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CONTINUITY OF PSEUDO-DIFFERENTIAL OPERATORS ON BESSEL AND BESOV SPACES

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Abstract. We study the continuity of pseudo-differential operators on Bessel potential spaces $H_p^s(\mathbb{R}^n)$, and on the corresponding Besov spaces $B_p^{s,q}(\mathbb{R}^n)$. The modulus of continuity ω we use is assumed to satisfy

$$\sum_{j\geq 0}[\omega(2^{-j})\Omega(2^j)]^2 < \infty$$

where Ω is a suitable positive function.

Introduction. Several authors studied the continuity of pseudo-differential operators (ψ.d.o.) on Bessel potential spaces H_p^s where the modulus of continuity ω (a positive, nondecreasing and concave function on $[0,\infty)$) satisfies

$$(1) \qquad \sum_{j\geq 0}\left[2^{\varepsilon j}\omega\left(2^{-j}\right)\right]^2 < \infty, \qquad (0 < \varepsilon = s - [s] < 1)\,.$$

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In the present work, we obtain an improvement by taking a more natural condition

$$\sum_{j\geq 0}\left[\omega\left(2^{-j}\right)\Omega\left(2^{j}\right)\right]^{2}<\infty, \tag{2}$$

where $\Omega:[0,\infty)\to[0,\infty)$ is a suitable function.

We consider the ψ.d.o. with a symbol satisfying

$$|\partial_\xi^\alpha\partial_x^\beta\sigma\left(x,\xi\right)|\leq C\left(1+|\xi|\right)^{m-|\alpha|+\delta|\beta|} \tag{3}$$

and

$$|\partial_\xi^\alpha\partial_x^\beta\sigma\left(x+h,\xi\right)-\partial_\xi^\alpha\partial_x^\beta\sigma\left(x,\xi\right)|\leq C\left(1+|\xi|\right)^{m-|\alpha|+\delta|\beta|}\omega(|h|\,|\xi|^\delta)\Omega\left(|\xi|^\varrho\right), \tag{4}$$

where $\delta\geq 0$, $\varrho\geq 0$, $m\geq 0$, $N\in\mathbb{N}$, $C=C_{\alpha,\beta}>0$, $(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n$ and $|\beta|\leq N$.

It is well known that if $\varrho=1$, then (2) and

$$\forall c>1,\ \exists A_c>0,\ \ (t/c\leq u\leq ct)\Rightarrow\Omega\left(u\right)\leq A_c\Omega(t), \tag{5}$$

imply the L^p estimates of such operators (for $\delta=0$ see [9] and for $0\leq\delta<1$ see [3]). Also, when $\delta=0$, $N=[s]$ and $\Omega\equiv 1$, the condition (1) implies that every operator with a symbol satisfying (3) and (4) is bounded from H_p^{s+m} into H_p^s (see [2]).

This paper is organized as follows. In Section 2, we first improve the result for H_p^s−continuity of ψ.d.o. with the help of the following condition

$$(\forall\nu>0,\ \exists C_\nu>0,\ \forall t\geq 1)\Rightarrow\int_1^t\frac{\Omega^2\left(u\right)}{u^{\nu+1}}du\leq C_\nu\frac{\Omega^2\left(t\right)}{t^\nu}. \tag{6}$$

Then, we discuss to which extent condition (2) is optimal. In Section 3, we study the corresponding continuity on Besov spaces $B_p^{s,q}$.

We conclude this section with some examples concerning conditions (5) and (6).

Example 1. (a) $\Omega\left(t\right)=t^r$, $\quad r>\dfrac{1+\nu}{2}$, $\quad$ (*we remark that* (5) *is evidently satisfied for any* $r>0$), $\quad A_c=c^r$ *and* $c>1$.

(b) $\Omega\left(t\right)=\exp\left(\log t\right)^r$ *if* $t\geq c_0$, $\quad\Omega\left(t\right)=0$ *if* $t<c_0, 0<r<1$,
$c_0=\max\left(1,\exp\left(\dfrac{1+\nu}{2r}\right)^{1/(r-1)}\right)$, $\quad A_c=\exp\left(\log c\right)^r$ *and* $c>1$.

(*c*) $\Omega(t) = t^p (\log t)^r$ *if* $t \geq e$, $\Omega(t) = 0$ *if* $t < e, r > 0$, $p > \frac{1+\nu}{2}$, $A_c = 2^r c^p ((\log c)^r + 1)$ *and* $c > 1$.

(*d*) $\Omega(t) = t^p (\log t)^r$ *if* $e \leq t \leq e^q$, $\Omega(t) = 0$ *if* $t \notin [e, e^q]$, $r > \frac{1+\nu}{2} - p > 0$, $q = 2r(1+\nu-2p)^{-1}$, $A_c = 2^r c^p ((\log c)^r + 1)$ *and* $c > 1$.

1. Some notations. The following definitions and notations will be used throughout this article. We assume that all functions, spaces, etc... are defined on the Euclidean space $\mathbb{R}^n$. We set $C^\infty(\mathbb{R}^n) = C^\infty$, $L^p(\mathbb{R}^n) = L^p$, etc. Let ϕ and ψ, satisfy $\phi \in C^\infty$, $\operatorname{supp}\phi \subset \left\{\xi \in \mathbb{R}^n,\ 2^{-1} \leq |\xi| \leq 2\right\}$, $\psi \in C^\infty$, $\operatorname{supp}\psi \subset \{\xi \in \mathbb{R}^n,\ |\xi| \leq 2\}$ and $\psi(0) = 1$. We fix a partition of unity

$$\psi(\xi) + \sum_{k=1}^{\infty} \phi(2^{-k}\xi) = 1, \quad (\xi \in \mathbb{R}^n), \tag{7}$$

and define the convolution operators Δ_k $(k = 1, 2, \ldots)$ and Q_j $(j = 0, 1, 2, \ldots)$ with symbols $\phi\left(2^{-k}\xi\right)$ and $\psi\left(2^{-j}\xi\right)$, respectively.

For $0 < \varrho \leq 1$ and $N \in \mathbb{N}$, we denote by $\Lambda_N = \Lambda(\varrho, N, \omega, \Omega)$ the space of all sequences (m_j) with the following properties

$$\left(m_j^{(\beta)}\right)_j \subset L^\infty, \quad \left|m_j^{(\beta)}(x+h) - m_j^{(\beta)}(x)\right| \leq C\omega(|h|)\,\Omega\left(2^{\varrho j}\right), \tag{8}$$

where $|\beta| \leq N$.

The ψ.d.o. with a symbol σ is defined by the formula

$$op_\sigma f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix.\xi} \sigma(x,\xi)\,\widehat{f}(\xi)\,d\xi, \qquad (f \in \mathcal{S}, \quad x \in \mathbb{R}^n)$$

where $\widehat{f} = \mathcal{F}f$ denotes the Fourier transform of f and $\mathcal{F}^{-1}f$ its inverse. Also, we denote by $\Sigma_N = \Sigma(\delta, \varrho, m, N, \omega, \Omega)$ the collection of all ψ.d.o. with symbols satisfying (3) and (4).

Let us now recall the definition of Bessel potential and Besov spaces. For more details about equivalent norms, embeddings, etc., see [1], [5], [6] and [8].

Definition 1. *For* $s \in \mathbb{R}$, $1 < p < \infty$, *the Bessel potential spaces are*

$$H_p^s = \left\{ f \in \mathcal{S}' : \left\| \left(|Q_0 f|^2 + \sum_{j \geq 1} 4^{sj} |\Delta_j f|^2 \right)^{1/2} \right\|_p < \infty \right\}.$$

For $s \in \mathbb{R}$, $1 \le p,\ q \le \infty$, *the Besov spaces are*

$$B_p^{s,q} = \left\{ f \in \mathcal{S}' : \left(\|Q_0 f\|_p + \sum_{j \ge 1} 2^{qsj} \|\Delta_j f\|_p^q \right)^{1/q} < \infty \right\}.$$

By C, we will denote a constant which can change value at each occurrence. If $1 \le p \le \infty$, p' is the conjugate exponent, given by $p' = p\,(p-1)^{-1}$. As usual, the expression $[\gamma]$ denotes the greatest integer less than or equal to γ.

2. H_p^s-continuity. The following theorem is the principal result of this work. In it, we prove that the condition (2) is sufficient for the continuity.

Theorem 1. *Let* $0 \le \delta \le 1 - \varrho < 1$, $s \in \mathbb{R}^+ \setminus \mathbb{N}$, $m \ge 0$, $1 < p < \infty$ *and* $N \in \mathbb{N}$. *Suppose that* (2), (5) *and* (6) *hold. If* $s > \delta N$, *then every* ψ*.d.o. of* Σ_N *is bounded from* H_p^{s+m} *into* H_p^s.

The lemmas we use in proving this result are the following.

Lemma 1. *Let* $0 \le \delta \le 1$, $\varrho \ge 0$ *and* $N \in \mathbb{N}$. *If* $(\chi_j) \in \Lambda_N$, *then we have*

$$\left\| \Delta_k(\chi_j(2^{j\delta}.)) \right\|_\infty \le C 2^{(j\delta - k)N} \omega(2^{j\delta - k}) \Omega(2^{\varrho j}),$$

where C *is independent of* j *and* k.

P r o o f. By Taylor's development one has

$$\begin{aligned} \chi_j(x-y) &= \sum_{|\beta|<N} \frac{(-y)^\beta}{\beta!}\, \chi_j^{(\beta)}(x) + \\ &\quad + N \sum_{|\beta|=N} \frac{(-y)^\beta}{\beta!} \int_0^1 (1-t)^{N-1} \chi_j^{(\beta)}(x-ty)\, dt = \\ &= \sum_{|\beta|\le N} \frac{(-y)^\beta}{\beta!}\, \chi_j^{(\beta)}(x) + R_j(x,y), \end{aligned}$$

where

$$R_j(x,y) = N \sum_{|\beta|=N} \frac{(-y)^\beta}{\beta!} \int_0^1 (1-t)^{N-1} \left(\chi_j^{(\beta)}(x-ty) - \chi_j^{(\beta)}(x) \right) dt.$$

By (8) and the concavity of ω we get

$$|R_j(x,y)| \leq C|y|^N \omega(|y|)\Omega\left(2^{\varrho j}\right).$$

Since $0 \notin \operatorname{supp}\phi$ one has

$$2^{nk}\int(-y)^\beta \mathcal{F}^{-1}(\phi)(2^k y)dy = (i2^{-k})^{|\beta|}\phi^{(\beta)}(0) = 0.$$

Therefore

$$\begin{aligned}
\left|\Delta_k(\chi_j(2^{j\delta}.))(x)\right| &= \\
&= \left|\int \mathcal{F}^{-1}(\phi)(y)\,R_j(2^{j\delta}x, 2^{j\delta-k}y)dy\right| \leq \\
&\leq C2^{(j\delta-k)N}\Omega\left(2^{\varrho j}\right)\int\left|\mathcal{F}^{-1}(\phi)(y)\right||y|^N\omega(2^{j\delta-k}|y|)dy \leq \\
&\leq C2^{(j\delta-k)N}\omega(2^{j\delta-k})\Omega\left(2^{\varrho j}\right)\int\left|\mathcal{F}^{-1}(\phi)(y)\right||y|^N(1+|y|)\,dy.
\end{aligned}$$

Hence we obtain the result. □

Lemma 2. *Let $\eta > 0$, $0 \leq \delta \leq 1-\varrho < 1$, $N \in \mathbb{N}$ and $s > \delta N$. Suppose that (2), (5) and (6) hold. Then there exists a constant $C > 0$, such that for all sequences $(\chi_j) \in \Lambda_N$ and (f_j) with $\operatorname{supp}\widehat{f_j} \subset \{\xi \in \mathbb{R}^n,\ |\xi| \leq \eta 2^j\}$, we have*

$$\left\|\sum_{j\geq 0}\chi_j(2^{\delta j}.)\,f_j\right\|_{H_p^s} \leq C\left\|\left(\sum_{j\geq 0}4^{sj}|f_j|^2\right)^{1/2}\right\|_p.$$

P r o o f. Let us recall the following property of H_p^s: For $s > 0$, we have

$$\left\|\sum_{j\geq 0}g_j\right\|_{H_p^s} \leq C\left\|\left(\sum_{j\geq 0}4^{sj}|g_j|^2\right)^{1/2}\right\|_p, \tag{9}$$

where $\operatorname{supp}\widehat{g}_j \subset \left\{\xi \in \mathbb{R}^n, |\xi| \leq b2^j\right\}$, with $b > 0$ (see [1], [6] or [7]).

Now, by using (7) with $2^{-j}\xi$, we obtain $\chi_j = Q_j\chi_j + \sum_{k=j+1}^{\infty}\Delta_k\chi_j$, thus

$$\sum_{j\geq 0}\chi_j(2^{\delta j}.)f_j = u_1 + u_2, \tag{10}$$

where

$$u_1 = \sum_{j\geq 0} f_j\ Q_j\chi_j \quad \text{and} \quad u_2 = \sum_{k\geq 1}\sum_{j=0}^{k-1} f_j\ \Delta_k\chi_j.$$

To estimate $\|u_1\|_{H_p^s}$, we take into account that the function $\mathcal{F}^{-1}(f_jQ_j\ \chi_j)$ has a support in the ball $|\xi| \leq (\eta+2)\, 2^j$ and apply (9) and the inequality

$$|Q_j\chi_j\,(x)| \leq C\left\|\mathcal{F}^{-1}\psi\right\|_1 \sup_{j\geq 0}\|\chi_j\|_\infty . \tag{11}$$

To estimate $\|u_2\|_{H_p^s}$, we use that the support of $\mathcal{F}^{-1}\left(\sum_{j=0}^{k-1} f_j\ \Delta_k\chi_j\right)$ is in the ball $|\xi| \leq \left(\frac{\eta}{2}+2\right)2^k$. Then (9) and Lemma 1 imply that $\|u_2\|_{H_p^{s+N(1-\delta)}}$ is bounded by

$$C\left\|\left(\sum_{k\geq 1}4^{(s+N(1-\delta))k}\left\{\sum_{j=0}^{k-1}2^{(j\delta-k)N}\omega(2^{\delta j-k})\Omega(2^{\varrho j})\,|f_j|\right\}^2\right)^{1/2}\right\|_p . \tag{12}$$

The monotonicity of ω, Schwarz's inequality and (6) (since $s>\delta N$) imply that (12) is bounded by

$$C\left(\sum_{k\geq 1}4^{(s-\delta N)k}\omega^2(2^{(\delta-1)k})\sum_{j=0}^{k-1}4^{j(\delta N-s)}\Omega^2\left(2^{\varrho j}\right)\right)^{1/2}\left\|\left(\sum_{l\geq 0}4^{sl}\,|f_l|^2\right)^{1/2}\right\|_p =$$

$$= C\left(\sum_{k\geq 1}4^{(s-\delta N)k}\omega^2\left(2^{(\delta-1)k}\right)\times\right.$$

$$\left.\times\sum_{j=0}^{k-1}(2^{\varrho})^{2j(\delta N-s)/\varrho}\,\Omega^2\left(2^{\varrho j}\right)\right)^{1/2}\left\|\left(\sum_{l\geq 0}4^{sl}\,|f_l|^2\right)^{1/2}\right\|_p \leq$$

$$\leq C'\left(\sum_{k\geq 1}\omega^2(2^{(\delta-1)k})\Omega^2(2^{\varrho k})\right)^{1/2}\left\|\left(\sum_{l\geq 0}4^{sl}\,|f_l|^2\right)^{1/2}\right\|_p .$$

The condition $0 \leq \delta \leq 1 - \varrho < 1$ allows one to apply (2) which implies

$$\begin{aligned} \|u_2\|_{H_p^{s+N(1-\delta)}} &\leq C' \left(\sum_{k\geq 1} \omega^2 (2^{-\varrho k}) \Omega^2 (2^{\varrho k}) \right)^{1/2} \left\| \left(\sum_{l\geq 0} 4^{sl} |f_l|^2 \right)^{1/2} \right\|_p \leq \\ &\leq C'' \left\| \left(\sum_{l\geq 0} 4^{sl} |f_l|^2 \right)^{1/2} \right\|_p \end{aligned}$$

and it remains to use the inclusion $H_p^{s+N(1-\delta)} \subset H_p^s$. □

P r o o f o f T h e o r e m 1.

Step 1. We begin with some preparation. Take $\varphi \in \mathrm{C}^\infty$ such that $\operatorname{supp} \varphi \subset \{\xi \in \mathbb{R}^n, \ |\xi| \leq 1\}$ and $\varphi(\xi) = 1$ for $|\xi| \leq 1/2$.

We decompose σ into

$$\begin{aligned} \sigma(x,\xi) &= \varphi(\xi)\,\sigma(x,\xi) + (1-\varphi(\xi))\,\sigma(x,\xi) \\ &= \tau(x,\xi) + \lambda(x,\xi). \end{aligned}$$

Let θ be a real function in C^∞ such that $\operatorname{supp}\theta \subset \{\xi \in \mathbb{R}^n, 2^{-1} \leq |\xi| \leq 2\}$ and $\sum_{j\geq 0} \left(\theta\left(2^{-j}\xi\right)\right)^2 = 1$. We set

$$\sigma_j(x,\xi) = 2^{-jm}\theta(\xi)\,\lambda\left(2^{-\delta j}x, 2^j\xi\right) \tag{13}$$

and write

$$\sigma_j(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iu.\xi} \left(1 + |u|^2\right)^{-L/2} \chi_{j,u}(x)\, du \tag{14}$$

where

$$\chi_{j,u}(x) = \int_{2^{-1}\leq|\xi|\leq 2} e^{-iu.\xi} (1-\Delta_\xi)^{L/2} \sigma_j(x,\xi)\, d\xi$$

and L is a natural number satisfying $L \geq n+1$.

Now, for $|\beta| \leq N$, since $(1-\Delta_\xi)^{L/2} \partial_x^\beta \sigma_j(x,\xi)$ is a linear combination of terms of the form

$$2^{j(|\alpha|-\delta|\beta|-m)}\theta^{(\gamma)}(\xi)\,\partial_\xi^\alpha \partial_x^\beta \lambda(2^{-\delta j}x, 2^j\xi), \qquad (L = |\alpha| + |\gamma|),$$

we obtain from (3) that

$$\left|\chi_{j,u}^{(\beta)}(x)\right| \leq C \sum_{L=|\alpha|+|\gamma|} \int_{\frac{1}{2}\leq|\xi|\leq 2} \left|\theta^{(\gamma)}(\xi)\right| \left(2^{-j}+|\xi|\right)^{m-|\alpha|+\delta|\beta|} d\xi \leq C_L'.$$

Similarly, (4) yields

$$\begin{aligned}\left|\chi_{j,u}^{(\beta)}(x+h)-\chi_{j,u}^{(\beta)}(x)\right| \quad &\leq \quad C \sum_{L=|\alpha|+|\gamma|} \int_{\frac{1}{2}\leq|\xi|\leq 2} |\theta^{(\gamma)}(\xi)|\,\omega\left(|h|\,|\xi|^{\delta}\right) \times \\ &\times\Omega(|2^j\xi|^{\varrho})\left(2^{-j}+|\xi|\right)^{m-|\alpha|+\delta|\beta|} d\xi.\end{aligned}$$

Next, using (5), the monotonicity and concavity of ω, we obtain that the right-hand side of the last inequality is bounded by $C_L''\omega(|h|)\,\Omega\left(2^{j\varrho}\right)$. The constants C_L' and C_L'' are independent of j and u. Therefore $(\chi_{j,u})$ is bounded in Λ_N uniformly with respect to j and u.

We continue our construction of $\lambda(x,\xi)$. Equations (13) and (14) imply

$$\begin{aligned}\lambda(x,\xi) \quad &= \quad \sum_{j\geq 0} 2^{jm}\theta\left(2^{-j}\xi\right)\sigma_j(2^{\delta j}x, 2^{-j}\xi) = \\ &= \quad (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|u|^2)^{-(n+1)/2}\lambda_u(x,\xi)\,du\end{aligned}$$

where

$$\lambda_u(x,\xi) = \sum_{j\geq 0} 2^{jm}\theta_u\left(2^{-j}\xi\right)\chi_{j,u}(2^{\delta j}x)$$

and

$$\theta_u(\xi) = (2\pi)^{-n}(1+|u|^2)^{(n+1-L)/2}e^{iu.\xi}\theta(\xi).$$

It is easy to verify that

$$\sup_{u\in\mathbb{R}^n}\left(\|\theta_u^{(\alpha)}\|_\infty\right) \leq C, \quad (|\alpha| \leq L-n-1).$$

Step 2. For every $f \in S$ we have the decomposition

$$op_\sigma f = op_\tau f + \int_{\mathbb{R}^n}\left(1+|u|^2\right)^{-(n+1)/2} op_{\lambda_u}(f)\,du. \tag{15}$$

We shall estimate, in H_p^s-norm, each of the two terms in the right-hand side of (15).

We begin with the following observation. If $\widehat{g}_j = v\left(2^{-j}.\right)\widehat{g}$, where $v \in C^\infty$ and $\operatorname{supp} v \subset \left\{\xi \in \mathbb{R}^n,\ b^{-1} \leq |\xi| \leq b\right\}$ with $b > 1$, then there exists a constant $C_v > 0$ such that

$$\left\| \left(\sum_{j\geq 0} 4^{sj} |g_j|^2 \right)^{1/2} \right\|_p \leq C_v \|g\|_{H_p^s}. \tag{16}$$

(See [1]).

It follows immediately from Lemma 2 and (16) that

$$\sup_{u\in\mathbb{R}^n} \left(\|op_{\lambda_u} f\|_{H_p^s} \right) \leq C \|f\|_{H_p^{s+m}}.$$

We now set

$$op_\tau f(x) = \int_{\mathbb{R}^n} \left(1 + |u|^2\right)^{-2n} a_u(x) f(x+u)\, du$$

where the family $(a_u)_{u\in\mathbb{R}^n}$ of continuous functions is defined by the formula

$$a_u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iu.\xi} (I - \Delta_\xi)^{2n} \tau(x,\xi)\, d\xi.$$

By (3), one obtains

$$\sup_{u\in\mathbb{R}^n} \left(\|\partial_x^\beta a_u(\cdot)\|_\infty \right) \leq C, \quad (|\beta| \leq N),$$

and this leads to

$$\|op_\tau f\|_p \leq C \left(\sup_{u\in\mathbb{R}^n} \|a_u\|_\infty \right) \|f\|_p. \tag{17}$$

On the other hand, since $\phi\left(2^{-j}\xi\right)\varphi(\xi) = 0$ one has

$$\Delta_j (op_\tau f)(\xi) = 0, \quad \text{if } j \geq 1. \tag{18}$$

Using this equality and Young's inequality we obtain

$$\|op_\tau f\|_{H_p^s} \leq \left\|\mathcal{F}^{-1}\psi\right\|_1 \|op_\tau f\|_p. \tag{19}$$

Since $s > 0$, we can apply Schwarz's inequality:

$$\|f\|_p = \left\| \left(Q_0 + \sum_{j\geq 1} \Delta_j \right) f \right\|_p \leq \left(\sum_{j\geq 0} 4^{-sj} \right)^{1/2} \|f\|_{H_p^s}.$$

Finally, we combine the last inequality, the inclusion $H_p^{s+m} \subset H_p^s$ and (17) to verify that (19) is majorized by $C \|f\|_{H_p^{s+m}}$, as desired. □

In the following theorem, we demonstrate that condition (2) is necessary as well. We remark that for $\delta = m = s = 0$ and $\varrho = 1$ such a result was proved by Bourdaud [3].

Theorem 2. *Let $0 \le \delta < 1$, $s \in \mathbb{R}^+ \setminus \mathbb{N}$, $m \ge 0$, $1 < p < \infty$ and $N \in \mathbb{N}$ be such that $s > \delta N$. Suppose that*

$$\sum_{k \ge 0} \left[\omega(2^{-k}) \Omega(2^k) \right]^2 = \infty.$$

Then there exists an operator op_τ of Σ_N and a function $g \in H_p^{s+m}$ such that $op_\tau g \notin H_p^s$.

P r o o f. Consider the symbol

$$\tau(x,\xi) = (1+|\xi|)^m \sum_{j \ge 0} 2^{-j\delta N} \omega(2^{(\delta-1)j}) \Omega\left(2^{\varrho j}\right) \exp\left(i 2^j x_1\right),$$

where $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. It is easy to see that op_τ is in Σ_N. Indeed, we multiply τ by a partition of unity $\sum_{k=1}^{\infty} \theta\left(2^{-k}\xi\right) = 1$ for $|\xi| \ge \frac{1}{2}$, where $\theta \in \mathrm{C}^\infty$ with $\operatorname{supp} \theta \subset \left\{\xi \in \mathbb{R}^n,\ 2^{-1} \le |\xi| \le 2\right\}$, so

$$\tau(x,\xi) = (1+|\xi|)^m \sum_{k \ge 1} m_k(x) \theta(2^{-k}\xi)$$

with $m_k(x) = \sum_{j=0}^{k-1} 2^{-j\delta N} \omega\left(2^{(\delta-1)j}\right) \Omega\left(2^{\varrho j}\right) \exp\left(i 2^j x_1\right)$. We suppose furthermore that

$$\sum_{j=0}^{k} \omega\left(2^{-j}\right) \le C_0 \omega(2^{-k}) \quad \text{and} \quad \sup_{j \ge 0} \omega\left(2^{-j}\right) \Omega\left(2^j\right) < \infty. \tag{20}$$

These inequalities and (4) give necessary estimates of m_k, i.e. $(m_k) \in \Lambda_N$.

Assume now that ω does not satisfy (20). It is sufficient to replace $\omega\left(2^{(\delta-1)j}\right)$ by $\widetilde{\omega}\left(2^{(\delta-1)j}\right)$ in the expression of τ, where $\widetilde{\omega}$ is a modulus of continuity such that

$$\sum_{j=0}^{k} \widetilde{\omega}\left(2^{-j}\right) \le C_0 \omega(2^{-k}), \quad \sup_{j \ge 0} \widetilde{\omega}\left(2^{-j}\right) \Omega\left(2^j\right) < \infty$$

and

$$\sum_{j\geq 0}\left[\widetilde{\omega}\left(2^{-j}\right)\Omega\left(2^{j}\right)\right]^{2}=\infty.$$

Now, let $g=\mathcal{F}^{-1}\left((1+|.|)^{-m}\widehat{\kappa}\right)$, where $\kappa\in\mathcal{S}$ be such that $\|\kappa\|_{p}\neq 0$ and $\operatorname{supp}\widehat{\kappa}\subset\left\{\xi\in\mathbb{R}^{n},\ |\xi|\leq\frac{1}{4}\right\}$. Since

$$\mathcal{F}\left(\kappa\exp\left(i2^{j}x_{1}\right)\right)\subset\left\{\xi\in\mathbb{R}^{n},\frac{3}{4}2^{j}\leq|\xi|\leq\frac{5}{4}2^{j}\right\}$$

then by (16) we obtain

$$\|op_{\tau}g\|_{H_{p}^{s}}\geq C\,\|\kappa\|_{p}\left\{\sum_{j\geq 0}4^{(s-\delta N)j}\left[\omega\left(2^{-j}\right)\Omega\left(2^{j}\right)\right]^{2}\right\}^{1/2}=\infty.\qquad\square$$

3. $B_{p}^{s,q}$-continuity. We establish now the corresponding result for $B_{p}^{s,q}$.

We use the two following conditions. Let $1\leq q\leq\infty$

$$\sum_{k\geq 0}\left[\omega(2^{-k})\Omega(2^{k})\right]^{q}<\infty,\tag{21}$$

$$(\forall\nu>0,\ \exists C_{\nu}>0,\ \forall t\geq 1)\Rightarrow\int_{1}^{t}\frac{\Omega^{q'}(u)}{u^{\nu+1}}du\leq C_{\nu}\frac{\Omega^{q'}(t)}{t^{\nu}}.\tag{22}$$

Theorem 3. *Let $0\leq\delta\leq 1-\varrho<1$, $s\in\mathbb{R}^{+}\setminus\mathbb{N}$, $m\geq 0$, $1\leq p,q\leq\infty$ and $N\in\mathbb{N}$. Suppose that (5), (21) and (22) hold. If $s>\delta N$, then every ψ.d.o. op_{σ} of Σ_{N} is bounded from $B_{p}^{s+m,q}$ into $B_{p}^{s,q}$.*

The crucial step in the proof of Theorem 3 is the following lemma.

Lemma 3. *Let $\eta>0$, $0\leq\delta\leq 1-\varrho<1$, $N\in\mathbb{N}$ and $s>\delta N$. Suppose that (5), (21) and (22) hold. Then there exists a constant $C>0$, such that for all sequences $(\chi_{j})\in\Lambda_{N}$ and (f_{j}) with $\operatorname{supp}\widehat{f_{j}}\subset\{\xi\in\mathbb{R}^{n},\ |\xi|\leq\eta 2^{j}\}$, we have*

$$\left\|\sum_{j\geq 0}\chi_{j}(2^{\delta j}.)\,f_{j}\right\|_{B_{p}^{s,q}}\leq C\left(\sum_{j\geq 0}2^{sqj}\,\|f_{j}\|_{p}^{q}\right)^{1/q}.$$

P r o o f. We use decomposition (10) and the fact that the inequality

$$\left\| \sum_{j\geq 0} g_j \right\|_{B_p^{s,q}} \leq C \left(\sum_{j\geq 0} 2^{sqj} \|g_j\|_p^q \right)^{1/q} \tag{23}$$

holds for all $s > 0$ and any sequence (g_j) such that $\operatorname{supp} \widehat{g}_j \subset \{\xi \in \mathbb{R}^n, |\xi| \leq b2^j\}$, with $b > 1$. (See [1] or [6]).

Estimate of u_1. It is sufficient to apply (23) and (11).

Estimate of u_2. Owing to (23) and Lemma 1 we can obtain

$$\|u_2\|^q_{B_p^{s+N(1-\delta),q}} \leq \leq C \sum_{k\geq 1} 2^{(s+N(1-\delta))kq} \left(\sum_{j=0}^{k-1} 2^{(j\delta-k)N} \omega\left(2^{\delta j-k}\right) \Omega\left(2^{\varrho j}\right) \|f_j\|_p \right)^q . \tag{24}$$

By using the monotonicity of ω and Hölder's inequality in ℓ^q, we get that the right-hand side of (24) is bounded by

$$\sum_{k\geq 1} 2^{(s-\delta N)kq} \omega^q(2^{(\delta-1)k}) \left(\sum_{j=0}^{k-1} 2^{(\delta N-s)jq'} \Omega^{q'}\left(2^{\varrho j}\right) \right)^{q/q'} \sum_{l\geq 0} 2^{sql} \|f_l\|_p^q .$$

Now, since $0 \leq \delta \leq 1 - \varrho < 1$ and taking into account (22), we get that (24) is bounded by the needed expression. It remains to use the embedding $B_p^{s+N(1-\delta),q} \subset B_p^{s,q}$. □

P r o o f o f T h e o r e m 3. As in Step 2 of proof of Theorem 1, we will use (15). We first get

$$\sup_{u\in\mathbb{R}^n} \left(\|op_{\lambda_u} f\|_{B_p^{s,q}} \right) \leq C \|f\|_{B_p^{s+m,q}} .$$

This estimate is obtained by applying the following observation. For all $s > 0$ we have

$$\left(\sum_{j\geq 0} 2^{sqj} \|g_j\|_p^q \right)^{1/q} \leq C \|g\|_{B_p^{s,q}}$$

where the sequence (g_j) is the same as in (16). (See [1]).

Also, by (18) we have

$$\|op_\tau f\|_{B_p^{s,q}} \leq \left\|\mathcal{F}^{-1}\psi\right\|_1 \|op_\tau f\|_p .$$

We finish the proof by using (17), the embeddings $B_p^{s,q} \subset L^p$ and $B_p^{s+m,q} \subset B_p^{s,q}$. $\square$

The proof of the next result is based on an argument similar to the one of Theorem 2. For this reason we do not go into detail.

Theorem 4. *Let* $0 \leq \delta < 1$, $s \in \mathbb{R}^+ \setminus \mathbb{N}$, $m \geq 0$, $1 \leq p, q \leq \infty$ *and* $N \in \mathbb{N}$ *be such that* $s > \delta N$. *Suppose that*

$$\sum_{k \geq 0} \left[\omega\left(2^{-k}\right) \Omega\left(2^{k}\right) \right]^q = \infty.$$

Then there exists an operator op_τ *of* Σ_N *and a function* $g \in B_p^{s+m,q}$ *such that* $op_\tau g \notin B_p^{s,q}$.

Remark 1. In the case $\delta = 0$, $N = [s]$ and $\Omega \equiv 1$ the proof of Theorems 3 and 4 is given in [4].

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