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# PARTIAL QUASI-BILATERAL GENERATING FUNCTION INVOLVING SOME SPECIAL FUNCTIONS

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Abstract. A group-theoretic method of obtaining more general class of generating functions from a given class of partial quasi-bilateral generating functions involving Hermite, Laguerre and Gegenbaur polynomials are discussed.

**1. Introduction and preliminaries.** The usual generating relation involving one special function is called linear or unilateral generating relation. By the term usual bilateral generating function, we mean a function $G(x, z, w)$ which can be expanded in powers of $w$ by the following relation:

$$G(x, z, w) = \sum_{n=0}^{\infty} a_n w^n f_n(x) g_n(z)$$

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where $a_n$ is quite arbitrary that is independent of $x$ and $z$ and $f_n(x)$, $g_n(z)$ are two different special functions. In particular, when two special functions are same that is $f_n \equiv g_n$, we call the generating relation as bilinear generating relation.

Unlike the usual bilateral generating relations [1], we shall introduce the concept of partial quasi-bilateral generating relation [2] for Hermite, Laguerre and Gegenbaur polynomials.

**Definition 1.1.** *By the term partial quasi-bilateral generating relation for two classical polynomials, we mean the relation:*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n S_{m+n}^{(\alpha)}(x) T_p^{(m+n)}(z) \tag{1.1.1}$$

*where the coefficients $a_n$'s are quite arbitrary and $S_{m+n}^{(\alpha)}(x)$, $T_p^{(m+n)}(z)$ are two particular special functions.*

The object of this note is to consider the operator

$$R_1 = xy^2\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y} + (1+m-2x^2)y^2$$

which raises the index of the Hermite polynomial by two and to establish a group theoretic method for obtaining a more general class of generating relation from a given class of partial quasi-bilateral generating relations involving Hermite, Laguerre and Gegenbaur polynomials when suitable one-parameter continuous transformations group can be constructed for those special functions.

## 2. Main results.

**a) Index raising operator for Hermite polynomial $H_{m+n}(x)$.** Here we would like to prove $R_1[H_{m+n}(x)y^n] = -\frac{1}{2}H_{m+n+2}(x)y^{n+2}$ where $R_1 = xy^2\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y} + (1+m-2x^2)y^2$ which will help us in proving our main theorems.

Now

$$\begin{aligned}
&R_1[H_{m+n}(x)y^n] \\
&\quad = \left[xy^2\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y} + (1+m-2x^2)y^2\right][H_{m+n}(x)y^n] \\
&\quad = xy^{n+2}\frac{\partial}{\partial x}(H_{m+n}(x)) + ny^3y^{n-1}H_{m+n}(x) + (1+m-2x^2)y^{n+2}H_{m+n}(x)
\end{aligned}$$

$$= y^{n+2}[x.2(m+n)H_{m+n-1}(x) + nH_{m+n}(x) + (1+m-2x^2)H_{m+n}(x)]$$
$$= y^{n+2}[2x(m+n)H_{m+n-1}(x) + (1+m+n-2x^2)H_{m+n}(x)]$$
$$= y^{n+2}[(1+m+n)H_{m+n}(x) - x(2xH_{m+n}(x) - 2(m+n)H_{m+n-1}(x))]$$
$$= y^{n+2}[(1+m+n)H_{m+n}(x) - xH_{m+n+1}(x)]$$
$$= -\frac{1}{2}y^{n+2}[2xH_{m+n+1}(x) - 2(m+n+1)H_{m+n}(x)]$$
$$= -\frac{1}{2}H_{m+n+2}(x)y^{n+2}$$

So the result is proved. If we slightly modify the operator $R_1$ as $R_2 = xy^2\frac{\partial}{\partial x}+y^3\frac{\partial}{\partial y}+(1-2x^2)y^2$, then we can similarly prove $R_2[H_{m+n}(x)y^{m+n}] = -\frac{1}{2}H_{m+n+2}(x)y^{m+n+2}$.

It may be worthy of mention here that in the derivation of generating functions for Hermite polynomial by group-theoretic method, one has to consider two operators which raise and lower the index of the Hermite polynomial by unity in order to generate a Lie Algebra [3]. But the operator $R_1$ (or $R_2$) raises the index of the Hermite polynomials by two.

**b) Group-theoretic method.** The group-theoretic method in the derivation of a more general class of generating relations from a given class of partial quasi-bilateral generating relations involving Hermite, Laguerre and Gegenbaur polynomials are discussed below.

Let us consider the partial quasi-bilateral generating relation involving Hermite and Laguerre polynomials of the form:

(2.1.1) $$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) L_p^{(m+n)}(z)$$

where $a_n$'s are quite arbitrary and $H_{m+n}(x)$, $L_p^{(m+n)}(z)$ are Hermite and Laguerre polynomials of order $(m+n)$ and $p$ respectively.

Now we seek the following two operators

$$R_1 = xy^2\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial y} + (1-2x^2)y^2 \text{ and } R_2 = t\frac{\partial}{\partial z} - t$$

such that

$$R_1[H_{m+n}(x)y^{m+n}] = -\frac{1}{2}H_{m+n+2}(x)y^{m+n+2}$$

and

$$R_2[L_p^{(m+n)}(z)t^{m+n}] = (-1)L_p^{(m+n+1)}(z)t^{m+n+1}$$

where

$$\exp(wR_1)f(x,y) = (1-2wy^2)^{-\frac{1}{2}} \exp\left(-\frac{2wx^2y^2}{1-2wy^2}\right) f\left(\frac{x}{\sqrt{1-2wy^2}}, \frac{y}{\sqrt{1-2wy^2}}\right)$$

and $\exp(vR_2)f(z,t) = \exp(-vt)f(z+vt,t)$.
Now multiplying both sides of (2.1.1) by $w^m$, we get

$$(2.1.2) \qquad w^m G(x,z,w) = \sum_{n=0}^{\infty} a_n w^{m+n} H_{m+n}(x) L_p^{(m+n)}(z)$$

We now replace '$w$' by '$wvyt$' in (2.1.2) and obtain

$$(2.1.3.) \quad (wvyt)^m G(x,z,wvyt) = \sum_{n=0}^{\infty} a_n (wvyt)^{m+n} H_{m+n}(x) L_p^{(m+n)}(z)$$

$$= \sum_{n=0}^{\infty} a_n (wv)^{m+n} (H_{m+n}(x) y^{m+n})(L_p^{(m+n)}(z) t^{m+n}).$$

We now operate [4] both sides of (2.1.3) by $\exp(wR_1)\ \exp(vR_2)$ and as a result of it, the relation (2.1.3) reduces to

$$(2.1.4) \qquad (1-2wy^2)^{-\frac{1}{2}} \exp\left(-\frac{2wx^2y^2}{1-2wy^2}\right) \exp(-vt)(wv)^m \left(\frac{yt}{\sqrt{1-2wy^2}}\right)^m$$

$$G\left(\frac{x}{\sqrt{1-2wy^2}}, z+vt, \frac{wvyt}{\sqrt{1-2wy^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \left(\frac{(wR_1)^s}{s!} H_{m+n}(x) y^{m+n}\right)\left(\frac{(vR_2)^r}{r!} L_p^{(m+n)}(z) t^{m+n}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^{s+r} a_n \frac{w^{m+n+s} v^{m+n+r}}{2^s s! r!} (H_{m+n+2s}(x) y^{m+n+2s})(L_p^{(m+n+r)}(z) t^{m+n+r}).$$

Putting $y = t = 1$ in (2.1.4) we get:

$$(1-2w)^{-\frac{1}{2}} \exp\left(-\frac{2wx^2}{1-2w}\right) \exp(-v)(wv)^m \left(\frac{1}{\sqrt{1-2w}}\right)^m G\left(\frac{x}{\sqrt{1-2w}}, z+v, \frac{wv}{\sqrt{1-2w}}\right) \tag{2.1.5}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+r} a_n \frac{w^{m+n+s} v^{m+n+r}}{2^s s! r!} H_{m+n+2s}(x) L_p^{(m+n+r)}(z).$$

Equating the coefficient of $(wv)^m$ from both sides of (2.1.5) we get

$$(1-2w)^{-\frac{m+1}{2}} \exp\left(-\frac{2wx^2}{1-2w} - v\right) G\left(\frac{x}{\sqrt{1-2w}}, z+v, \frac{wv}{\sqrt{1-2w}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+r} a_n \frac{w^{n+s} v^{n+r}}{2^s s! r!} H_{m+n+2s}(x) L_p^{(m+n+r)}(z).$$

Thus we have the following theorem:

**Theorem 2.1.** *If there exist the following partial quasi-bilateral generating functions for Hermite polynomial $H_{m+n}(x)$ and Laguerre polynomial $L_p^{(m+n)}(z)$, of order $(m+n)$ and $p$ of the form:*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) L_p^{(m+n)}(z),$$

*where $a_n$ is quite arbitrary that is independent of $x$ and $z$, then the following more general generating relation holds:*

$$(1-2w)^{-\frac{m+1}{2}} \exp\left(-\frac{2wx^2}{1-2w} - v\right) G\left(\frac{x}{\sqrt{1-2w}}, z+w, \frac{wv}{\sqrt{1-2w}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{s+r} a_n \frac{w^{n+s} v^{n+r}}{2^s s! r!} H_{m+n+2s}(x) L_p^{(m+n+r)}(z), \text{ where } |w| < \frac{1}{2}.$$

**Theorem 2.2.** *If there exist the following partial quasi-bilateral generating functions for Hermite polynomial $H_{m+n}(x)$ and Gegenbaur polynomial*

$C_p^{m+n}(z)$ *of order* $(m+n)$ *and* $p$ *of the form:*

$$G(x,z,w)=\sum_{n=0}^{\infty}a_nw^nH_{m+n}(x)C_p^{(m+n)}(z),$$

*where* $a_n$ *is quite arbitrary that is independent of* $x$ *and* $z$*, then the following more general generating relation holds:*

$$(1-2w)^{-\frac{m+1}{2}}(1-2v)^{-\frac{p+2m}{2}}\exp\left(-\frac{2wx^2}{1-2w}\right)$$

$$G\left(\frac{x-w}{\sqrt{1-2w}},\frac{z}{\sqrt{1-2v}},\frac{wv}{(1-2v)\sqrt{1-2w}}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^sa_n\frac{2^{r-s}w^{n+s}v^{n+r}}{s!r!}H_{m+n+2s}(x)C_p^{(m+n+r)}(z),$$

*where* $|w|<\dfrac{1}{2}$ *and* $|v|<\dfrac{1}{2}$.

P r o o f. Let us consider the following partial quasi-bilateral generating relation involving Hermite polynomial $H_{m+n}(x)$ and Gegenbaur polynomial $C_p^{m+n}(x)$ of the form:

$$G(x,z,w)=\sum_{n=0}^{\infty}a_nw^nH_{m+n}(x)C_p^{(m+n)}(z). \tag{2.2.1}$$

Multiplying both sides of (2.2.1), by $w^m$ we get

$$w^mG(x,z,w)=\sum_{n=0}^{\infty}a_nw^{m+n}H_{m+n}(x)C_p^{(m+n)}(z). \tag{2.2.2}$$

Now replacing '$w$' by '$wvyt$' in (2.2.2) we get:

$$(wvyt)^mG(x,z,wvyt)=\sum_{n=0}^{\infty}a_n(wv)^{m+n}(H_{m+n}(x)y^{m+n})(C_p^{(m+n)}t^{m+n}). \tag{2.2.3}$$

We now seek the following two operators

$$R_1=xy^2\frac{\partial}{\partial x}+y^3\frac{\partial}{\partial y}+(1-2x^2)y^2\quad\text{and}\quad R_2=zt\frac{\partial}{\partial z}+2t^2\frac{\partial}{\partial t}+pt$$

such that

$$R_1[H_{m+n}(x)y^{m+n}] = -\frac{1}{2}H_{m+n+2}(x)y^{m+n+2}$$

and

$$R_2[C_p^{(m+n)}(z)t^{m+n}] = 2(m+n)C_p^{(m+n+1)}(z)t^{m+n+1}$$

where

$$\exp(wR_1)f(x,y) = (1-2wy^2)^{-\frac{1}{2}}\exp\left(-\frac{2wx^2y^2}{1-2wy^2}\right)f\left(\frac{x}{\sqrt{1-2wy^2}}, \frac{y}{\sqrt{1-2wy^2}}\right)$$

and

$$\exp(vR_2)f(z,t) = (1-2vt)^{-\frac{1}{2}}f\left(\frac{z}{\sqrt{1-2vt}}, \frac{t}{1-2vt}\right).$$

We now operate both sides of (2.2.3) by $\exp(wR_1)\exp(vR_2)$ and as a result of it, the relation (2.2.3) reduces to

$$(2.2.4) \qquad (1-2wy^2)^{-\frac{1}{2}}(1-2vt)^{-\frac{p}{2}}\exp\left(-\frac{2wx^2y^2}{1-2wy^2}\right)(wv)^m$$

$$\left(\frac{yt}{(1-2vt)\sqrt{1-2wy^2}}\right)^m G\left(\frac{x}{\sqrt{1-2wy^2}}, \frac{z}{\sqrt{1-2vt}}, \frac{wvyt}{(1-2vt)\sqrt{1-2wy^2}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^{m+n}\left(\frac{(wR_1)^s}{s!}H_{m+n}(x)y^{m+n}\right)\left(\frac{(vR_2)^r}{r!}C_p^{(m+n)}(z)t^{m+n}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s a_n \frac{2^r(m+n)_r.w^{m+n+s}v^{m+n+r}}{2^s s!r!}$$

$$(H_{m+n+2s}(x)y^{m+n+2s})(C_p^{(m+n+r)}(z)t^{m+n+r})$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s a_n \frac{2^{r-s}(m+n)_r.w^{m+n+s}v^{m+n+r}}{s!r!}$$

$$(H_{m+n+2s}(x)y^{m+n+2s})(C_p^{(m+n+r)}(z)t^{m+n+r})$$

Now putting $y = t = 1$ in (2.2.4) and equating the coefficient of $(wv)^n$ from both sides of (2.2.4), we get:

$$(1-2w)^{-\frac{m+1}{2}}(1-2v)^{-\frac{p+2m}{2}}\exp\left(-\frac{2wx^2}{1-2w}\right)G\left(\frac{x}{\sqrt{1-2w}}, \frac{z}{\sqrt{1-2v}}, \frac{wv}{(1-2v)\sqrt{1-2w}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s a_n \frac{2^{r-s}(m+n)_r w^{n+s} v^{n+r}}{s!r!} H_{m+n+2s}(x) C_p^{(m+n+r)}(z),$$

whenever

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) C_p^{(m+n)}(z) \quad \text{and } |w| < \frac{1}{2},\ |v| < \frac{1}{2}.$$

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