Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **27** (2001), 159-170

Serdica
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

ON SOME RESULTS RELATED TO KÖTHE'S CONJECTURE

Agata Smoktunowicz

Communicated by V. Drensky

Abstract. The Köthe conjecture states that if a ring R has no nonzero nil ideals then R has no nonzero nil one-sided ideals. Although for more than 70 years significant progress has been made, it is still open in general. In this paper we survey some results related to the Köthe conjecture as well as some equivalent problems.

Introduction. All rings and algebras throughout this paper are associative. For the convenience of reader, the notation of various objects is assembled here: Let F be a field and A be an F-algebra. A is said to be algebraic if any element of A is a root of a polynomial with coefficients from F. If there exist a number n such that any element is a root of a polynomial of degree n then A is an agebraic algebra of bounded degree. An F-algebra A is locally finite if every finitely generated subalgebra of A is finite dimensional. A ring (or an algebra) R is called nil if for every $r \in R$ there exists an $n = n(r)$ such that $r^n = 0$. We say

2000 *Mathematics Subject Classification:* 16N40, 16-02.
Key words: associative ring, nil ideal, Jacobson radical.

that R is nil of bounded index if there is an n such that $r^n = 0$ for each $r \in R$. R is nilpotent if there is an n such that $r_1 \dots r_n = 0$ for each $r_1, \dots, r_n \in R$. A ring R is locally nilpotent if every finitely generated subring of R is nilpotent.

The nil radical $N(R)$ (Köthe's upper nil radical) is the sum of all nil ideals in R. $N(R)$ is the largest nil ideal in R. The locally nilpotent radical $L(R)$ (Levitzki radical) is the sum of all locally nilpotent ideals in R. (Baer's) lower nil radical $Nil_*(R)$ is the smallest semiprime ideal in R, and is equal to the intersection of all the prime ideals in R. $Nil_*(R) = 0$ if and only if R has no non-zero nilpotent ideals. A ring R is called Jacobson radical (or quasi-regular) if for every $r \in R$ there is $r' \in R$ such that $r + r' + rr' = 0$. The Jacobson radical of a ring R is the largest quasi-regular ideal of R. It is denoted by $J(R)$. For every ring R, $Nil_*(R) \subseteq N(R) \subseteq J(R)$.

1. The Köthe conjecture. The purpose of this paper is to survey some results connected with the Köthe conjecture. The following important question was asked in 1930 by Köthe:

Question 1 (Köthe's problem [23]). *If a ring R has no nonzero nil ideals, does it follow that R has no nonzero nil one-sided ideals?*

This Question remains open. Köthe himself conjectured that the answer is into affirmative wich is now known as the Köthe conjecture. There are many assertions equivalent to the Köthe conjecture (e.g. [12]).

Theorem 2. *The following statements are equivalent to the Köthe's conjecture:*

1 *(*[3], [25]*) The sum of two right nil ideals in any ring is nil.*

2 *(Krempa* [24] *and Sands* [34] *independently) For every nil ring R, the ring of* 2×2 *matrices over R is nil.*

3 *(Krempa* [24] *and Sands* [34] *independently) For every nil ring R, the ring of* $n \times n$ *matrices over R is nil.*

4 (Krempa [24]) *For every nil ring R the polynomial ring $R[x]$ in one indeterminate is Jacobson radical.*

There is another famous problem connected with the Köthe problem.

Question 3 (Kurosch's Problem [31]). *Let R be a finitely generated associative algebra such that every element of R is algebraic. Is the algebra R finite dimensional as a vector space over F?*

This famous problem has two special cases. The first case is

Question 4 (Kurosch's Problem for Division rings [31], [21]). *Let R be a finitely generated associative algebra such that every element of R is algebraic. If R is a division algebra is R finite dimensional as a vector space over F?*

Kurosch's problem for division rings is still open in general, but it is answered affirmatively for F uncountable [31], F finite, and F having only finite algebraic field extensions (in particular F algebraically closed). The last follows from the Levitzki-Shirshov theorem, see e.g. [21], [11].

Theorem 5 (Levitzki-Shirshov). *Any algebraic algebra of bounded degree is locally finite.*

When F is a finite field we have

Theorem 6 (Jacobson [25]). *Let D be an algebraic division algebra over a finite field F. Then D is commutative and is therefore an algebraic field extension of F.*

Later, a more general theorem was proved

Theorem 7 (Jacobson-Herstein, [25]). *Let R be a ring such that, for any $a, b \in R$, there exists an integer $n = n(a, b) > 1$ such that $(ab - ba)^n = ab - ba$. Then R is commutative.*

The second case of Kurosch's problem is Levitzki's Problem:

Question 8. *If N is a finitely generated nil algebra without 1, is N nilpotent?*

Golod solved Kurosch's problem in 1964.

Theorem 9 (Golod 1964). *For every field F there exists a nil F-algebra R which is not locally nilpotent.*

Golod used the group $1 + R$, when F has positive characteristic, to get a counter-example to the General Burnside Problem. For information about Burnside Problems see e.g. [46].

2. Artinian and Noetherian rings. We say that the Köthe conjecture holds for a ring R (is true for a ring R) if the ideal generated by every nil left ideal of R is nil.

Köthe's conjecture has been shown to be true for the class of right Artinian rings. Levitzki's Theorem implies the truth of Köthe's conjecture for the larger class of right Noetherian rings.

Theorem 10 (Levitzki [25], [31]). *Let R be a right Noetherian ring. Then every nil one-sided ideal of R is nilpotent.*

Herstein conjectured that the following generalization of Levitzki's Theorem should hold:

Question 11 (Herstein's conjecture [16], [17], [42]). *Suppose that $I \subseteq J$ are left ideals of a left Noetherian ring R such that J is nil over I, in the sense that for each $a \in J$ there exists a natural number n such that $a^n \in I$. Does $J^m \subseteq I$ for some m?*

Herstein's conjecture is open in general. Some interesting results connected with Herstein's conjecture may be found in [42].

There is another famous conjecture open in general:

Question 12 (Jacobson conjecture [31]). *Let $J = J(R)$ be the Jacobson radical of a ring R. Suppose that R is both left and right Noetherian. Does $\bigcap_{i=1}^{\infty} J^i = 0$?*

If the ring R is only left Noetherian then the conjecture is not true, as proved by Herstein. A discussion on results related to Jacobson's conjecture appears in [31, pp. 405–407]. The following result is connected with Levitzki's theorem:

Theorem 13 (Utumi [25]). *Assume that R satisfies ACC for left annihilators $ann_r(a) = \{x \in R : xa = 0\}$, where $a \in R$. Then any non-zero nil right ideal (resp. left) contains a nonzero nilpotent right (resp. left) ideal.*

Younghua [44] introduced the concept of Köthe subsets. A subset M of R is called a Köthe set if there exists a maximal left nil ideal L such that $M = (L + LR + N(R)) - N(R)$. Recall that $N(R)$ is the nil radical of R. In particular he proved:

Theorem 14. *Let R be a ring. Köthe's conjecture holds for R if and only if for every Köthe subset M of R, R satisfies ACC for left annihilators $ann(a) = \{x \in R : xa = 0\}$ where $a \in M$.*

The Jacobson radical has many interesting properties, for example

Theorem 15 ([31], [25]). *Suppose R is an algebra over F and $K \supseteq F$ is an algebraic field extension. Then $J(R) \otimes_F K \subseteq J(R \otimes_F K)$. Moreover for each natural n the ring of $n \times n$ matrices with coefficients in $J(R)$ is Jacobson radical.*

The following open question is connected with this result:

Question 16 (Question 9, [26]). *Let A be a nil algebra over a field F and let $F \subseteq K$ be a finite field extension. Is the algebra $A \otimes_F K$ nil?*

If the Köthe problem has an affirmative solution then the answer to Question 16 is affirmative. It is natural to ask

Question 17. *Is Question 16 equivalent to the Köthe conjecture?*

The Köthe problem is important also because of the structural theorem of Amitsur:

Theorem 18 (Amitsur, [1]). *For every ring R, the Jacobson radical $J(R[x])$ of the polynomial ring $R[x]$ in an indeterminate x over R is equal to $I[x]$ for a nil ideal I of R.*

The original proof of this theorem (given by Amitsur) shows also that

Theorem 19. *For every ring R, the nil radical $N(R[x])$ of the polynomial ring $R[x]$ in an indeterminate x over R is equal to $\overline{I}[x]$ for some nil ideal $\overline{I}$ of R.*

3. Algebras over uncountable fields. There are other classes of rings for which Köthe's conjecture is true. If for example the Jacobson radical of a ring R is nil then the Köthe conjecture is true. This follows from the fact that the Jacobson radical of a ring R contains every quasi-invertible left ideal of R and every nil left ideal is quasi-invertible. There are three important theorems of Amitsur:

Theorem 20 ([43]). *Let R be an algebra R over a field F such that either R is algebraic over F, or $\dim_F(R) < \mathrm{card}F$. Then the Jacobson radical of R is nil.*

Theorem 21 ([2]). *If R is an algebra over an uncountable field, then the Jacobson radical of the polynomial ring $R[x]$ is nil.*

Theorem 22 ([2]). *If R is a nil algebra over an uncountable field, then the polynomial ring $R[x_1, \ldots, x_n]$ over R in n commuting indeterminates is also nil.*

The situation is different for countable fields, as shown by the author:

Theorem 23 ([37]). *For every countable field F there is a nil F-algebra R (generated by four elements) such that the polynomial ring in one indeterminate over R is not nil.*

There are some generalizations of this theorem:

Theorem 24 ([38]). *For every countable field F and for each n there is an algebra R over F such the free associative R-algebra $R[x_1, \dots, x_n]$ i.e., the polynomial ring $R[x_1, \dots, x_n]$ in n commuting indeterminates over R, is Jacobson radical and not nil.*

For the case $n = 1$ see joint paper of the author with Puczyłowski [41].

4. The Brown-McCoy radical. Recall that for a given ring R the Brown-McCoy radical of R is defined as the intersection of all ideals I of R such that R/I is a simple ring with 1. In particular a ring is Brown-McCoy radical if and only if it cannot be homomorphically mapped onto a ring with 1, or equivalently, onto a simple ring with 1. The Brown-McCoy radical of a ring R contains the Jacobson radical of a ring R.

Theorem 25 ([28]). *Let R be a ring such that $R[x]$ can be homomorphically mapped onto a ring with 1. Then there is a homomorphism $f : R[x] \to P$ onto a simple ring P with 1, such that $f(rx) = 1$ for some $r \in R$.*

Hence for every nil ring R, the polynomial ring $R[x]$ is Brown-McCoy radical [28]. Note that this theorem is related to the Köthe problem, which is equivalent to asking whether for every nil ring R the polynomial ring $R[x]$ is Jacobson radical. Later it was proved [5] that for a nil ring R the polynomial ring $R[x]$ in one indeterminate over R cannot be homomorphically mapped onto a ring with a nonzero idempotent.

It can be shown that if for some n the polynomial ring in n noncommuting indeterminates over R, can be homomorphically mapped onto a ring with 1 then for some natural m the polynomial ring in m commuting indeterminates over R can be homomorhically mapped onto a ring with 1 [40]. In [26] some open questions concerning Brown-McCoy radical are stated. We recall one of them

Question 26 ([26], [28]). *If R is a nil ring does it follow that for each $n > 2$ the polynomial ring $R[x_1, \dots, x_n]$ over R in n commuting indeterminates is Brown-McCoy radical?*

Question 26 is answered affirmatively for $n = 2$ ([40]). For other interesting questions connected with nil rings see [7], [43].

5. Special kinds of algebras.

PI-algebras: An algebra A over a field F is said to satisfy a polynomial identity (or be a PI-algebra) if there is a nonzero polynomial $f(x_1, \dots, x_n)$ in the free algebra $F[x_1, x_2, \ldots]$ such that $f(r_1, \ldots, r_n) = 0$ for all $r_1 \ldots, r_n \in A$. In particular each commutative algebra is a PI-algebra. Köthe's conjecture is true for the class of PI-algebras over an arbitrary field F. For some related results see [15].

Klein [22] proved that if R is a nil ring of bounded index then $R[x]$ is a nil ring of bounded index. It is known ([32], [25]) that if R is a PI-algebra over an arbitrary field F then $Nil_*(R) = N(R)$. The nilpotence of the Jacobson radical of an affine (finitely generated as an algebra) PI-algebra over a field of characteristic 0 was established by Rasmyslov [29] and Kemer [20] and in the most general setup by Braun [8].

Theorem 27 (Rasmyslov-Kemer-Braun)**.** *If R is a finitely generated PI-algebra over a Noetherian commutative ring then the Jacobson radical of R is a nilpotent ideal of R.*

Monomial algebras: Let F be a field, Z a set, $\langle Z \rangle$ the free semigroup on Z with unit element, and $F\langle Z \rangle$ the free F-algebra on Z. Given any subset $Y \subseteq \langle Z \rangle$, the factor algebra $A = F\langle Z \rangle / Y$ is called the monomial algebra defined by the set Y of words of $\langle Z \rangle$.

Belov and Gateva-Ivanova [6] proved that the Jacobson radical of a finitely generated monomial algebra over a field is nil. Later Beidar and Fong [4] proved that the Jacobson radical of a monomial algebra over a field is locally nilpotent. In the case of zero characteristic it was earlier proved in [18]. Hence the Köthe conjecture is true for monomial algebras. On the other hand, Zelmanov proved that the locally nilpotent radical of a finitely generated monomial algebra need not be (Baer's) lower nil radical [45].

Finitely presented algebras: Let F be a field, let R be a finitely generated free algebra and let I be the ideal of R generated by a finite number of elements of R. Then the factor algebra R/I is called a finitely presented algebra. There are many open questions in this area:

Question 28 (Amitsur, [27])**.** *If A is a finitely presented algebra, does it follow that the Jacobson radical of A is nil?*

Question 29. *If A is a finitely presented algebra, does it follow that the nil radical of A is locally nilpotent? If A is nil, is A nilpotent?*

Sum of all nil left ideals of a ring: Given a ring R let $\overline{N}(R)$ denote sum of all nil left ideals of R. $\overline{N}(R)$ is a two-sided ideal of R. Indeed, if $a \in R$ and Ra is nil then aR is nil. Thus $\overline{N}(R)$ is also the sum of all nil right ideals of R and $N(R) \subseteq \overline{N}(R) \subseteq J(R)$ ([30], [10]). The Köthe question is equivalent to

Question 30. *Let R be a ring. Does $N(R) = \overline{N}(R)$?*

Rowen [30] studied connections between Köthe's problem and infinite upper triangular matrices satisfying special conditions. He also asked the following questions:

Question 31 (Rowen, [30]). *Let R be a ring. If $N(R) = 0$ does $\bigcap_{i=1}^{\infty}(\overline{N}(R))^i = 0$? If $\bigcap_{i=1}^{\infty}(\overline{N}(R))^i = 0$ does $\overline{N}(R) = N(R)$?*

There are some open questions of Andrunakiewicz:

Question 32 (Andrunakiewicz, [10]). *Can the factor ring $R/\overline{N}(R)$ have a non-zero left nil ideal?*

Question 33 (Andrunakiewicz, [10]). *Let R be a ring without a nil left ideals. Is there a prime ideal of R such that R/P has no nil left ideals?*

If the Köthe conjecture is true then the answers to Questions 31, 33 are in the affirmative.

Simple radical rings: A ring is called simple if it has no proper two sided ideals and $R^2 \neq 0$. It was asked by Levitzki [43], Jacobson [10], Kaplansky [19] if there is a simple nil ring. An example of a simple ring which is Jacobson radical was found by Sąsiada in 1961 [35], [36]. Ryabukhin proved the following theorem:

Theorem 34 ([33]). *A simple nil algebra over a field F exists if and only if there is a natural number $m \geq 2$ and an ideal N of the free F-algebra A in $2m+1$ indeterminates $a, x_1, \ldots, x_m, y_1, \ldots, y_m$ such that A/N is nil, $a \notin N$ and $a - \sum_{i=1}^{m} x_i a^2 y_i \in N$.*

Recently the author has constructed examples of simple nil rings:

Theorem 35 ([39]). *For every countable field F there is a simple associative nil algebra over F.*

A natural open question still remains, namely

Question 36. *If R is an algebra over an uncountable field, is R not simple?*

Almost nilpotent rings: An associative ring R is said to be almost nilpotent if every non-zero (two-sided) ideal of R strictly contains some power of R [9]. "Almost nilpotent" and "nil" are two independent generalizations of "nilpotent". France-Jackson, Heyman, de la Rosa [13] considered the "almost nilpotent" version of Köthe's problem, namely: If L is a nonzero almost nilpotent left ideal of a ring R, is it true that the two-sided ideal $L + LR$ generated by L in R is almost nilpotent? They gave a counterexample.

REFERENCES

[1] S. A. Amitsur. Radicals of polynomial rings. *Canad. J. Math.* **8** (1956), 335–361.

[2] S. A. Amitsur. Algebras over infinite fields. Proc. Amer. Math. Soc. **7** (1956), 35–48.

[3] S. A. Amitsur. Nil radicals; Historical notes and some new results. In: Colloq. Math. Soc. Janos Bolyai 6: Rings, Modules and Radicals, Keszthely (Hungary) 1971, 47–65.

[4] K. I. Beidar, Y. Fong. On radicals of monomial algebras. *Comm. Algebra* **26**, *12* (1998), 3913–3919.

[5] K. I. Beidar, Y. Fong, E. R. Puczyłowski. Polynomial rings over nil rings cannot be homomorphically mapped onto a rings with nonzero idempotents. *J. Algebra* **238** (2001), 389–399.

[6] A. Belov, T. Gateva-Ivanova. Radicals of monomial algebras. In: Proceedings of First International Tainan-Moscow Algebra Workshop (Tainan 1994), Walter de Gruyter, Berlin–New York, 159–169.

[7] G. M. Bergman. Radicals, tensor product and algebraicity. *Israel Math. Conf. Proc.* **1** (1989), 150–192.

[8] A. J. Braun. The nilpotency of the radical in a finitely generated PI-ring. *J. Algebra* **89** (1984), 375–396.

[9] N.J.Divinsky. Rings and Radicals. Geirge Allen and Unwin Ltd., London, 1965.

[10] Dnestrowskaya Tetrad (the Dniester Notebook) Neresennyje Problemy Teorii Kolets i Modulej. Nowosibirsk, 1969.

[11] V. Drensky. Free Algebras and PI-Algebras. Graduate Course in Algebra, Springer-Verlag, Singapore, 2000.

[12] J. W. Fisher, J. Krempa. "R^G is nil implies R is Nil" is equivalent to the "Koethe Conjecture". *Houston J. Math.* **9**, *2* (1983), 177–180.

[13] H. France-Jackson, G. A. P. Heyman, B. de la Rosa. Nil problems revisited for almost nilpotent rings. *Quaest. Math.* **12** (1989), 281–287.

[14] E. S. Golod, I. R. Shafarevich. On the class field tower. *Izv. Akad. Nauk SSSR Mat. Ser.* **28** (1964), 261–272 (in Russian).

[15] O. Hadas. On a bounded index version of Koethe's conjecture. *Proc. Royal Irish Acad.* **96A** (1996), 91–94.

[16] I. N. Herstein. A theorem of left Noetherian rings. *J. Math. Anal. Appl.* **15** (1966), 91–96.

[17] I. N. Herstein. A non-nilpotent type of theorem. In: Aspects of Mathematics and Its Applications. North-Holland, Amsterdam, 1986.

[18] E. Jespers, E. R. Puczyłowski. The Jacobson and Brown-McCoy radicals of rings graded by free groups. *Comm. Algebra* **19** (1991), 551–558.

[19] I. Kaplansky. "Problems in the theory of rings" revisited, *Amer. Math. Monthly* **77** (1970) 445–454.

[20] A. R. Kemer. Capelly identities and the nilpotence of the radical of a finitely generated PI-algebra. *Dokl. Akad. Nauk SSSR* **225** (1980), 793–797 (in Russian); English transl. *Soviet Math. Dokl.* **22** (1980), 750–753.

[21] V. K. Kharchenko. Simple, prime and semiprime rings. Handbook of Algebra, Vol. 1, North-Holland, Amsterdam 1996, 761–812.

[22] A. A. Klein. Rings with bounded index of nilpotence. *Contemporary Math.* **13** (1982), 151–154.

[23] G.Köethe. Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollstanding irreduzibel ist. *Math. Z.* **32** (1930), 161–186.

[24] J. Krempa. Logical connections among some open problems concerning nil rings. *Fund. Math.* **76** (1972), 121–130.

[25] T. Y. Lam. A First Course in Noncommutative Rings. Graduate Texts in Mathematics, **131**. Springer-Verlag, New York, 1999.

[26] E. R. Puczyłowski. Some questions concerning radicals of associative rings. *Colloq. Math. Soc. János Bolyai* vol **61**: Theory of Radicals, Szekszard. Hungary, 1991, 209–227.

[27] E. R. Puczyłowski. Some results and questions on nil rings. 15th School of Algebra (Canela 1998), Mat. Contemp. 16 (1999), 256–280.

[28] E. R. Puczyłowski, A. Smoktunowicz. On maximal ideals and the Brown-McCoy radical of polynomial rings. *Comm. Algebra* **26** (1998), 2473–2482.

[29] Yu. P. Razmyslov. On the Jacobson radical in PI algebras. *Algebra i Logika* **13** (1974), 337–360 (in Russian); English trans. *Algebra and Logic* **13** (1974), 192–204.

[30] L. H. Rowen. Kothe's conjecture. *Israel Math. Conf. Proc.* **1** (1989), 193–203.

[31] L. H. Rowen. Ring Theory, vol. I. Academic Press, New York, 1988.

[32] L. H. Rowen. Polynomial Identities of Ring Theory. Academic Press, New York, 1980.

[33] Ju. M. Ryabukhin. On problem of the existence of a simple nil ring. *Sibirsk. Math. Z.* **10** (1969) 950–956 (in Russian).

[34] A. D. Sands. Radical and Morita contex. J. Algebra **24** (1973), 335–345.

[35] E. Sąsiada. Solution of the problem of the existence of a simple radical ring. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom.Phys.* **9** (1961), 257.

[36] E. Sąsiada, P. Cohn. An example of a simple radical ring. *J. Algebra* **5** (1967), 373–377.

[37] A. Smoktunowicz. Polynomial rings over nil rings need not be nil. *J. Algebra* **233** (2000), 427–436.

[38] A. Smoktunowicz. Amitsur's conjecture on polynomial rings in n commuting indeterminates, submitted.

[39] A. Smoktunowicz. A simple nil ring exists. *Comm. Algebra*, to appear.

[40] A. Smoktunowicz. Some results on the Brown-McCoy radical, submitted.

[41] A. Smoktunowicz, E.R.Puczyłowski. A polynomial ring that is Jacobson radical and not nil. *Israel J. Math.*, to appear.

[42] J. T. Stafford. A nil implies nilpotent theorem for left ideals. *J. Algebra* **133** (1990), 545-549.

[43] F. A. Szász. Radicals of Rings. Akademiami Kiado, Budapest, 1981.

[44] Xu Yonghua. On the Koethe problem and the nilpotent problem. *Scienta Sinica (Series A)* **XXVI**, *9* (1983), 901–908.

[45] E. I. Zelmanov. An example of a finitely generated prime ring. *Sibirsk Mat. Z.* **20** (1979), 423 (in Russian); English trans. *Siberian Math. J.* **20** (1979), 303–304.

[46] E. I. Zelmanov. Nil Rings and Periodic Groups. KMS Lecture Notes in Mathematics, Seul, 1992.

Department of Mathematics
Yale University
New Haven
CT 06520-8283, USA

Received July 7, 2001