ and the DSP is solvable for given conjugacy classes, then such $(p+1)$-tuples are unique up to conjugacy, see [1] and [8] for the multiplicative version; from this result one easily deduces the uniqueness in the additive version.

3) For $\kappa = 2$ the coexistence of irreducible and reducible $(p+1)$-tuples of matrices M_j is impossible, see [1], Theorem 1.1.2, or [4], Theorem 18. One can easily deduce from this fact that the same is true for matrices A_j.

Recall that the variety $\mathcal{V}$ was defined in Subsection 1.1..

Remarks 17. 1) If $\mathcal{V}$ is nonempty and if the eigenvalues are generic, then it contains only irreducible $(p+1)$-tuples, it is smooth and its dimension equals $1 - \kappa + n^2$, see [4].

2) If on some stratum of $\mathcal{V}$ the centralizer is trivial, then the stratum is smooth and its dimension equals $1 - \kappa + n^2$; we call this dimension the *expected dimension* of $\mathcal{V}$, see [4], Proposition 2.

In the present paper we consider examples of varieties $\mathcal{V}$ for conjugacy classes satisfying the conditions of Theorem 8. The first of them (see Section 2) is with $\kappa = 2$, n odd and $p = 3$. We discuss its stratified structure and we show that it contains a stratum (on which the centralizer is non-trivial) of dimension $n^2 + (n-3)/2$, i.e. exceeding the expected one by $(n-1)/2$, as well as strata of dimensions $n^2 + s - 1$ for $s = 1, 2, \ldots, (n-1)/2$. And it contains strata of expected dimension $n^2 - 1$ on which the centralizer is trivial.

The second example (see Section 3) is one with $\kappa = 2$ where the variety $\mathcal{V}$ is not connected (but its closure is).

In Section 4 we show that for $(-\kappa)$ arbitrarily high there exist examples of varieties $\mathcal{V}$ in which certain strata where the centralizer is non-trivial do not belong to the closures of the strata where it is trivial and are of dimension higher than the expected one. This provides a negative answer to a question stated in [4].

2. An example with $p = 3$.

2.1. Description of the example. Consider for $p = 3$, $n = 2k+1$, $k \in \mathbf{N}^*$, the PMV $\Lambda(k) = ((k+1,k),(k+1,k),(k+1,k),(k+1,k))$ (the matrices A_j are presumed diagonalizable; a similar example can be given for matrices M_j

as well). Denote the respective eigenvalues of the matrices A_j with such a PMV by λ_j, μ_j (λ_j is of multiplicity $k+1$, one has $\lambda_j \neq \mu_j$, $j = 1,2,3,4,$). One has $d_j = 2k(k+1)$, see Remark 14, hence, $\kappa = 2$. The PMV $\Lambda(k)$ satisfies the conditions of Theorem 8 – one has $\Psi(\Lambda(k)) = \Lambda(k-1)$ (see Remark 14) and the iterations of Ψ stop at a quadruple of JNFs of size 1.

We assume that $\sum\limits_{j=1}^{4} \lambda_j = 0$ (A) is the only non-genericity relation satisfied by the eigenvalues (note that it implies $\sum\limits_{j=1}^{4} \mu_j = 0$).

Proposition 18. *Any quadruple of matrices A_j like above whose sum is 0 is up to conjugacy block upper-triangular, with diagonal blocks of sizes 1 or 2. The diagonal blocks of size 1 equal either $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ or $(\mu_1, \mu_2, \mu_3, \mu_4)$. The restriction of A_j to a diagonal block of size 2 has eigenvalues λ_j, μ_j.*

The propositions from this subsection are proved in the next ones.

Example 19. *There exist irreducible quadruples of 2×2-matrices B_j whose sum is 0 and with eigenvalues λ_j, μ_j:*

$$B_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \mu_1 \end{pmatrix}, \; B_2 = \begin{pmatrix} \lambda_2 & -1 \\ 0 & \mu_2 \end{pmatrix}, \; B_3 = \begin{pmatrix} \lambda_3 & 0 \\ u & \mu_3 \end{pmatrix}, \; B_4 = \begin{pmatrix} \lambda_4 & 0 \\ -u & \mu_4 \end{pmatrix}$$

where $u \in \mathbf{C}^*$.

Proposition 20. 1) *The variety Π of quadruples of diagonalizable 2×2-matrices B_j with eigenvalues λ_j, μ_j and such that $B_1 + \cdots + B_4 = 0$ is connected.*

2) *Its subvariety Π_0 consisting of all such irreducible quadruples is also connected.*

Example 21. *For $l \in \mathbf{N}^*$ there exist upper-triangular quadruples of $(2l+1) \times (2l+1)$-matrices H_j with zero sum, with trivial centralizers and with eigenvalues λ_j, μ_j of multiplicity $l+1$, l (the matrix I is $l \times l$):*

$$H_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 I & 0 \\ 0 & 0 & \mu_1 I \end{pmatrix}, \; H_2 = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 I & I \\ 0 & 0 & \mu_2 I \end{pmatrix}, \; H_3 = \begin{pmatrix} \lambda_3 I & 0 & I \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \mu_3 I \end{pmatrix}$$

(pay attention to the block-decomposition of H_3 which is different from the one of H_1 and H_2). If for $Z \in gl(2l+1, \mathbf{C})$ one has $[Z, H_1] = [Z, H_2] = [Z, H_3] = 0$, then $[Z, H_1] = 0$ implies that Z is block-diagonal, with diagonal blocks of sizes $(l+1) \times (l+1)$ and $l \times l$. One deduces then from $[Z, H_2] = [Z, H_3] = 0$ that Z is scalar (the details are left for the reader).

Remark 22. Examples similar to the above one can be given for other permutations of the eigenvalues on the diagonal as well (e.g. when all μ_j come first followed by all λ_j). For some permutations there exist no examples of such upper-triangular quadruples with trivial centralizers (e.g. when the first and the last eigenvalues on the diagonal are equal – in this case the matrices commute with $E_{1,2l+1}$).

Definition 23. *An irreducible quadruple of diagonalizable 2×2-matrices with eigenvalues (λ_j, μ_j) whose sum is 0 is said to be of* type B. *(Example* 19 *shows such a quadruple.)*

An upper-triangular quadruple with trivial centralizer of diagonalizable $h \times h$-matrices ($h = 2l + 1$) with eigenvalues λ_j, μ_j of multiplicities $l + 1$, l is said to be of type H_h. *(Example* 21 *shows such a quadruple.)*

A stratum of $\mathcal{V}$ the quadruples of which up to conjugacy are block-diagonal, with s diagonal blocks of type B defining non-equivalent representations and with one diagonal block of type H_{n-2s} is said to be of type HB_s.

Proposition 24. 1) *A stratum of type HB_s is locally a smooth algebraic variety of dimension $n^2 + s - 1$.*

2) *It is globally connected.*

Proposition 25. *The variety $\mathcal{V}$ from the example is connected.*

Conclusive remarks. As we saw, the stratum of type HB_0 (Example 21) consists of quadruples with trivial centralizers and is of dimension $n^2 - 1$ (the expected one) while the one of type $HB_{(n-1)/2}$ is of dimension $n^2+(n-3)/2$. All intermediate dimensions are attained on the strata HB_s, see Proposition 24. For $s > 0$ they consist of quadruples with non-trivial centralizers (they are block-diagonal up to conjugacy). Except the strata of type HB_s there are other strata of $\mathcal{V}$ with non-trivial centralizer, e.g. such on which the representation is a direct sum of some representations of type B and a representation with a non-trivial centralizer as mentioned in Remark 22.

2.2. Proof of Proposition 18. 1^0. It is clear that there exist diagonal quadruples of matrices A_j like above whose sum is 0 (their first $k + 1$ diagonal entries equal λ_j and the last k ones equal μ_j, see the non-genericity relation (A)). It follows from 3) of Remarks 16 that there exist no irreducible such quadruples.

2^0. Any reducible quadruple can be conjugated to a block upper-triangular form. The restriction of the quadruple to each diagonal block B is presumed to define an irreducible representation.

If the size l of the block is odd and > 1, then the minimal possible value of κ for this block is 2 and it is attained only when for each j the multiplicities of λ_j and μ_j as eigenvalues of $A_j|_B$ equal $(s + 1, s)$ or $(s, s + 1)$ where $l = 2s + 1$. (To prove this one can use Remark 14.) The absence of non-genericity relations other than (A) implies that the multiplicity of λ_j (and, hence, the one of μ_j) is one and the same for all j. However, the existence of diagonal quadruples of matrices A_j of size l with such multiplicities of λ_j, μ_j implies that such blocks B do not exist.

3^0. If the size of B is $2m$, $m \in \mathbf{N}^*$, then the minimal possible value of κ is 0 and it is attained only when the multiplicities of λ_j and μ_j as eigenvalues of $A_j|_B$ equal (m, m) (the easy computation is left for the reader). Such blocks B exist only for $m = 1$, see [6].

4^0. If the size of B is $2m$, $m \in \mathbf{N}^*$, and if $\kappa = 2$, then this can happen only if for three of the indices j the multiplicities of λ_j and μ_j as eigenvalues of $A_j|_B$ equal (m, m) and for the fourth one (say, for $j = 4$) they equal $(m-1, m+1)$ or $(m+1, m-1)$ (we leave the proof for the reader again). This together with (A) implies that $\lambda_4 = \mu_4$ which is impossible. Hence, such blocks B do not exist.

Hence, only blocks of size 1 and of size 2 are possible to occur on the diagonal (for the ones of size 2 see 3^0). The fact that (A) is the only non-genericity relation implies that the blocks of size 1 equal either $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ or $(\mu_1, \mu_2, \mu_3, \mu_4)$.

The proposition is proved. $\square$

2.3. Proof of Proposition 20. 1^0. Prove 1). Denote by c_j^* the conjugacy class of the matrix B_j. Denote by τ the quantity $\mathrm{tr}(B_1 + B_2)$. By varying the matrices B_1 and B_2 (resp. B_3 and B_4) within their conjugacy classes one can obtain as their sum $S = B_1 + B_2$ (resp. as $-(B_3 + B_4)$) any non-scalar matrix from the set $\Delta(\tau)$ of 2×2-matrices with trace equal to τ.

Indeed, if $S_{1,2} = g \neq 0$, then set $B_1 = \begin{pmatrix} \lambda_1 & 0 \\ u & \mu_1 \end{pmatrix}$, $B_2 = \begin{pmatrix} h & g \\ w & \lambda_2 + \mu_2 - h \end{pmatrix}$. One fixes first h to obtain the necessary entry $S_{1,1}$. One has $g \neq 0$, hence, there exists a unique w satisfying the condition $\det(B_2) = \lambda_2\mu_2$; after this one chooses u to obtain the necessary entry $S_{2,1}$.

2^0. If $S_{1,2} = 0$, then one can conjugate S by some matrix $Y \in GL(2, \mathbf{C})$ to obtain the condition $S_{1,2} \neq 0$, find the matrices B_1 and B_2 like above and then conjugate them (and S) by Y^{-1}. This is possible to do because S is not scalar.

The sets $\Delta(\tau)$ and $\Delta(\tau) \backslash \{\tau I/2\}$ being connected so is the variety Π. Indeed, one has $\Pi = \{(B_1, B_2, B_3, B_4) | B_j \in c_j^*, B_1 + B_2 = -(B_3 + B_4)\}$.

3^0. Prove 2). If the quadruple of matrices B_j is reducible, then so is the couple B_1, B_2, hence, the eigenvalues of $B_1 + B_2$ equal either $(\lambda_1 + \lambda_2, \mu_1 + \mu_2)$ or $(\lambda_1 + \mu_2, \mu_1 + \lambda_2)$.

The subset $\Delta^0(\tau)$ of $\Delta(\tau)$ defined by the condition the eigenvalues of a matrix from $\Delta(\tau)$ to equal either $(\lambda_1 + \lambda_2, \mu_1 + \mu_2)$ or $(\lambda_1 + \mu_2, \mu_1 + \lambda_2)$ is a proper subvariety of the smooth irreducible variety $\Delta(\tau)$. Therefore the connectedness of Π_0 is proved just like the one of Π, by replacing $\Delta(\tau)$ by $\Delta(\tau)\backslash\Delta^0(\tau)$. $\Box$

2.4. Proof of Proposition 24. 1^0. Prove 1). Denote by Σ the variety of block-diagonal quadruples whose first s diagonal blocks are of type B and the last block *up to conjugacy* is of type H_{n-2s}. Each of the first s blocks defines a smooth variety of dimension 5, see Remarks 17. The last block defines a smooth variety of dimension $(n-2s)^2-1$. Hence, $\dim\,\Sigma = 5s+(n-2s)^2-1$. To deduce $\dim\,(HB_s)$ from $\dim\,\Sigma$ one has to add to $\dim\,\Sigma$ the dimension of a transversal T to the group of infinitesimal conjugations preserving the block-diagonal form of the quadruple. This is the group $\mathcal{G}$ of block-diagonal matrices (which are deformations of I) with the same sizes of the diagonal blocks as the ones of the quadruple (we leave the proof of this statement for the reader; use the fact that the diagonal blocks define non-equivalent representations the first s of which of type B and the last of type H_{n-2s}). Hence, $\dim\,\mathcal{G} = 4s+(n-2s)^2$, $\dim\,T = n^2-4s-(n-2s)^2$ and $\dim\,\mathcal{V} = n^2-4s-(n-2s)^2+5s+(n-2s)^2-1 = n^2+s-1$.

The stratum of type HB_s is locally diffeomorphic to $\Sigma\times T$, hence, it is smooth.

2^0. Prove 2). The variety Π_0 of quadruples of matrices of type B is connected, see Proposition 20. It is smooth as well, hence, it is irreducible. Hence, the cartesian product of s copies of Π_0 is connected; if one deletes from it the subvariety on which two of the representations are equivalent, then the resulting variety is still connected.

Hence, the variety Σ is connected. The connectedness of Σ and the one of $GL(n,\mathbf{C})$ imply the one of the stratum of type HB_s. $\Box$

2.5. Proof of Proposition 25. Every quadruple of matrices A_j from $\mathcal{V}$ can be conjugated to a block upper-triangular form with diagonal blocks of sizes 1 or 2 (Proposition 18). Conjugate the quadruple by a suitable one-parameter family of diagonal matrices to make the entries of all blocks above the diagonal tend to 0 while preserving the diagonal blocks. The limit quadruple (denoted by $(A_1',\ldots,A_4')$) also belongs to $\mathcal{V}$. Indeed, the restriction of A_j' to each diagonal block of size 2 is diagonalizable, hence, A_j' is diagonalizable, the eigenvalues and their multiplicities are the same as for A_j and the sum of the matrices A_j' is 0.

After this deform continuously the blocks of size 2 so that they become diagonal (by Proposition 20 this is possible). The resulting quadruple is diagonal. It is a direct sum of $k+1$ quadruples $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and of k quadruples $(\mu_1,\mu_2,\mu_3,\mu_4)$. It is unique up to conjugacy and can be reached by continuous deformation from any quadruple of $\mathcal{V}$. Hence, $\mathcal{V}$ is connected. $\Box$

3. The variety $\mathcal{V}$ is not always connected. We illustrate the title of the section by the following

Example 26. *Consider the case $n = 2$, $p = 2$, the conjugacy classes c_1 and c_2 being diagonalizable, with eigenvalues $\pi, 2$ and $1-\pi, -1$, the conjugacy class c_3 consisting of the non-scalar matrices with eigenvalues $-1, -1$. Hence, $\kappa = 2$ and the triple of conjugacy classes satisfies the conditions of Theorem 8 (to be checked directly). A priori $\mathcal{V}$ contains at least the following two components (denoted by $\mathcal{V}_1$ and $\mathcal{V}_2$). In $\mathcal{V}_1$ the triples of matrices A_j equal (up to conjugacy)*

$$A_1 = \begin{pmatrix} \pi & 1 \\ 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1-\pi & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

In $\mathcal{V}_2$ they equal (up to conjugacy)

$$A_1 = \begin{pmatrix} \pi & 0 \\ 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1-\pi & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.$$

The variety $\mathcal{V}$ contains no irreducible triples, see 3) of Remarks 16. Hence, every triple from $\mathcal{V}$ is triangular up to conjugacy but not diagonal (otherwise A_3 must be scalar). Hence, $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$.

On the other hand, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ because the eigenvalues with which each matrix acts on the invariant subspace are different for the two components. Hence $\mathcal{V}$ is disconnected.

In the above example, however, the closure of $\mathcal{V}$ is connected. Indeed, consider the matrices

$$A_1'(\varepsilon) = \begin{pmatrix} \pi & \varepsilon \\ 0 & 2 \end{pmatrix}, \quad A_2'(\varepsilon) = \begin{pmatrix} 1-\pi & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3'(\varepsilon) = \begin{pmatrix} -1 & -\varepsilon \\ 0 & -1 \end{pmatrix},$$

where $\varepsilon \in (\mathbf{C}, 0)$. For $\varepsilon \neq 0$ this is a triple of matrices from $\mathcal{V}_1$, for $\varepsilon = 0$ this is a triple from its closure (but not from $\mathcal{V}$ because A_3 is scalar). In the same way, for $\varepsilon \neq 0$ the matrices

$$A_1''(\varepsilon) = \begin{pmatrix} \pi & 0 \\ \varepsilon & 2 \end{pmatrix}, \quad A_2''(\varepsilon) = \begin{pmatrix} 1-\pi & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3''(\varepsilon) = \begin{pmatrix} -1 & 0 \\ -\varepsilon & -1 \end{pmatrix}$$

belong to $\mathcal{V}_2$, for $\varepsilon = 0$ they belong to its closure but not to $\mathcal{V}$ and one has $A_j'(0) = A_j''(0)$.

In the above example the disconnectedness of $\mathcal{V}$ seems to result from the class c_3 not being closed. It would be interesting to prove or disprove that the closure of $\mathcal{V}$ is always connected.

4. Another example.

4.1 Description of the example. *For values of $(-\kappa)$ arbitrarily big there exist examples when a component of $\mathcal{V}$ does not lie in the closure of the* *union of its components on which the centralizer is trivial, and is of dimension higher than the expected one.*

Indeed, consider the following example. Suppose that $p > 3$ and that the $(p+1)$ conjugacy classes c_j (or C_j) are diagonalizable, each MV being of the form $(m_j, 1, 1, \ldots, 1)$, $3 \leq m_j \leq n-1$. Hence, $r_j = n - m_j$. Suppose that $r_1 + \cdots + r_{p+1} = 2n - 2$.

Lemma 27. *The $(p+1)$-tuple of conjugacy classes c_j or C_j like above satisfies the conditions of Theorem* 8.

Indeed, one has $n_1 = n-2$ and applying Ψ once one obtains a $(p+1)$-tuple of conjugacy classes satisfying condition (ω_{n-2}), see Remark 14. □

Denote by μ_j the eigenvalue of A_j (or of M_j) of multiplicity m_j. Suppose that there holds the only non-genericity relation $\mu_1 + \cdots + \mu_{p+1} = 0$ $(*)$ (resp. $\mu_1 \ldots \mu_{p+1} = 1$).

Remark 28. There exists no irreducible $(p+1)$-tuple of matrices $A_j \in c_j$ whose sum is 0, see Remark 12. Indeed, condition (δ_n) from Remark 12 does not hold – set $b_j = \mu_j$ and recall that $(*)$ holds; then $\mathrm{rk}(A_j - b_j I) = r_j$ and $r_1 + \ldots + r_{p+1} = 2n - 2 < 2n$. A similar remark holds for matrices M_j as well.

Define the conjugacy classes $c_j^* \subset gl(n-2, \mathbf{C})$ and $c_j' \subset gl(n-1, \mathbf{C})$ (resp. $C_j^* \subset GL(n-2, \mathbf{C})$ and $C_j' \subset GL(n-1, \mathbf{C})$) as obtained from c_j (resp. from C_j) by keeping the distinct eigenvalues the same and by decreasing the multiplicity of μ_j by 2 and by 1; the JNFs defined by the conjugacy classes c_j^*, C_j^*, c_j' and C_j' are diagonal. Hence, the sum (resp. the product) of all eigenvalues of the classes c_j^* and c_j' (resp. C_j^* and C_j') counted with the multiplicities equals 0 (resp. 1).

Condition (ω_{n-2}) holds for the classes c_j^* or C_j^* while condition (ω_{n-1}) holds for the classes c_j' or C_j'. By Theorem 2 from [5] (we need $p > 3$ to apply it), the DSP is solvable for the classes c_j^* (resp. C_j^*) and c_j' (resp. C_j'). Denote by $H_j \in c_j^*$ (resp. $H_j \in C_j^*$) and $G_j \in c_j'$ (resp. $G_j \in C_j'$) matrices with sum equal to 0 (resp. with product equal to I) whose $(p+1)$-tuple is irreducible.

Proposition 29. 1) *There exist $(p+1)$-tuples of $n \times n$-matrices with trivial centralizers, whose sums equal* 0 *(or whose products equal I) and blocked as follows:*

$$A_j \text{ (or } M_j) = \begin{pmatrix} H_j & R_j & Q_j \\ 0 & \mu_j & 0 \\ 0 & 0 & \mu_j \end{pmatrix} \quad \text{or} \quad A_j \text{ (or } M_j) = \begin{pmatrix} \mu_j & 0 & T_j \\ 0 & \mu_j & S_j \\ 0 & 0 & H_j \end{pmatrix}.$$

2) *Any $(p+1)$-tuple with trivial centralizer of matrices $A_j \in c_j$ or $M_j \in C_j$ is up to conjugacy block upper-triangular, with all diagonal blocks but one being equal, of size one, the restriction of A_j or M_j to such a block being equal to μ_j.*

The different block is first or last on the diagonal. The number of diagonal blocks is ≥ 3.

The proposition is proved in the next subsection.

Consider the stratum $\mathcal{U} \subset \mathcal{V}$ of $(p+1)$-tuples of matrices which up to conjugacy are of the form $\tilde{G}_j = \begin{pmatrix} G_j & 0 \\ 0 & \mu_j \end{pmatrix}$.

Lemma 30. *A point of the stratum* $\mathcal{U}$ *does not belong to the closure of any of the strata on which the centralizer is trivial.*

Indeed, the matrix algebra generated by the $(p+1)$-tuples of matrices defined by a point of the stratum $\mathcal{U}$ contains a matrix with distinct eigenvalues (the $(p+1)$-tuple of matrices G_j is irreducible and defines a representation not equivalent to $(\mu_1, \ldots, \mu_{p+1})$) while each matrix from an algebra defined by a point of any stratum of $\mathcal{V}$ where the centralizer is trivial has an eigenvalue of multiplicity ≥ 2, see 2) of Proposition 29. $\square$

Proposition 31. 1) *One has* $\dim \mathcal{U} = 3(n-1)^2 + 1 - \sum_{j=1}^{p+1} r_j^2$.

2) *The dimension of each of the two strata of* $\mathcal{V}$ *(denoted by* $\mathcal{W}_1$, $\mathcal{W}_2$*) of the* $(p+1)$*-tuples of matrices which up to conjugacy are like the ones from* 1) *of Proposition* 29 *is the expected one; it equals* $3(n-1)^2 - \sum_{j=1}^{p+1} r_j^2 = \dim \mathcal{U} - 1$.

The proposition is proved in Subsection 4.3. It implies that $\dim \mathcal{U} > \dim \mathcal{W}_i$, i.e. $\dim \mathcal{U}$ is greater than the expected dimension.

4.2. Proof of Proposition 29. We prove the proposition only in the case of matrices A_j and for the left $(p+1)$-tuple of matrices given in 1) of the proposition leaving for the reader the proof in the other cases – it can be performed in a similar way. We prove part 1) in $1^0 - 2^0$ and part 2) in $3^0 - 4^0$.

1^0. Denote by H an irreducible $(p+1)$-tuple of matrices like in the proposition as well as the representation defined by it and by μ the $(p+1)$-tuple $(\mu_1, \ldots, \mu_{p+1})$.

One has $\delta := \dim \mathrm{Ext}^1(H, \mu) = \dim \mathrm{Ext}^1(\mu, H) = 2$. Indeed, $\delta = \dim(\mathcal{L}/\mathcal{N})$ where

$$\mathcal{L} = \{(L_1, \ldots, L_{p+1}) \mid L_j = (H_j - \mu_j I)X_j,\ L_1 + \cdots + L_{p+1} = 0\},$$

$$\mathcal{N} = \{((H_1 - \mu_1 I)X, \ldots, (H_{p+1} - \mu_{p+1} I)X)\}$$

where the matrices X_j and X are $(n-2) \times 1$.

The dimension of the space of matrices of the form $(H_j - \mu_j I)Y =$ (where Y is $(n-2) \times 1$) equals r_j. The condition $L_1 + \cdots + L_{p+1} = 0$ is equivalent to $n-2$ linearly independent conditions (their linear independence follows easily from the fact that the representation H is irreducible and not equivalent to the one-dimensional representation μ). Hence, $\dim \mathcal{L} = r_1 + \cdots + r_{p+1} - n + 2 = n$. The same kind of argument shows that $\dim \mathcal{N} = n-2$ which implies that $\delta = 2$.

2^0. It follows from 1^0 that one can construct two linearly indepent $(p+1)$-tuples of $(n-2)\times 1$-matrices belonging to the space $\mathcal{L}/\mathcal{N}$ – these are the $(p+1)$-tuples of matrices R_j and Q_j. Show that the centralizer of the thus constructed $(p+1)$-tuple of matrices is trivial. Denote a matrix from the centralizer by $Z = \begin{pmatrix} K & B & C \\ D & e & f \\ U & v & w \end{pmatrix}$ where K is of the size of H_j etc.

The commutation relations imply $UH_j - \mu_j U = 0$, $j = 1, \ldots, p+1$. It follows from H and μ being non-equivalent and H being irreducible that $U = 0$. But then $DH_j - \mu_j D = 0$, $j = 1, \ldots, p+1$, and in the same way one obtains $D = 0$.

Hence, one has $[K, H_j] = 0$, $j = 1, \ldots, p+1$, which implies that $K = \alpha I$, $\alpha \in \mathbf{C}$ (recall that H is irreducible).

This in turn implies that

$$(H_j - \mu_j I)B + (e-\alpha)R_j + vQ_j = 0, \quad (H_j - \mu_j I)C + fR_j + (w-\alpha)Q_j = 0.$$

The definition of the matrices R_j and Q_j implies that $B = C = 0$, $v = f = 0$, $e = w = \alpha$. Hence, the centralizer is trivial. This proves 1) of the proposition.

3^0. Prove 2). Recall that there exists no irreducible $(p+1)$-tuple of matrices $A_j \in c_j$ whose sum is 0 (Remark 28) and that $(*)$ is the only non-genericity relation satisfied by the eigenvalues. Hence, every $(p+1)$-tuple of matrices $A_j \in c_j$ whose sum is 0 is up to conjugacy block upper-triangular and all diagonal blocks but one (denoted by D) are of size 1 and the restrictions of A_j to them equal μ_j. (The block D can also be of size 1 but in this case $A_j|_D \neq \mu_j$.)

4^0. If the first and the last diagonal blocks are equal, then the centralizer of the $(p+1)$-tuple is non-trivial – it contains the matrix $E_{1,n}$. So assume that the first diagonal block is different from all others (the case when this is the last block can be treated in a similar way). If there are only two diagonal blocks (D of size $n-1$ and μ of size 1), then one has $\dim \mathrm{Ext}^1(D,\mu) = 0$ (this follows from $r_1 + \cdots + r_{p+1} = 2n-2$) and, hence, the representation defined by the matrices A_j is a direct sum. Hence, if the centralizer of the $(p+1)$-tuple is trivial, then there are at least three diagonal blocks. □

4.3. Proof of Proposition 31. We prove the proposition only for matrices A_j, for matrices M_j the proof is similar.

1^0. Prove 1). The dimension of the variety of irreducible $(p+1)$-tuples of matrices $G_j \in c'_j$ with zero sum equals $u' = \left(\sum_{j=1}^{p+1} d(c'_j)\right) - ((n-1)^2-1)$, see Remarks 17. Hence, the dimension of block-diagonal matrices $\tilde{G}_j$ whose sum is 0 equals u'. To obtain $\dim \mathcal{U}$ one has to add to u' the dimension of a transversal T at I to the subgroup of $GL(n, \mathbf{C})$ of block-diagonal matrices with diagonal blocks of sizes $n-1$ and 1, the only ones conjugation with which preserves the block-diagonal form of the $(p+1)$-tuple. One has $d(c'_j) = (n-1)^2 - r_j - (n-1-r_j)^2 = (2n-3)r_j - r_j^2$ (see Remark 14) and $\dim T = 2n-2$. Hence,

$$\dim \mathcal{U} = (2n-3)\sum_{j=1}^{p+1} r_j - \sum_{j=1}^{p+1} r_j^2 - ((n-1)^2-1) + 2n-2 = 3(n-1)^2 + 1 - \sum_{j=1}^{p+1} r_j^2.$$

2^0. Prove 2). The dimension of the variety of $(p+1)$-tuples of matrices $H_j \in c_j^*$ whose sum is 0 equals $u^* = \left(\sum_{j=1}^{p+1} d(c_j^*)\right) - ((n-2)^2-1)$ (computed like u', by changing $n-1$ to $n-2$). Hence, this is the dimension of the variety of $n \times n$-matrices which are block-diagonal, with diagonal blocks equal to H_j, μ_j, μ_j. Note that $d(c_j^*) = (n-2)^2 - r_j - (n-2-r_j)^2 = (2n-5)r_j - r_j^2$ (Remark 14).

3^0. The dimension of each of the two varieties of $(p+1)$-tuples of matrices like in 1) of Proposition 29 equals $u^* + 2\sum_{j=1}^{p+1} r_j - 2(n-2) = u^* + 2n$. Indeed, each of the matrices Q_j, R_j or T_j, S_j belongs to a linear space of dimension r_j (this is the image of the linear operator $(.) \mapsto (H_j - \mu_j I)(.)$ or $(.) \mapsto (.)(H_j - \mu_j I)$ acting on $\mathbf{C}^{n-2}$). One has to subtract $2(n-2)$ because $\sum R_j = \sum Q_j = 0$ and $\sum T_j = \sum S_j = 0$.

4^0. One can consider the matrices from 1) of Proposition 29 like block upper-triangular, with two diagonal blocks the lower of which is of size 2 and is scalar. The subgroup of $GL(n, \mathbf{C})$ conjugation with which preserves this form is the subgroup of block upper-triangular matrices with diagonal blocks of sizes $n-2$ and 2. Hence, a transversal at I to it is of dimension $2(n-2)$ and one has

$$\dim \mathcal{W}_i = u^* + 2n + 2(n-2) = u^* + 4n - 4 =$$

$$= (2n-5)\sum_{j=1}^{p+1} r_j - \sum_{j=1}^{p+1} r_j^2 - ((n-2)^2-1) + 4n - 4 = 3(n-1)^2 - \sum_{j=1}^{p+1} r_j^2. \quad \square$$

REFERENCES

[1] N. M. Katz. Rigid local systems. Annals of Mathematics, Studies Series, Study **139**, Princeton University Press, 1995.

[2] V. P. Kostov. On the Deligne-Simpson problem. *C. R. Acad. Sci. Paris,* Série I **329** (1999), 657–662.

[3] V. P. Kostov. On the Deligne-Simpson problem. Electronic preprint math.AG/0011013.

[4] V. P. Kostov. Some examples related to the Deligne-Simpson problem (with Appendix by Ofer Gabber). In: Proceedings of the Second International Conference on Geometry, Integrability and Quantization, Sts. Constantine and Elena, Bulgaria, June 7-15, 2000, (Eds Ivailo M. Mladenov, Gregory L. Naber) ISBN 954-90618-2-5, 208–227, electronic preprint math.AG/0011015.

[5] V. P. Kostov. On the existence of monodromy groups of fuchsian systems on Riemann's sphere with unipotent generators. *Journal of Dynamical and Control Systems* **2** , *1* (1996) 125–155.

[6] V. P. Kostov. The Deligne-Simpson problem for zero index of rigidity. In: Perspectives in Complex Analysis, Differential Geometry and Mathematical Physics (Eds S. Dimiev, K. Sekigawa) Proceedings du Fifth Colloquium on Vector Fields and Complex Structures, St. Constantin (Bulgaria), September 2000, World Scientific Publisher, 1–20, electronic preprint math.AG/0011107.

[7] V. P. Kostov. On some aspects of the Deligne-Simpson problem. Electronic preprint math.AG/0005016.

[8] C. T. Simpson. Products of matrices. In: Differential Geometry, Global Analysis and Topology, Canadian Math. Soc. Conference Proceedings **12**, AMS, Providence 1992, 157–185.

Université de Nice
Laboratoire de Mathématiques
Parc Valrose, 06108 Nice Cedex 2, France
e-mail: `kostov@math.unice.fr`

Received February 26, 2001