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ANALOG OF FAVARD'S THEOREM FOR POLYNOMIALS CONNECTED WITH DIFFERENCE EQUATION OF 4-TH ORDER.

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Abstract. Orthonormal polynomials on the real line $\{p_n(\lambda)\}_{n=0}^{\infty}$ satisfy the recurrent relation of the form: $\lambda_{n-1}p_{n-1}(\lambda)+\alpha_n p_n(\lambda)+\lambda_n p_{n+1}(\lambda)=\lambda p_n(\lambda),\ n=0,1,2,\dots,$ where $\lambda_n>0, \alpha_n\in R,\ n=0,1,\dots;\lambda_{-1}=p_{-1}=0,\lambda\in C$.

In this paper we study systems of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ which satisfy the equation: $\alpha_{n-2}p_{n-2}(\lambda)+\overline{\beta_{n-1}}p_{n-1}(\lambda)+\gamma_n p_n(\lambda)+\beta_n p_{n+1}(\lambda)+\alpha_n p_{n+2}(\lambda)=\lambda^2 p_n(\lambda), n=0,1,2,\dots,$ where $\alpha_n>0,\ \beta_n\in C,\ \gamma_n\in R$, $n=0,1,2,\dots,\ \alpha_{-1}=\alpha_{-2}=\beta_{-1}=0,\ p_{-1}=p_{-2}=0,\ p_0(\lambda)=1$, $p_1(\lambda)=c\lambda+b,\ c>0,\ b\in C,\ \lambda\in C$.

It is shown that they are orthonormal on the real and the imaginary axes in the complex plane: $\displaystyle\int_{R\cup T}(p_n(\lambda),p_n(-\lambda))d\sigma(\lambda)\overline{\begin{pmatrix} p_m(\lambda)\\ p_m(-\lambda)\end{pmatrix}}=\delta_{n,m}$, $n,m=\overline{0,\infty};T=(-\infty,\infty)$ with respect to some matrix measure $\sigma(\lambda)=\begin{pmatrix}\sigma_1(\lambda) & \sigma_2(\lambda)\\ \sigma_3(\lambda) & \sigma_4(\lambda)\end{pmatrix}$.

Also the Green formula for difference equation of 4-th order is built.

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Let us consider the recurrent relation for the system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ of the form:

$$\alpha_{n-2}p_{n-2}(\lambda) + \overline{\beta_{n-1}}p_{n-1}(\lambda) + \gamma_n p_n(\lambda) + \beta_n p_{n+1}(\lambda) + \alpha_n p_{n+2}(\lambda) = \lambda^2 p_n(\lambda), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $\alpha_n > 0$, $\beta_n \in C$, $\gamma_n \in R$, $n = 0, 1, 2, \ldots$, $\alpha_{-1} = \alpha_{-2} = \beta_{-1} = 0$, $p_{-1} = p_{-2} = 0$, $p_0(\lambda) = 1$, $p_1(\lambda) = c\lambda + b$, $c > 0$, $b \in C$, $\lambda \in C$.

This relation can be written in the matrix form:

$$\begin{pmatrix} \gamma_0 & \beta_0 & \alpha_0 & 0 & 0 & 0 & . \\ \overline{\beta_0} & \gamma_1 & \beta_1 & \alpha_1 & 0 & 0 & . \\ \alpha_0 & \overline{\beta_1} & \gamma_2 & \beta_2 & \alpha_2 & 0 & . \\ 0 & \alpha_1 & \overline{\beta_2} & \gamma_3 & \beta_3 & \alpha_3 & . \\ . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ . \end{pmatrix} = \lambda^2 \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ . \end{pmatrix}$$

or

$$J_5 p = \lambda^2 p,$$

where J_5 is five-diagonal, symmetric, semi-infinite matrix and p is vector of polynomials.

Let us study the properties of these systems of polynomials. Note, that orthonormal polynomials on the real line belong to this class of polynomials. For them the following equation is fulfilled: $J_3 p = \lambda p$ with three-diagonal, symmetric, semi-infinite matrix and from this immediately it follows, that $J_3^2 p = \lambda^2 p$ and J_3^2 – is five-diagonal, symmetric, semi-infinite matrix (Question: how larger is the considered class than the class of real polynomials? For the answer see [1, Theorem 5, p. 272]).

Definition ([1, p. 265–266])**.** *Find matrix measure* $\tilde{\sigma}(\lambda) = \begin{pmatrix} \tilde{\sigma}_1(\lambda) & \tilde{\sigma}_2(\lambda) \\ \tilde{\sigma}_3(\lambda) & \tilde{\sigma}_4(\lambda) \end{pmatrix}$, $\lambda \in C$; $\tilde{\sigma}_i(\lambda) : C \to C$ *is piecewise continuous mapping on the real and the imaginary axis,* $i = \overline{1,4}$, *such that*

1) $\tilde{\sigma}(\lambda)$ *is symmetric, monotonically increasing matrix function:*

$$\tilde{\sigma}_1(\lambda) = \overline{\tilde{\sigma}_1(\lambda)}, \tilde{\sigma}_4(\lambda) = \overline{\tilde{\sigma}_4(\lambda)}, \tilde{\sigma}_2(\lambda) = \overline{\tilde{\sigma}_3(\lambda)};$$

$$\tilde{\sigma}(\lambda_2) \geq \tilde{\sigma}(\lambda_1),\ \lambda_2 \geq \lambda_1,\ \lambda_1, \lambda_2 \in R$$

$$\tilde{\sigma}(\lambda_2) \geq \tilde{\sigma}(\lambda_1),\ \frac{\lambda_2}{i} \geq \frac{\lambda_1}{i},\ \lambda_1, \lambda_2 \in (-i\infty, i\infty)$$

2) $\displaystyle\int_{R \cup T} (\lambda^k, (-\lambda)^k) d\tilde{\sigma}(\lambda) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = s_k,\ k = \overline{0, \infty},$

$$\int_{R\cup T}(\lambda^{k-1},(-\lambda)^{k-1})d\tilde{\sigma}(\lambda)\overline{\begin{pmatrix}\lambda\\-\lambda\end{pmatrix}} = m_k,\ k=\overline{1,\infty},$$

where $\{s_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ *are fixed sequences of complex numbers;*

We'll call this problem ***generalized symmetric moments problem.***

Definition ([1, p. 266])**.** *We call a pair of sequences* $\{s_k, m_{k+1}\}_{k=0}^{\infty}$, $s_k \in C, m_{k+1} \in C,\ k=\overline{0,\infty}$ **symmetric** *if*

$$\overline{s_{2k+1}} = m_{2k+1};$$

$$\overline{s_{2k}} = s_{2k}, \overline{m_{2k+2}} = m_{2k+2}, \quad k=\overline{0,\infty}.$$

We call a pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}$, $s_k \in C$, $m_{k+1} \in C$, $k=\overline{0,\infty}$ **positive** *one if*

$$\begin{bmatrix} s_0 & s_1 & .\ . & s_k\\ m_1 & m_2 & .\ . & m_{k+1}\\ s_2 & s_3 & .\ . & s_{k+2}\\ . & . & .\ . & .\\ m_k & m_{k+1} & .\ . & m_{2k}\end{bmatrix} > 0,\ k=2l+1;\ \begin{bmatrix} s_0 & s_1 & .\ . & s_k\\ m_1 & m_2 & .\ . & m_{k+1}\\ s_2 & s_3 & .\ . & s_{k+2}\\ . & . & .\ . & .\\ s_k & s_{k+1} & .\ . & s_{2k}\end{bmatrix} > 0, k=2l$$

$$l=\overline{0,\infty}.$$

The following theorem holds true:

Theorem 1 ([1, Theorem 6, p. 274])**.** *Let moments problem in general form be given. For the existance of a problem's solution* $\sigma(\lambda)$ *(with an infinite number of increasing points) it is necessary and sufficient the pair of sequences* $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ *to be symmetric and positive.*

Zolotarev suggested to investigate systems of polynomials, which satisfy (1). He also gave the usefull notes. The next theorem is an analog of Favard's theorem [2, Theorem 1.5, p. 60]. It gives the orthonormality properties for the systems from (1).

Theorem 2. *Let the system of polynomials* $\{p_n(\lambda)\}_{n=0}^{\infty}$ *satisfy (1). Then these polynomials are orthonormal on the real and the imaginary axes in the complex plane:*

$$\int_{R\cup T}(p_n(\lambda),p_n(-\lambda))d\sigma(\lambda)\overline{\begin{pmatrix}p_m(\lambda)\\p_m(-\lambda)\end{pmatrix}} = \delta_{n,m},\ n,m=\overline{0,\infty}; T=(-\infty,\infty),$$

where $\sigma(\lambda) = \begin{pmatrix} \sigma_1(\lambda) & \sigma_2(\lambda) \\ \sigma_3(\lambda) & \sigma_4(\lambda) \end{pmatrix}$ *is symmetric, non-decreasing matrix-function with infinite number of points of increasing:* $\overline{\sigma_1(\lambda)} = \sigma_1(\lambda)$, $\overline{\sigma_4(\lambda)} = \sigma_4(\lambda)$, $\overline{\sigma_2(\lambda)} = \sigma_3(\lambda)$; $\sigma(\lambda_2) \geq \sigma(\lambda_1)$, $\lambda_2 \geq \lambda_1$, $\lambda_1, \lambda_2 \in R$, $\sigma(\lambda_2) \geq \sigma(\lambda_1)$, $\dfrac{\lambda_2}{i} \geq \dfrac{\lambda_1}{i}$, $\lambda_1, \lambda_2 \in T$, $T = (-i\infty, i\infty)$.

P r o o f. Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ be given system of polynomials. We define a functional $\sigma(u,v)$, where $u, v \in P$, P is a space of all polynomials. If $u(\lambda), v(\lambda) \in P$, than we can write: $u(\lambda) = \sum\limits_{i=0}^{n} a_i p_i(\lambda)$, $v(\lambda) = \sum\limits_{j=0}^{m} b_j p_j(\lambda)$, $a_i \in C$, $i = \overline{0,n}$, $b_j \in C, j = \overline{0,m}$, ($a_n \neq 0$, $b_m \neq 0$), $n, m \in N$. Note, that for arbitrary polynomials $u(\lambda), v(\lambda)$ such decomposition is always possible because the polynomials $p_n(\lambda)$ have positive leading coefficients. Also, for arbitrary $u(\lambda), v(\lambda)$ such representation is unique. By definition

$$\sigma(u,v) = \sum_{i=0}^{k} a_i \overline{b_i},\ k = \min\{n,m\}$$

This functional is bilinear. Also, it satisfies the relation:

$$\overline{\sigma(u,v)} = \sigma(v,u).$$

Note, that for the polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ we have:

$$\sigma(p_n(\lambda), p_m(\lambda)) = \delta_{n,m},\ n,m = 0,1,\ldots \tag{2}$$

Let us show that this functional has the property:

$$\sigma(\lambda^2 u, v) = \sigma(u, \lambda^2 v),\ u, v \in P. \tag{3}$$

Really, if $u(\lambda) = \sum\limits_{i=0}^{n} a_i p_i(\lambda)$, $v(\lambda) = \sum\limits_{j=0}^{m} b_j p_j(\lambda)$, $a_i \in C$, $i = \overline{0,n}$, $b_j \in C$, $j = \overline{0,m}$, ($a_n \neq 0$, $b_m \neq 0$), $n, m \in N$, then

$$\sigma(\lambda^2 u, v) = \sigma\left(\lambda^2 \sum_{i=0}^{n} a_i p_i(\lambda), \sum_{j=0}^{m} b_j p_j(\lambda)\right) = \sigma\left(\sum_{i=0}^{n} \lambda^2 a_i p_i(\lambda), \sum_{j=0}^{m} b_j p_j(\lambda)\right) =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \overline{b_j} \sigma(\lambda^2 p_i(\lambda), p_j(\lambda))$$

If we prove that

$$\sigma(\lambda^2 p_i(\lambda), p_j(\lambda)) = \sigma(p_i(\lambda), \lambda^2 p_j(\lambda)),\ i,j = 0,1,\ldots, \tag{4}$$

then

$$\sigma(\lambda^2 u, v) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i\overline{b_j}\sigma(p_i(\lambda), \lambda^2 p_j(\lambda)) = \sigma\left(\sum_{i=0}^{n} a_i p_i(\lambda), \sum_{j=0}^{m} \lambda^2 b_j p_j(\lambda)\right) =$$

$$= \sigma(u, \lambda^2 v)$$

and relation (3) will be proved. But

$$\sigma(\lambda^2 p_i(\lambda), p_j(\lambda)) = \sigma(\alpha_{i-2}p_{i-2}(\lambda)+\overline{\beta_{i-1}}p_{i-1}(\lambda)+\gamma_i p_i(\lambda)+\beta_i p_{i+1}(\lambda)+\alpha_i p_{i+2}(\lambda),$$

$$p_j(\lambda)) = \alpha_{i-2}\delta_{i-2,j} + \overline{\beta_{i-1}}\delta_{i-1,j} + \gamma_i\delta_{i,j} + \beta_i\delta_{i+1,j} + \alpha_i\delta_{i+2,j},\ i, j = 0, 1, 2, \ldots$$

$$\sigma(p_i(\lambda), \lambda^2 p_j(\lambda)) = \sigma(p_i(\lambda), \alpha_{j-2}p_{j-2}(\lambda) + \overline{\beta_{j-1}}p_{j-1}(\lambda) + \gamma_j p_j(\lambda) + \beta_j p_{j+1}(\lambda)+$$

$$+\alpha_j p_{j+2}(\lambda)) = \alpha_{j-2}\delta_{i,j-2} + \beta_{j-1}\delta_{i,j-1} + \gamma_j\delta_{i,j} + \overline{\beta_j}\delta_{i,j+1} + \alpha_j\delta_{i,j+2} =$$

$$= \alpha_i\delta_{i,j-2} + \beta_i\delta_{i,j-1} + \gamma_i\delta_{i,j} + \overline{\beta_{i-1}}\delta_{i,j+1} + \alpha_{i-2}\delta_{i,j+2} = \alpha_i\delta_{i+2,j} + \beta_i\delta_{i+1,j}+$$

$$+\gamma_i\delta_{i,j} + \overline{\beta_{i-1}}\delta_{i-1,j} + \alpha_{i-2}\delta_{i-2,j},\ i, j = 0, 1, 2, \ldots$$

This implies the correctness of (3).

Let

$$s_k = \sigma(\lambda^k, 1),\ k = 0, 1, \ldots$$

$$m_k = \sigma(\lambda^{k-1}, \lambda),\ k = 1, 2, \ldots$$

Then

$$\overline{s_{2k}} = \overline{\sigma(\lambda^{2k}, 1)} = \sigma(1, \lambda^{2k}) = \sigma(\lambda^{2k}, 1) = s_{2k},$$

$$\overline{s_{2k+1}} = \overline{\sigma(\lambda^{2k+1}, 1)} = \sigma(1, \lambda^{2k+1}) = \sigma(\lambda^{2k}, \lambda) = m_{2k+1},\ k = 0, 1, \ldots;$$

$$\overline{m_{2k}} = \overline{\sigma(\lambda^{2k-1}, \lambda)} = \sigma(\lambda, \lambda^{2k-1}) = \sigma(\lambda^{2k-1}, \lambda) = m_{2k},\ k = 1, 2, \ldots$$

Hence, the pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}$, $s_k \in C$, $m_{k+1} \in C$, $k = \overline{0, \infty}$ is symmetric.

Note that

$$\sigma(u, u) > 0,\ u \in P,\ u \neq 0.$$

Using this we have

$$0 < \sigma\left(\sum_{k=0}^{n} a_k\lambda^k, \sum_{i=0}^{n} a_i\lambda^i\right) = \sum_{k,i=0}^{n} a_k\overline{a_i}\sigma(\lambda^k, \lambda^i) =$$

$$= \begin{cases} \sum_{k=0}^{n} \left[\sum_{j=0}^{n/2} a_k \overline{a_{2j}} s_{k+2j} + \sum_{j=0}^{n/2-1} a_k \overline{a_{2j+1}} m_{k+2j+1} \right], & \text{if } n \text{ is even} \\ \sum_{k=0}^{n} \left[\sum_{j=0}^{(n-1)/2} a_k \overline{a_{2j}} s_{k+2j} + \sum_{j=0}^{(n-1)/2} a_k \overline{a_{2j+1}} m_{k+2j+1} \right], & \text{if } n \text{ is odd} \end{cases}$$

$$a_i \in C, i = \overline{0, n}, a_n \neq 0, \ n \in N$$

It follows that the pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ is positive.

Let $\sigma(\lambda) = \begin{pmatrix} \sigma_1(\lambda) & \sigma_2(\lambda) \\ \sigma_3(\lambda) & \sigma_4(\lambda) \end{pmatrix}$ be a solution of corresponding generalized symmetric problem of moments. Let us consider a functional:

$$\hat{\sigma}(u, v) = \int_{R \cup T} (u(\lambda), u(-\lambda)) d\sigma(\lambda) \overline{\begin{pmatrix} v(\lambda) \\ v(-\lambda) \end{pmatrix}}, \ u, v \in P.$$

Let $u(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$, $v(\lambda) = \sum_{i=0}^{m} b_i \lambda^i$, $a_k \in C, k = \overline{0, n}$, $b_i \in C$, $i = \overline{0, m}$. Then

$$\hat{\sigma}(u, v) = \hat{\sigma}\left(\sum_{k=0}^{n} a_k \lambda^k, \sum_{i=0}^{m} b_i \lambda^i \right) = \sum_{k=0}^{n} \sum_{i=0}^{m} a_k \overline{b_i} \hat{\sigma}(\lambda^k, \lambda^i).$$

The following equality holds true:

$$\hat{\sigma}(\lambda^k, \lambda^i) = \sigma(\lambda^k, \lambda^i), \ k, i = 0, 1, \ldots$$

Actually,

$$\hat{\sigma}(\lambda^k, \lambda^{2j}) = \hat{\sigma}(\lambda^{k+2j}, 1) = s_{k+2j} = \sigma(\lambda^{k+2j}, 1) = \sigma(\lambda^k, \lambda^{2j}),$$

$$\hat{\sigma}(\lambda^k, \lambda^{2j+1}) = \hat{\sigma}(\lambda^{k+2j}, \lambda) = m_{k+2j+1} = \sigma(\lambda^{k+2j}, \lambda) = \sigma(\lambda^k, \lambda^{2j+1}),$$

$$k, j = 0, 1, 2, \ldots$$

This implies that

$$\hat{\sigma}(u, v) = \sum_{k=0}^{n} \sum_{i=0}^{m} a_k \overline{b_i} \sigma(\lambda^k, \lambda^i) = \sigma(u, v), \ u, v \in P.$$

Using the orthonormality relation (2) we obtain that polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ satisfy the required property of orthonormality and this completes the proof. $\square$

We now turn to the difference equation of the type:

(5) $\alpha_{n-2}y_{n-2} + \overline{\beta_{n-1}}y_{n-1} + \gamma_n y_n + \beta_n y_{n+1} + \alpha_n y_{n+2} = \lambda^2 y_n(\lambda),\ n = 2, 3, \ldots,$

where $\alpha_n > 0$, $\beta_n \in C$, $\gamma_n \in R$, $n = 0, 1, 2, \ldots$; $\lambda \in C$ is a parameter, $y = \begin{pmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ y_n \\ \cdot \end{pmatrix}$ is solution vector.

Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ be the system of polynomials such that: $p_0(\lambda) = 1$, $p_1(\lambda) = c\lambda + b$, $c > 0$, $b \in C$, $\lambda \in C$, the polynomials $p_2(\lambda)$, $p_3(\lambda)$ can be found from equalities:

$$\gamma_0 p_0(\lambda) + \beta_0 p_1(\lambda) + \alpha_0 p_2(\lambda) = \lambda^2 p_0(\lambda),$$

$$\overline{\beta_0} p_0(\lambda) + \gamma_1 p_1(\lambda) + \beta_1 p_2(\lambda) + \alpha_1 p_3(\lambda) = \lambda^2 p_1(\lambda),\ \gamma_0, \gamma_1 \in R, \beta_0 \in C,$$

and $\{p_n(\lambda)\}_{n=0}^{\infty}$ is solution of (5).

This system of polynomials satisfy (1). Let $\sigma(u, v), u, v \in P$ be a functional defined as in the proof of Theorem 2. Let us write the first 4 polynomials in the sequence $\{p_n(\lambda)\}_{n=0}^{\infty}$:

$$p_0(\lambda) = 1, p_1(\lambda) = \mu_1\lambda + \nu_1, p_2(\lambda) = \mu_2\lambda^2 + \nu_2\lambda + \eta_2, p_3(\lambda) = \mu_3\lambda^3 + \nu_3\lambda^2 + \eta_3\lambda + \xi_3,$$

where $\mu_k > 0, \nu_k \in C, k = \overline{1, 3},\ \eta_2, \eta_3, \xi_3 \in C$.

Consider the following systems of polynomials:

$$p_n^+(\lambda) = \frac{p_n(\lambda) + p_n(-\lambda)}{2},\ p_n^-(\lambda) = \frac{p_n(\lambda) - p_n(-\lambda)}{2\lambda},$$

$$f_n(\lambda) = \sigma_u\left(\frac{p_n(u) - up_n^-(\lambda) - p_n^+(\lambda)}{u^2 - \lambda^2}, 1\right),$$

$$g_k(\lambda) = \sigma_u\left(\frac{p_n(u) - up_n^-(\lambda) - p_n^+(\lambda)}{u^2 - \lambda^2}, p_1(u)\right),$$

(6) $$n = 0, 1, \ldots.$$

The $f_n(\lambda), g_n(\lambda)$ can also be written in the form:

$$f_n(\lambda) = \sigma_u\left(\frac{p_n^+(u) - p_n^+(\lambda)}{u^2 - \lambda^2}, 1\right) + \sigma_u\left(u\frac{p_n^-(u) - p_n^-(\lambda)}{u^2 - \lambda^2}, 1\right),$$

$$g_n(\lambda) = \sigma_u\left(\frac{p_n^+(u) - p_n^+(\lambda)}{u^2-\lambda^2}, p_1(u)\right) + \sigma_u\left(u\frac{p_n^-(u) - p_n^-(\lambda)}{u^2-\lambda^2}, p_1(u)\right),$$

$$n = \overline{0,\infty}.$$

Now we can construct the fundamental system of solutions of (5) (see [3]):

Theorem 3. *Let us consider the difference equation* (5)*. The systems of polynomials* $\{p_n^+(\lambda)\}_{n=0}^\infty$*,* $\{p_n^-(\lambda)\}_{n=0}^\infty$*,* $\{f_n(\lambda)\}_{n=0}^\infty$*,* $\{g_n(\lambda)\}_{n=0}^\infty$ *form the fundamental system of solutions of* (5)*. The initial conditions are as follows:*

$$\begin{cases} p_0^+(\lambda) = 1 \\ p_1^+(\lambda) = \nu_1 \\ p_2^+(\lambda) = \mu_2\lambda^2 + \eta_2 \\ p_3^+(\lambda) = \nu_3\lambda^2 + \xi_3 \end{cases}; \quad \begin{cases} p_0^-(\lambda) = 0 \\ p_1^-(\lambda) = \mu_1 \\ p_2^-(\lambda) = \nu_2 \\ p_3^-(\lambda) = \mu_3\lambda^2 + \eta_3 \end{cases};$$

$$\begin{cases} f_0(\lambda) = 0 \\ f_1(\lambda) = 0 \\ f_2(\lambda) = \mu_2 \\ f_3(\lambda) = -\dfrac{\mu_3\nu_1}{\mu_1} + \nu_3 \end{cases}; \quad \begin{cases} g_0(\lambda) = 0 \\ g_1(\lambda) = 0 \\ g_2(\lambda) = 0 \\ g_3(\lambda) = \dfrac{\mu_3}{\mu_1} \end{cases}. \tag{7}$$

Proof. The systems $\{p_n^\pm(\lambda)\}_{n=0}^\infty$ are solutions of (5) because they are linear combination of solutions of this equation. The conditions (7) are easily verified.

Substituting $f_n(\lambda)$ in the left-hand side of (5) we have:

$$\sigma_u\left(\frac{u^2p_n^+(u) - \lambda^2p_n^+(\lambda)}{u^2-\lambda^2}, 1\right) + \sigma_u\left(u\frac{u^2p_n^-(u) - \lambda^2p_n^-(\lambda)}{u^2-\lambda^2}, 1\right) = \sigma_u(p_n^+(u), 1) +$$

$$+\lambda^2\sigma_u\left(\frac{p_n^+(u) - p_n^+(\lambda)}{u^2-\lambda^2}, 1\right) + \sigma_u(up_n^-(u), 1) + \lambda^2\sigma_u\left(u\frac{p_n^-(u) - p_n^-(\lambda)}{u^2-\lambda^2}, 1\right) =$$

$$= \lambda^2 f_n(\lambda) + \sigma_u(p_n^+(u) + up_n^-(u), 1) = \lambda^2 f_n(\lambda) + \sigma_u(p_n(u), 1) = \lambda^2 f_n(\lambda),$$

$$n = 1, 2, \ldots$$

Analogously for polynomials $g_k(\lambda)$ we have:

$$\sigma_u\left(\frac{u^2p_n^+(u) - \lambda^2p_n^+(\lambda)}{u^2-\lambda^2}, p_1(u)\right) + \sigma_u\left(u\frac{u^2p_n^-(u) - \lambda^2p_n^-(\lambda)}{u^2-\lambda^2}, p_1(u)\right) =$$

$$= \lambda^2 g_n(\lambda) + \sigma_u(p_n(u), p_1(u)) = \lambda^2 g_n(\lambda), \quad n = 2, 3, \ldots.$$

It follows that the systems $\{f_n(\lambda)\}_{n=0}^{\infty}$, $\{g_n(\lambda)\}_{n=0}^{\infty}$ are solutions of (5).

Using formula (6) and conditions (7) for $p_n^{\pm}(\lambda), n=\overline{0,3}$ we obtain:

$$f_0(\lambda)=g_0(\lambda)=0,$$

$$f_1(\lambda)=g_1(\lambda)=0,$$

$$f_2(\lambda)=\sigma_u(\mu_2,1)=\mu_2,\ \ g_2(\lambda)=\sigma_u(\mu_2,p_1(u))=\mu_2\sigma_u(1,p_1(u))=0,$$

$$f_3(\lambda)=\sigma_u(\nu_3,1)+\sigma_u(u\mu_3,1)=\nu_3+\mu_3\sigma_u(u,1)=\nu_3+\mu_3\sigma_u\left(\frac{p_1(u)-\nu_1}{\mu_1},1\right)=$$

$$=\nu_3+\frac{\mu_3}{\mu_1}(\sigma_u(p_1(u),1)-\sigma_u(\nu_1,1))=\nu_3-\frac{\mu_3\nu_1}{\mu_1},$$

$$g_3(\lambda)=\sigma_u(\nu_3,p_1(u))+\sigma_u(u\mu_3,p_1(u))=\mu_3\sigma_u\left(\frac{p_1(u)-\nu_1}{\mu_1},p_1(u)\right)=\frac{\mu_3}{\mu_1}.$$

The linear independence of solutions $\{p_n^{\pm}(\lambda)\}_{n=0}^{\infty}$, $\{f_n(\lambda)\}_{n=0}^{\infty}$, $\{g_n(\lambda)\}_{n=0}^{\infty}$ is evident from (7). The proof is complete. $\square$

Let $\{y_n\}_{n=0}^{\infty},\{v_n\}_{n=0}^{\infty}$ be solutions of equation (5) corresponding to parameter values λ and ξ respectively:

$$\lambda^2 y_n(\lambda)=\alpha_{n-2}y_{n-2}(\lambda)+\overline{\beta_{n-1}}y_{n-1}(\lambda)+\gamma_n y_n(\lambda)+\beta_n y_{n+1}(\lambda)+\alpha_n y_{n+2}(\lambda),$$

$$\xi^2 v_n(\xi)=\alpha_{n-2}v_{n-2}(\xi)+\overline{\beta_{n-1}}v_{n-1}(\xi)+\gamma_n v_n(\xi)+\beta_n v_{n+1}(\xi)+\alpha_n v_{n+2}(\xi),$$
$$n=2,3,\ldots$$

Then

$$\lambda^2 y_n(\lambda)\overline{v_n(\xi)}=\alpha_{n-2}y_{n-2}(\lambda)\overline{v_n(\xi)}+\overline{\beta_{n-1}}y_{n-1}(\lambda)\overline{v_n(\xi)}+\gamma_n y_n(\lambda)\overline{v_n(\xi)}+$$
$$+\beta_n y_{n+1}(\lambda)\overline{v_n(\xi)}+\alpha_n y_{n+2}(\lambda)\overline{v_n(\xi)},$$

$$\overline{\xi}^2 y_n(\lambda)\overline{v_n(\xi)}=\alpha_{n-2}y_n(\lambda)\overline{v_{n-2}(\xi)}+\beta_{n-1}y_n(\lambda)\overline{v_{n-1}(\xi)}+\gamma_n y_n(\lambda)\overline{v_n(\xi)}+$$
$$+\overline{\beta_n}y_n(\lambda)\overline{v_{n+1}(\xi)}+\alpha_n y_n(\lambda)\overline{v_{n+2}(\xi)},\ \ n=2,3,\ldots$$

Subtracting the second equality from the first one we have:

$$(\lambda^2-\overline{\xi}^2)y_n(\lambda)\overline{v_n(\xi)}=\alpha_{n-2}(y_{n-2}(\lambda)\overline{v_n(\xi)}-y_n(\lambda)\overline{v_{n-2}(\xi)})+\overline{\beta_{n-1}}y_{n-1}(\lambda)\overline{v_n(\xi)}-$$
$$-\beta_{n-1}y_n(\lambda)\overline{v_{n-1}(\xi)}+\beta_n y_{n+1}(\lambda)\overline{v_n(\xi)}-\overline{\beta_n}y_n(\lambda)\overline{v_{n+1}(\xi)}+\alpha_n(y_{n+2}(\lambda)\overline{v_n(\xi)}-$$
$$-y_n(\lambda)\overline{v_{n+2}(\xi)})=-A_{n-2}(\lambda,\xi)-B_{n-1}(\lambda,\xi)+B_n(\lambda,\xi)+A_n(\lambda,\xi),$$

where $A_n(\lambda,\xi) = \alpha_n(y_{n+2}(\lambda)\overline{v_n(\xi)} - y_n(\lambda)\overline{v_{n+2}(\xi)})$, $B_n(\lambda,\xi) = \beta_n y_{n+1}(\lambda)\overline{v_n(\xi)} - \overline{\beta_n} y_n(\lambda)\overline{v_{n+1}(\xi)}$.

Summing up over n we obtain:

$$(\lambda^2 - \overline{\xi}^2)\sum_{n=k}^{m} y_n(\lambda)\overline{v_n(\xi)} = B_m(\lambda,\xi) + A_m(\lambda,\xi) + A_{m-1}(\lambda,\xi) - A_{k-2}(\lambda,\xi) -$$

$$- B_{k-1}(\lambda,\xi) - A_{k-1}(\lambda,\xi),\ 2 \le k \le m.$$

So, we have proved the following theorem:

Theorem 4. *Let $\{y_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ be solutions of equation* (5) *corresponding to parameters λ and ξ respectively. Then the following formula holds:*

$$(\lambda^2 - \overline{\xi}^2)\sum_{n=k}^{m} y_n(\lambda)\overline{v_n(\xi)} = \alpha_m(y_{m+2}(\lambda)\overline{v_m(\xi)} - y_m(\lambda)\overline{v_{m+2}(\xi)}) +$$

$$+\alpha_{m-1}(y_{m+1}(\lambda)\overline{v_{m-1}(\xi)} - y_{m-1}(\lambda)\overline{v_{m+1}(\xi)}) + \beta_m y_{m+1}(\lambda)\overline{v_m(\xi)} - \overline{\beta_m} y_m(\lambda)\overline{v_{m+1}(\xi)} -$$

$$-\alpha_{k-2}(y_k(\lambda)\overline{v_{k-2}(\xi)} - y_{k-2}(\lambda)\overline{v_k(\xi)}) - \alpha_{k-1}(y_{k+1}(\lambda)\overline{v_{k-1}(\xi)} - y_{k-1}(\lambda)\overline{v_{k+1}(\xi)}) -$$

$$-(\beta_{k-1} y_k(\lambda)\overline{v_{k-1}(\xi)} - \overline{\beta_{k-1}} y_{k-1}(\lambda)\overline{v_k(\xi)}),$$

$$2 \le k \le m.$$

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