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## POROSITY AND VARIATIONAL PRINCIPLES

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ABSTRACT. We prove that in some classes of optimization problems, like lower semicontinuous functions which are bounded from below, lower semicontinuous or continuous functions which are bounded below by a coercive function and quasi-convex continuous functions with the topology of the uniform convergence, the complement of the set of well-posed problems is  $\sigma$ -porous. These results are obtained as realization of a theorem extending a variational principle of Ioffe-Zaslavski.

**1. Introduction.** The Weierstrass theorem claims that a lower semicontinuous function attains its minimal value if it has closed level subsets and at least one of them is nonempty and compact. What happens if there is lack of compactness? A general positive answer is clearly impossible. Then given a (complete) metric space  $X$  and a family  $A$  of real-valued functions in  $X$ , it is interesting to measure the size of the set of those minimization problems which possess some good properties, like existence of a solution or also uniqueness and

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stability with respect to perturbations. Usually a set is considered big if its complement is of the first Baire category, or, whenever a measure in  $A$  is defined, if its complement has measure zero.

In literature many particular classes of problems have been studied from the point of view of the Baire category. There are many results providing the genericity of the existence of a solution (and even well-posedness, which is a stronger property) in this sense. One of the first says that, if we consider the family of continuous and bounded functions defined on a complete metric space, with the topology of the uniform convergence, the set of well-posed minimization problems is generic, i.e. it contains a  $G_\delta$  dense set, cf. [10]. In other classes of functions we can mention the Deville-Godefroy-Zizler variational principle [3], the Ioffe-Zaslavski principle [8], and many others, see [1, 5, 6, 7, 9, 13], providing similar results.

From the point of view of the measure, only few results are available (see [15]). The two notions, the first Baire category and measure zero, do not agree in general. That is a set which is small with respect to one of these notions can be big in the other sense and viceversa. For example we can decompose  $\mathbb{R}$  into two complementary sets  $A$  and  $B$ , where  $A$  is of the first Baire category and  $B$  is of measure zero, see [12]. But there is a concept of smallness that unifies both: the notion of porosity (see the exact definition in Section 2), which has its roots in some early papers of Denjoy and was introduced in metric spaces by Zajíček (see e.g. the survey [16] and the thesis [11]). The advantage of this concept is that every such set is of the first Baire category and in finite dimensions is also of Lebesgue measure zero (even in this last setting, the class of the  $\sigma$ -porous sets is strictly smaller than the class of sets that are simultaneously of the first Baire category and of Lebesgue measure zero). Moreover, in any general Banach space there are first Baire category sets that are not  $\sigma$ -porous, cf. [16].

Smallness in this latter sense of the family of ill-posed problems has been proved in some classes of optimization problems, see [2, 4]. For example, in [4] a general approach was developed to prove  $\sigma$ -porosity of ill-posed problems in the classes of (continuous) optimization problems with a metric on the functions at least as strong as the uniform distance. In such a way a strengthening of the well-known Deville-Godefroy-Zizler principle [3] was obtained.

The main result of this paper is a strengthening of the Ioffe-Zaslavski principle [8]. From this we obtain, independently from Deville and Revalski, a strengthening of the Deville-Godefroy-Zizler principle. Moreover we prove that in some classes of functions, not covered by the Deville-Godefroy-Zizler principle, the set of well-posed minimization problems has  $\sigma$ -porous complement. We consider

four classes of functions: lower semicontinuous functions which are bounded from below, lower semicontinuous or continuous functions which are bounded from below by a coercive function and quasi-convex continuous functions. The metric on these classes is always the metric of the uniform distance. In the next section we recall the notions of well-posedness and porosity, and we state the main result. Sections 3 and Section 4 are dedicated to some applications.

**2. Main result.** In this section we prove a strengthening of the Ioffe-Zaslavski principle, a useful tool to prove that in several classes of functions the well-posed minimization problems form a big subset in the corresponding class. In order to do so, we recall the notion of porosity. Let  $(X, \rho)$  be a metric space. Denote by  $B(x, \varepsilon)$  the open ball in  $X$  centered at  $x$  with radius  $\varepsilon > 0$ .

**Definition 2.1.** *A set  $M \subset X$  is called porous in  $X$  if there are  $\bar{\lambda} \in (0, 1]$  and  $\bar{\varepsilon} > 0$  such that for any  $x \in M$  (or equivalently  $x \in X$ ) and  $\varepsilon \in (0, \bar{\varepsilon}]$  there is  $z \in X$  with the property:  $B(z, \bar{\lambda}\varepsilon) \subset B(x, \varepsilon) \setminus M$ .*

*A set  $M$  is  $\sigma$ -porous in  $X$  if it is a countable union of porous sets in  $X$ .*

We work here with a notion of porosity which is stronger than the original one introduced by Zajíček (see e.g. [16, 11]) where the constant  $\bar{\lambda}$  is allowed to depend also on  $x$ . Let us mention that a porous set in  $X$  is always nowhere dense in  $X$ . Hence, every  $\sigma$ -porous set in  $X$  is also of the first Baire category in  $X$ .

Following Ioffe and Zaslavski [8], we shall consider two complete metric spaces  $(X, \rho)$  and  $(A, d)$ , the first being called the *domain space* and the second the *data space*. We shall further assume that with every  $a \in A$  a lower semicontinuous function  $f_a$  on  $X$  is associated with values in  $\bar{\mathbb{R}} = [-\infty, +\infty]$  and none of these functions is identically  $+\infty$ . The following condition (H) is used to get genericity results in minimum problems:

(H) *There is a dense subset  $B \subseteq A$ , such that for any  $a \in B$ , any  $\varepsilon > 0$  and any  $\gamma > 0$  there exist a nonempty open set  $U \subset A$ ,  $\bar{x} \in X$ ,  $\alpha \in \mathbb{R}$  and  $\eta > 0$  such that for any  $b \in U$ :*

(i)  $d(a, b) < \varepsilon$  and  $\inf f_b > -\infty$ ;

(ii) *if  $z \in X$  is such that  $f_b(z) \leq \inf f_b + \eta$ , then  $\rho(z, \bar{x}) \leq \gamma$  and  $|f_b(z) - \alpha| \leq \gamma$ .*

We shall say that the problem of minimizing  $f_a$  is well-posed if it has a unique solution and the solution is stable, see [5, 14, 17]. Here is the precise definition.

**Definition 2.2.** Given  $a \in A$ , we say that the problem of minimizing  $f_a$  on  $X$  is well-posed with respect to data in  $A$  (or just with respect to  $A$ ) if:

- (1)  $\inf f_a$  is finite and attained at a unique point  $x_a \in X$ ;
- (2) for any sequence  $\{a_n\} \subset A$  converging to  $a$ ,  $\inf f_{a_n} > -\infty$  for all large  $n$  and any sequence  $\{w_n\} \subset X$  such that  $\lim_{n \rightarrow +\infty} (f_{a_n}(w_n) - \inf f_{a_n}) = 0$  converges to  $x_a$ ; moreover,  $\lim_{n \rightarrow +\infty} f_{a_n}(w_n) = \inf f_a$ .

A sequence as  $\{w_n\}$  above is called *asymptotically minimizing sequence*. Thus minimizing  $f_a$  is a well-posed problem amounts to saying that it has a unique solution towards which every asymptotically minimizing sequence converges.

The Ioffe-Zaslavski principle claims the following:

**Theorem 2.3.** Assume (H). Then the minimization problem for  $f_a$  is well-posed with respect to  $A$  for a generic  $a \in A$ . In other words, there is a dense  $G_\delta$  subset  $A' \subseteq A$  such that for any  $a \in A'$  the minimization problem for  $f_a$  is well-posed with respect to  $A$ .

To obtain a porosity result, we need to add an uniformity hypothesis in (H). Therefore our basic hypothesis becomes:

(K) There is a dense subset  $B \subseteq A$  satisfying the following: there exist  $\lambda \in (0, 1]$  and  $\varepsilon_0 > 0$  such that for any  $a \in B$  and any  $\varepsilon \in (0, \varepsilon_0]$  there exist  $e = e(a, \varepsilon) \in A$ ,  $\bar{x} = \bar{x}(a, \varepsilon) \in X$ ,  $\alpha = \alpha(a, \varepsilon) \in \mathbb{R}$  and  $\eta = \eta(a, \varepsilon) > 0$  with  $B(e, \lambda\varepsilon) \subseteq B(a, \varepsilon)$  and  $\forall b \in B(e, \lambda\varepsilon)$ :

- (i)  $\inf f_b > -\infty$ ;
- (ii) if  $z \in X$  is such that  $f_b(z) \leq \inf f_b + \eta$ , then  $\rho(z, \bar{x}) \leq \varepsilon$  and  $|f_b(z) - \alpha| \leq \varepsilon$ .

The following is the main result of the paper.

**Theorem 2.4.** Assume (K). Then there exists a subset  $A' \subseteq A$ , with  $\sigma$ -porous complement in  $A$ , such that for any  $a \in A'$  the minimization problem for  $f_a$  is well-posed with respect to  $A$ .

**Proof.** Take  $a \in B$ . From condition (K),  $\forall n \in \mathbb{N}$  such that  $1/n \leq \varepsilon_0$  there are  $e = e(a, n) \in A$ ,  $\bar{x} = \bar{x}(a, n) \in X$ ,  $\alpha = \alpha(a, n) \in \mathbb{R}$  and  $\eta = \eta(a, n) > 0$  such that  $\forall b \in B(e, \lambda/n)$  we have

$$(2.1) \quad d(a, b) < \frac{1}{n} \quad \text{and} \quad \inf f_b > -\infty$$

and, whenever  $f_b(z) \leq \inf f_b + \eta$ ,

$$(2.2) \quad \rho(z, \bar{x}) \leq \frac{1}{n} \quad \text{and} \quad |f_b(z) - \alpha| \leq \frac{1}{n}.$$

For each  $n \geq 1/\varepsilon_0$ , consider the set

$$A_n = \bigcup_{\substack{a \in B \\ m \geq n}} B\left(e, \frac{\lambda}{m}\right),$$

where  $e = e(a, m)$ . Let  $A' = \cap A_n$ . We now prove that the complement of  $A'$  in  $A$  is  $\sigma$ -porous in  $A$  and  $\forall a \in A'$  the minimization problem for  $f_a$  is well-posed with respect to  $A$ .

To prove that  $A \setminus A'$  is  $\sigma$ -porous, we prove that  $M_n = A \setminus A_n$  is porous. In fact

$$A \setminus A' = A \setminus \cap A_n = \cup(A \setminus A_n)$$

Now we show that the Definition 2.1 is verified for  $M_n$ , with the choice of  $\bar{\lambda} = \lambda/2$  and  $\bar{\varepsilon} = 1/n$ . Take  $c \in M_n \cap B$  and  $\varepsilon \leq \bar{\varepsilon}$ .  $\exists \bar{m} \geq n$  such that  $\varepsilon \in (1/(\bar{m}+1), 1/\bar{m}]$ . As  $c \in B$ , from (K) we have that there is a ball  $B(e, \lambda/(\bar{m}+1)) \subseteq B(c, 1/(\bar{m}+1))$ . From the definition of  $A_n$  we have:  $B(e, \lambda/(\bar{m}+1)) \subseteq A_n$ . It follows that

$$B(e, \bar{\lambda}\varepsilon) = B\left(e, \frac{\lambda}{2}\varepsilon\right) \subseteq B\left(e, \frac{\lambda}{\bar{m}+1}\right) \subseteq B\left(c, \frac{1}{\bar{m}+1}\right) \cap A_n \subseteq B(c, \varepsilon) \cap A_n.$$

Let  $c \in M_n \setminus B$  and  $\varepsilon \leq \bar{\varepsilon}$ .  $\exists \bar{m} \geq n$  such that  $\varepsilon \in (1/(\bar{m}+1), 1/\bar{m}]$ .  $B$  is a dense subset of  $A \Rightarrow \exists c_1 \in B$  such that

$$d(c, c_1) < \varepsilon - \frac{1}{\bar{m}+1} \Rightarrow B\left(c_1, \frac{1}{\bar{m}+1}\right) \subseteq B(c, \varepsilon).$$

As  $c_1 \in B$ ,  $\exists B(e, \lambda/(\bar{m}+1)) \subseteq B(c_1, 1/(\bar{m}+1)) \cap A_n$ . It follows that

$$B(e, \bar{\lambda}\varepsilon) = B\left(e, \frac{\lambda}{2}\varepsilon\right) \subseteq B\left(e, \frac{\lambda}{\bar{m}+1}\right) \subseteq B\left(c_1, \frac{1}{\bar{m}+1}\right) \cap A_n \subseteq B(c, \varepsilon) \cap A_n.$$

So,  $M_n$  is porous in  $A$ .

We now prove that the minimization problem for  $f_a$  is well-posed with respect to  $A$  for all  $a \in A'$ . So, let  $a \in A'$ . Then, there are two sequences  $\{a_n\} \subset B$  and  $\{k_n\} \subset N$ , with  $\lim_{n \rightarrow +\infty} k_n = +\infty$ , such that  $a \in B_n = B(e_n, \lambda/k_n)$ , where  $e_n = e(a_n, k_n)$ . Set  $x_n = \bar{x}(a_n, k_n)$ ,  $\alpha_n = \alpha(a_n, k_n)$  and  $\eta_n = \eta(a_n, k_n)$ . We assume without loss of generality that  $\lim_{n \rightarrow +\infty} \eta_n = 0$  decreasingly. Let  $z_n \in X$  be such that  $f_a(z_n) < \inf f_a + \eta_n$ . Then  $f_a(z_n) < \inf f_a + \eta_m$  if  $m \leq n$ , so by (2.2)  $\rho(z_n, x_m) \leq 1/m$ . It follows that  $\rho(z_n, z_{n+k}) \leq 2/m$ , for any  $n \geq m$  and any  $k \in N$ , that is,  $\{z_n\}$  is a Cauchy sequence. Set  $x_a = \lim_{n \rightarrow +\infty} z_n$ . As  $f_a$  is lower semicontinuous, we have  $f_a(x_a) \leq \liminf_{n \rightarrow +\infty} f_a(z_n) = \inf f_a$ . The element  $x_a \in X$  such that  $f_a(x_a) = \inf f_a$  is unique, indeed if there exists  $y_a \neq x_a$  s.t.  $f_a(y_a) =$

$\inf f_a$ , we would have a nonconvergent sequence  $\{z_n\} = \{x_a, y_a, x_a, y_a, \dots\}$ . We further note that by (2.2)  $f_a(x_a) = \lim_{n \rightarrow +\infty} \alpha_n$ .

To conclude, consider a sequence  $\{b_n\} \subset A$  converging to  $a$ , and let  $\{w_n\} \subset X$  be a sequence such that  $\lim_{n \rightarrow +\infty} (f_{b_n}(w_n) - \inf f_{b_n}) = 0$ . Set  $\xi_n = (f_{b_n}(w_n) - \inf f_{b_n})$ ,  $\forall n \in N$ . Choose  $n(m) \in N$  such that  $b_n \in B_m$  and  $\xi_n \leq \eta_m$  for  $n \geq n(m)$ . For such  $n$  we have by (2.2):  $\rho(w_n, x_m) \leq 1/m$  and  $|f_{b_n}(w_n) - \alpha_m| \leq 1/m$ . As  $\lim_{m \rightarrow +\infty} x_m = x_a$  and  $\lim_{m \rightarrow +\infty} \alpha_m = f_a(x_a)$ , it follows that  $\lim_{n \rightarrow +\infty} w_n = x_a$  and  $\lim_{n \rightarrow +\infty} f_{b_n}(w_n) = \inf f_a$ .

This completes the proof.  $\square$

**3. Porosity in the Deville-Godefroy-Zizler principle.** We prove that a strengthening to the porosity of the variational principle of Deville-Godefroy-Zizler [3] is a consequence of the Theorem 2.4. To prove this we consider a Banach space  $A$  of bounded continuous functions on a Banach space  $X$  with the following three properties:

- (a) the norm in  $A$  is not weaker than the topology of uniform convergence on  $X$ :  $\|a\|_A \geq \sup\{|a(x)| : x \in X\}$ ;
- (b)  $A$  contains compositions of its elements with translations and homotheties of  $X$  and  $\|a(t + \cdot)\|_A = \|a\|_A$  for each  $a \in A$  and each  $t \in X$ ;
- (c)  $A$  contains a bump function, that is to say, a function  $\varphi(x)$  supported on the unit ball and satisfying  $0 \leq \varphi(x) \leq 1 = \varphi(0)$ .

Under these assumptions, the variational principle of Deville-Godefroy-Zizler states that *for any proper l.s.c. and bounded from below function  $f$  on  $X$  the set of  $a \in A$  for which  $f + a$  is Tykhonov well-posed is a dense  $G_\delta$  subset of  $A$* . A minimization problem for  $f$  is Tykhonov well-posed if it has a unique solution  $x_0$  towards which every minimizing sequence converges, i.e.  $\exists ! x_0 \in X$  such that  $f(x_0) = \inf f$  and for any sequence  $\{x_n\} \subset X$  such that  $\lim_{n \rightarrow +\infty} f(x_n) = \inf f$  we have  $\lim_{n \rightarrow +\infty} x_n = x_0$ . Remark that if a problem is well-posed with respect to  $A$ , it is also Tykhonov well-posed. In the setting of the Deville-Godefroy-Zizler principle the converse is also true.

As an application of the Theorem 2.4 we have:

**Theorem 3.1.** *The set of  $a \in A$  for which  $f + a$  is well-posed with respect to  $A$  has a  $\sigma$ -porous complement in  $A$ .*

**Proof.** To prove the statement we set  $f_a = f + a$  and  $B = A$ . (i) of (K) is satisfied  $\forall a \in A$ . Take  $\lambda = 1/10$  and  $\varepsilon_0 = 1$ . Given  $a \in A$  and  $\varepsilon \in (0, 1]$ ,

choose

$$\psi(x) = \frac{\varphi(\varepsilon^{-1}x)}{\|\varphi(\varepsilon^{-1}x)\|_A} \quad \forall x \in X,$$

where  $\varphi$  is the bump function defined in (c). From condition (b) it follows that  $\psi \in A$ . Moreover, from (a),  $\|\varphi(\varepsilon^{-1}x)\|_A \geq \|\varphi(\varepsilon^{-1}x)\|_\infty = 1$ . Furthermore  $\|\psi\|_A = 1$ .

Let  $\bar{x} \in X$  s.t.  $f_a(\bar{x}) < \inf f_a + \varepsilon/5$ ,  $e(x) = a(x) - (4\varepsilon/5)\psi(x - \bar{x})$ ,  $\alpha = f_e(\bar{x})$  and  $\eta = \varepsilon/10$ .  $A$  is a Banach space  $\Rightarrow e \in A$ .  $B(e, \lambda) = B(e, \varepsilon/10) \subseteq B(a, \varepsilon)$ , indeed

$$(e - a)(x) = -\frac{4\varepsilon}{5}\psi(x - \bar{x}) \Rightarrow \|e - a\|_A = \frac{4\varepsilon}{5}\|\psi\|_A = \frac{4\varepsilon}{5},$$

it follows that for each  $b \in B(e, \varepsilon/10)$ :

$$\|b - a\|_A \leq \|b - e\|_A + \|e - a\|_A < \frac{\varepsilon}{10} + \frac{4\varepsilon}{5} = \frac{9\varepsilon}{10} < \varepsilon.$$

Let  $b \in B(e, \varepsilon/10)$  and let  $z \in X$  such that  $f_b(z) \leq \inf f_b + \eta$ :

$$\begin{aligned} \alpha - \frac{\varepsilon}{5} &= f_e(\bar{x}) - \frac{\varepsilon}{5} = f_a(\bar{x}) - \frac{4\varepsilon}{5} - \frac{\varepsilon}{5} < \inf f_a - \frac{4\varepsilon}{5} \leq f_a(z) - \frac{4\varepsilon}{5} \leq \\ &\leq f_e(z) < f_b(z) + \frac{\varepsilon}{10} \leq \inf f_b + \frac{\varepsilon}{5} \leq \inf f_e + \frac{3\varepsilon}{10} \leq \alpha + \frac{3\varepsilon}{10} = \\ &= f_a(\bar{x}) - \frac{4\varepsilon}{5} + \frac{3\varepsilon}{10} = f_a(\bar{x}) - \frac{\varepsilon}{5} - \frac{3\varepsilon}{10} < \inf f_a - \frac{3\varepsilon}{10}. \end{aligned}$$

It follows that  $f_e(z) < \inf f_a$  and  $|\alpha - f_b(z)| \leq \varepsilon$ . Indeed

$$\alpha - \frac{\varepsilon}{5} < f_b(z) + \frac{\varepsilon}{10} < \alpha + \frac{3\varepsilon}{5} \Rightarrow \alpha - f_b(z) < \frac{3\varepsilon}{10} \text{ and } f_b(z) - \alpha < \frac{\varepsilon}{5}.$$

Moreover  $\|z - \bar{x}\| \leq \varepsilon$ , because otherwise, by the definition of  $e$ :

$$e(z) = a(z) - \left(\frac{4\varepsilon}{5}\right)\psi(z - \bar{x}) = a(z) \Rightarrow f_a(z) = (f + a)(z) = (f + e)(z) = f_e(z)$$

a contradiction to the inequality above:  $f_e(z) < \inf f_a$ .  $\square$

**4. Further applications.** To provide other possible applications to the principle, we shall consider now some classes  $A$  of optimization problems. It should be observed that for the classes we consider in this section the principle of Deville-Godefroy-Zizler cannot be applied, as  $A$  is not a Banach space.

Let  $(X, \rho)$  be a complete metric space. Let  $A$  be a family of functions in  $X$  endowed with the topology of uniform convergence. A metric which induces this topology is the following one:

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in X\} \quad \text{with } f, g \in A.$$

The space  $A$  will be one of the following four ones:



1.  $A = \{f : X \rightarrow \mathbb{R} : f \text{ is lower semicontinuous and bounded from below}\}$ ;
2.  $A = \{f : X \rightarrow \mathbb{R} : f \text{ is lower semicontinuous and } f(x) \geq \psi(x), \forall x \in X\}$  with  $\psi$  a bounded from below coercive function in  $X$  (the latter means  $\psi(x) \rightarrow \infty$  if  $\rho(x, \theta_X) \rightarrow \infty$ , where  $\theta_X$  is a fixed element of  $X$ );
3.  $A = \{f : X \rightarrow \mathbb{R} : f \text{ is continuous and } f(x) \geq \psi(x), \forall x \in X\}$  with  $\psi$  a bounded from below coercive function in  $X$ ;
4.  $X$  is a real Banach space and  $A = \{f : X \rightarrow \mathbb{R} : f \text{ is continuous, quasi-convex and bounded from below}\}$ .

Observe that in all the above cases the space  $(A, d)$  is a complete metric space.

**Theorem 4.1.** *Let  $A$  be one of the four classes of functions we consider. Then, there is a subset  $A' \subseteq A$  with  $\sigma$ -porous complement in  $A$  such that  $\forall a \in A'$  the minimization problem for  $a$  is well-posed with respect to  $A$ .*

*Proof.* Let  $f_a = a, \forall a \in A$ . We prove that (K) is satisfied. Let  $\lambda = 1/8$  and  $\varepsilon_0 = 1$ . If we are in the first three spaces of functions (l.s.c., coercive and l.s.c., coercive and continuous) set  $B = \{f \in A : f \text{ attains a minimum}\}$ . If we are in the space of quasi-convex functions let  $B = \{f \in A : \exists c \in \mathbb{R} \text{ such that } \text{Int} f^c \neq \emptyset \text{ and } f^c(x) = c, \forall x \in f^c\}$ , with  $f^c = \{x \in X : f(x) \leq c\}$ .  $B$  is dense in  $A$ , because for any  $f \in A$  and any  $\varepsilon > 0, \exists g \in B$  such that  $d(f, g) < \varepsilon$ . For, set

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \text{lev}\left(f, \frac{\varepsilon}{2}\right) \\ \inf f + \frac{\varepsilon}{2} & \text{if } x \in \text{lev}\left(f, \frac{\varepsilon}{2}\right) \end{cases}$$

with  $\text{lev}(f, c) = \{x \in X : f(x) \leq \inf f + c\}$ . In all the above cases  $g \in B$  and  $d(f, g) < \varepsilon$ .

Let  $f \in B$  and  $\varepsilon \in (0, 1]$ . We can choose  $\bar{x} \in X$  with  $f(\bar{x}) = \inf f$  and  $\gamma \in (0, \varepsilon]$  such that, in the quasi-convex case,  $B(\bar{x}, \gamma) \subset \text{lev}(f, 0)$ . Consider the function

$$\bar{f}(x) = \begin{cases} f(x) + \frac{\varepsilon}{2} & \text{if } \rho(x, \bar{x}) > \gamma \\ f(x) + \frac{\varepsilon}{2\gamma} \rho(x, \bar{x}) & \text{if } \rho(x, \bar{x}) \leq \gamma \end{cases};$$

$\bar{f} \in A, d(\bar{f}, f) \leq \varepsilon$  and  $\inf \bar{f} = \bar{f}(\bar{x}) = \inf f$ . With  $e = \bar{f}, \eta = \varepsilon/4$  and  $\alpha = \inf \bar{f}$ , (K) is satisfied, as we now verify.

$$B(\bar{f}, \lambda\varepsilon) = B\left(\bar{f}, \frac{\varepsilon}{8}\right) \subset B(f, \varepsilon).$$

Let  $g \in B(\bar{f}, \varepsilon/8)$  and let  $z \in X$  such that  $g(z) \leq \inf g + \varepsilon/4$ . We prove that  $|g(z) - \alpha| \leq \varepsilon$  and  $\rho(z, \bar{x}) \leq \varepsilon$ . We have

$$\begin{cases} g(z) - \alpha = g(z) - \inf \bar{f} \leq \inf g - \inf \bar{f} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{8} + \frac{\varepsilon}{4} < \varepsilon \\ g(z) - \alpha = g(z) - \inf \bar{f} \geq \inf g - \inf \bar{f} \geq -\frac{\varepsilon}{8} > -\varepsilon \end{cases}.$$

Moreover

$$\bar{f}(z) < g(z) + \frac{\varepsilon}{8} \leq \inf g + \frac{3}{8}\varepsilon \leq \inf \bar{f} + \frac{\varepsilon}{2}.$$

If  $\rho(z, \bar{x}) > \varepsilon \geq \gamma$ , it follows that

$$\bar{f}(z) = f(z) + \frac{\varepsilon}{2} \geq \inf f + \frac{\varepsilon}{2} = \inf \bar{f} + \frac{\varepsilon}{2},$$

a contradiction.

(K) is satisfied and the theorem holds.  $\square$

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