GROUPS WITH THE MINIMAL CONDITION ON NON-“NILPOTENT-BY-FINITE” SUBGROUPS

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Abstract. We characterize the groups which do not have non-trivial perfect sections and such that any strictly descending chain of non-“nilpotent-by-finite” subgroups is finite.

0. Let $\mathcal{X}$ be a property pertaining to subgroups. One of approaches to study the structure of groups is to investigate the groups in which the set of non-$\mathcal{X}$-subgroups is small in some sense (or in other words, the groups which have many $\mathcal{X}$-subgroups). There is increasing interest in groups with many nilpotent subgroups. Problems of this type have been considered in several papers. For instance, the examples of non-nilpotent groups with nilpotent and subnormal proper subgroups (the so-called groups of Heineken-Mohamed type) were constructed by Heineken and Mohamed [15], Hartley [14], Menegazzo [16] and others. In this way Bruno [6–8], Bruno and Phillips [9] and Asar [4] have studied the minimal non-“nilpotent-by-finite” groups (i.e. non-“nilpotent-by-finite”

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groups with nilpotent-by-finite proper subgroups). Their results imply that any minimal non-"nilpotent-by-finite" group \( G \) is either a \( p \)-group in which any two proper subgroups generate a proper subgroup or \( G = V \rtimes H \) is a semidirect product of a normal subgroup \( V \) and a quasicyclic \( p \)-subgroup \( H \), where \( V \) is a special \( q \)-group, \( V' \) is centralized by \( H \) and \( V/V' \) is a minimal normal subgroup of \( G/V' \) (\( p \) and \( q \) are distinct primes).

One of the possible interpretations of the requirement that "many subgroups of \( G \) satisfy \( \mathcal{X} \)" is that "the set of subgroups in \( G \) not having \( \mathcal{X} \) satisfies the minimal condition". In this direction the aim of our article is to consider groups with the minimal condition on non-"nilpotent-by-finite" subgroups Min\(-NF\). We say that a group \( G \) satisfies Min\(-NF\) if for every strictly descending chain \( \{G_n \mid n \in \mathbb{N}\} \) of subgroups in \( G \) there exists a number \( n_0 \in \mathbb{N} \) such that \( G_n \) is a nilpotent-by-finite subgroup for any integer \( n \geq n_0 \). For the next we need the concept of an \( HM^*\)-group first introduced by Asar [3] in the class of \( p \)-groups with the normalizer condition. We extend this notion and call \( G \) an \( HM^*\)-group if its commutator subgroup \( G' \) is hypercentral and \( G/G' \) is a divisible Černikov \( p \)-group. Obviously any group of Heineken-Mohamed type and any minimal non-"nilpotent-by-finite" group are \( HM^*\)-groups and satisfy Min\(-NF\).

Notice that Černikov (see e.g. [11]) and Šunkov [20] have studied groups with the minimal condition on non-abelian subgroups; Phillips and Wilson [17] have investigated groups in which the set of non-"locally nilpotent" subgroups satisfies the minimal condition. Recently Dixon, Evans and Smith [12] have shown that a locally graded group with the minimal condition on non-nilpotent subgroups is either nilpotent or locally finite.

In this paper we characterize groups without non-trivial perfect sections and which satisfy Min\(-NF\). Namely, we prove:

**Theorem.** Let \( G \) be a group without non-trivial perfect sections. Then \( G \) satisfies Min\(-NF\) if and only if it is of one of the following types:

1. \( G \) is a nilpotent-by-finite group;
2. \( G \) contains a normal subgroup \( H \) of finite index such that

   \[
   H = H_0 \cdot H_1 \cdot \ldots \cdot H_n \ (n \geq 1),
   \]

   where \( H_i \) is an \( HM^*\)-group with nilpotent commutator subgroup \( H'_i = H' \leq H_0 \) (\( i = 1, \ldots, n \)), \( H_0 \) is a nilpotent group with divisible Černikov quotient group \( H_0/H' \) (in particular, \( H_0/H' \) is trivial) and, furthermore, if \( k \neq s \) (\( 1 \leq k, s \leq n \)), then \( \pi(H_k/H') \cap \pi(H_s/H') = \emptyset \).
Throughout this paper \( p \) is a prime and \( \mathbb{C}_{p^\infty} \) is the quasicyclic \( p \)-group. For a group \( G \), we denote by \( Z(G) \) the centre, by \( G', G'', \ldots, G^{(n)}, \ldots \) the members of the derived series, by \( \pi(G) \) the set of all primes which divide the orders of periodic elements in \( G \) and by \( \mathbb{Z}_{p^n}G \) the group ring of \( G \) over the ring \( \mathbb{Z}_{p^n} \) of integers modulo a prime power \( p^n \). Recall that a section of \( G \) is a group of the form \( S = H/N \) for some subgroups \( N \) and \( H \) of \( G \), where \( N \) is normal in \( H \). The group \( G \) is perfect if \( G' = G \).

We shall also use other standard terminology from [13] and [18].

1. In the sequel we shall need the following lemmas.

Lemma 1. Let \( G \) be a group satisfying Min-\( \overline{NF} \) and let \( A \) be a subgroup of \( G \). Then:

1. \( A \) satisfies Min-\( \overline{NF} \);
2. if \( A \) is normal in \( G \), then the quotient group \( G/A \) satisfies Min-\( \overline{NF} \);
3. if \( A \) is a normal non-“nilpotent-by-finite” subgroup, then \( G/A \) satisfies the minimal condition on subgroups.

The proof of the lemma is immediate.

As usual, \( R \) is called a right \( V \)-ring if each of its right ideals is an intersection of maximal right ideals of \( R \).

Lemma 2. Let \( G = A \rtimes B \) be a semidirect product of an infinite abelian subgroup \( A \) of exponent \( p^n \) \((n \geq 1)\) and a quasicyclic \( q \)-subgroup \( B \) with distinct primes \( p \) and \( q \). If \( G \) satisfies Min-\( \overline{NF} \), then \( G \) is a nilpotent group or, considered as a \( \mathbb{Z}_{p^n}B \)-module, the abelian group \( A = S_0 \oplus S_1 \oplus \cdots \oplus S_m \) \((m \geq 1)\) is a direct sum of simple \( \mathbb{Z}_{p^n}B \)-submodules \( S_1, \ldots, S_m \) and a finite \( \mathbb{Z}_{p^n}B \)-submodule \( S_0 \), where \( S_0 \leq Z(G) \).

Proof. Suppose that \( G \) is a non-“nilpotent-by-finite” group. Since \( A \) is abelian, it becomes a right module over the commutative (von Neuman) regular ring \( R = \mathbb{Z}_{p^n}B \) via the conjugation action on \( A \).

Let \( u \) be any non-zero element of the module \( A \) such that \( u \notin Z(G) \). Then the cyclic submodule \( uR \) is isomorphic to the right \( R \)-module \( R_0 = R/\text{ann}_R(u) \), where \( \text{ann}_R(u) = \{ r \in R \mid ur = 0 \} \). Moreover \( R_0 \) is a commutative (von Neuman) regular ring and each of its ideals is a right \( R \)-module. If \( R_0 \) is a non-simple module, then it contains a non-zero element \( b_1 \) such that \( b_1R_0 \) is a proper ideal in \( R_0 \). By Proposition 19.24 (d) of [13], \( b_1R_0 = e_1R_0 \) for some
idempotent $e_1$ and, furthermore, $R_0 = e_1 R_0 \oplus R_1$ is a direct sum (of rings) for some finitely generated ideal $R_1$ of $R_0$. If $R_1$ is a non-simple $R$-module, then it has a proper submodule $e_2 R_1$, where $e_2^2 = e_2$. Hence $R_1 = e_2 R_1 \oplus R_2$ is a direct sum for some finitely generated ideal $R_2$ of $R_0$. By a similar argument we obtain a strictly descending chain of $R$-submodules $R_0 > R_1 > \ldots$. This implies that $u R$ also contains a strictly descending chain of $R$-submodules $\{D_m \mid m \in \mathbb{N} \cup \{0\}\}$, where $D_0 = u R$ and $D_m$ is isomorphic to $R_m$. By our hypothesis there exists an integer $n$ such that $D_n B$ is a non-“nilpotent-by-finite” subgroup, while $D_{n+1} B$ is nilpotent-by-finite and, as a consequence, $D_{n+1} \leq Z(G)$.

It is also clear that $R_n$ is a cyclic $R$-module. Using the same idea, as above, we deduce that $D_n$ contains a strictly descending chain of $R$-submodules $\{D_{nm} \mid m \in \mathbb{N} \cup \{0\}\}$ (where $D_{n0} = D_n$) such that $D_{nk} B$ is not a nilpotent-by-finite subgroup for some integer $k$, while $D_{n,k+1} B$ is nilpotent-by-finite. Continuing in this manner we conclude that $A$ contains a simple $R$-submodule $S$. By the theorem of Kaplansky [13, Corollary 19.53], $R$ is a $V$-ring and $S$ is an injective $R$-module. Therefore $A = S \oplus A_1$ is a direct sum of $S$ and some submodule $A_1$. Repeating this argument we deduce that there exists an integer $l$ such that

$$A = A_1 \oplus \cdots \oplus A_l \oplus A_0$$

is a direct sum of simple $R$-submodules $A_1, \ldots, A_l$ and some $R$-submodule $A_0$, where $A_0 B$ is a nilpotent-by-finite subgroup. It is easily verified that $A_0 \leq Z(G)$ and so $A_0$ is finite. The result follows. \(\square\)

**Lemma 3.** Let $G$ be a non-perfect group with nilpotent-by-finite proper normal subgroups. Then $G$ satisfies Min-$\mathbf{NF}$ if and only if it is of one of the following types:

1. $G$ is a nilpotent-by-finite group;
2. $G$ is a minimal non-“nilpotent-by-finite” group;
3. $G = G' \times S$, $S \cong \mathbb{C}_p^\infty$, $G' = S_1 \times \cdots \times S_n$ $(n \geq 1)$ is a $p'$-subgroup and a direct product of finitely many Sylow $p_i$-subgroups $S_1, \ldots, S_n$, where $S_i$ is a nilpotent subgroup of exponent $p_i^{m_i}$ and of derived length $d_i$. Furthermore, for any $i$ there is an integer $k_i$ $(1 \leq k_i \leq d_i)$ such that $S$ acts trivially on $S_i^{(k_i)}$ and $S_i^{(l-1)}/S_i^{(l)} = A_0 \oplus A_1 \oplus \cdots \oplus A_m$ is a finite direct sum of simple $\mathbb{Z}_{p_i^{m_i}} S$-submodules $A_1, \ldots, A_m$ $(m \geq 1)$ and a finite $\mathbb{Z}_{p_i^{m_i}} S$-submodule $A_0$ under the conjugation action on $S_i$, where $A_0 \leq Z(S_i^{(l-1)} \times S_i^{(l)})$ $(i = 1, \ldots, n; l = 1, \ldots, k_i; k_i, m_i, d_i \in \mathbb{N}$ and $S_i^{(0)} = S_i)$;
(4) \( G = A \times S \), where \( S \) is a minimal non-“nilpotent-by-finite” \( p \)-group, \( A \) is a normal nilpotent \( p' \)-subgroup and \( G/S' \) is a group of type (3).

Proof. (\( \Rightarrow \)) This part of the proof is evident.

(\( \Leftarrow \)) First, if the quotient group \( G/G' \) contains two proper subgroups which generate it, then \( G = AB \) is a product of two nilpotent-by-finite proper normal subgroups \( A \) and \( B \). Since \( A \) (respectively \( B \)) contains an abelian \( G \)-invariant subgroup \( A_0 \) (respectively \( B_0 \)) of finite index, we conclude that \( A_0B_0 \) is a nilpotent normal subgroup of finite index in \( G \).

Now assume that any two proper subgroups of \( G/G' \) generate a proper subgroup in \( G/G' \). Hence \( G/G' \) is either a cyclic \( p \)-group (and in this case \( G \) is a nilpotent-by-finite group) or the quasicyclic \( p \)-group for some prime \( p \). Let \( G/G' \cong \mathbb{C}_{p^\infty} \). If \( D \) is a proper nilpotent \( G \)-invariant subgroup of finite index in \( G' \), then \( G/DG'' \) is an abelian group, which is a contradiction. Thus \( G' \) is a nilpotent subgroup. Inasmuch as \( G \) satisfies \( \text{Min-NF} \), it contains a subgroup \( S \) which is a non-perfect minimal non-“nilpotent-by-finite” group.

If \( S = G \), then \( G \) is of type (2). Therefore we may suppose that \( S \neq G \). Then \( G = G'/S \) and, by Theorem 2.5 of [9], \( S \) is a torsion subgroup. Let \( \overline{G} = G/G''(G' \cap S) = \overline{G'} \times \overline{S} \). It is easy to see that \( \overline{G} \) satisfies the minimal condition on normal subgroups \( \text{Min-n} \) and so, by the theorem of Baer [18, Theorem 5.25] and Theorem 2.1 of [9], \( G \) is a locally finite group. If \( \overline{G} \) is a \( p \)-group, then it is \( \tilde{\text{Černikov}} \) (see [18, p.156, Corollary 2]). From this it follows that \( \overline{G} \) is a nilpotent group and we reach a contradiction. Hence \( \overline{G} \) is a \( p' \)-subgroup. Our hypothesis and Theorem B of [5] give that \( G' \) is a \( \pi \)-subgroup for some finite set of primes \( \pi \) and, as a consequence, \( G = A \times Q \), where \( Q = S \) is either a minimal non-“nilpotent-by-finite” \( p \)-group or \( Q \cong \mathbb{C}_{p^\infty} \) and \( A \) is a \( p' \)-subgroup of \( G' \).

Assume that \( Q \) is a quasicyclic \( p \)-subgroup and \( A = S_1 \times \ldots \times S_n \) is a group direct product of the Sylow \( p_i \)-subgroups \( S_1, \ldots, S_n \). Obviously \( \exp(S_i) = p_i^{m_i} \) for some \( m_i \in \mathbb{N} \) \((i = 1, \ldots, n)\). Let \( d_i \) be the derived length of \( S_i \).

We have seen that there exists an integer \( m \) \((1 \leq m \leq d_i)\) such that \( S_i^{(m-1)}/Q \) is a non-“nilpotent-by-finite” subgroup, while \( S_i^{(m)} \) is a nilpotent-by-finite. Then \([S_i^{(m)}, Q] = 1\). By Lemma 2, \( S_i^{(l-1)}/S_i^{(l)} = A_0 \oplus A_1 \oplus \cdots \oplus A_k \) is a direct sum of simple \( \mathbb{Z}_{p_i^{m_i}} \)-submodules \( A_1, \ldots, A_k \) \((k \geq 1)\) and a finite \( \mathbb{Z}_{p_i^{m_i}} \)-submodule \( A_0 \) under the conjugation action on \( S_i \), where \( A_0 \leq Z(S_i^{(l-1)}Q/S_i^{(l)}) \) \((1 \leq l \leq m)\). Thus \( G \) is a group of type (3).

Finally, if \( Q \) is a minimal non-“nilpotent-by-finite” group, then it is not difficult to see that \( G/S' \) is a group of type (3). The lemma is proved. □
Corollary 4. Let $G$ be a group without non-trivial perfect sections. If $G$ satisfies $\text{Min-NF}$, then it is countable and locally finite.

Lemma 5. Let $G$ be an $HM^*$-group. Then $G$ satisfies $\text{Min-NF}$ if and only if $G'$ is a nilpotent subgroup.

Proof. ($\Leftarrow$) Let $\{K_n \mid n \in \mathbb{N}\}$ be any strictly descending chain of subgroups in $G$. Since $G/G'$ is Černikov, there is an integer $m$ such that $K_n \leq G'$ and so $G$ satisfies $\text{Min-NF}$.

($\Rightarrow$) If the commutator subgroup $G'$ is not nilpotent-by-finite, then, in view of Lemma 2, it has a subnormal non-“nilpotent-by-finite” subgroup $S$ with all proper normal subgroups nilpotent-by-finite. Then Lemma 3 yields that $S$ is a non-hypercentral subgroup and we obtain a contradiction. This implies that $G'$ is a nilpotent-by-finite subgroup and consequently it is nilpotent, as desired. □

Now we give some examples of $HM^*$-groups.

Examples. (i) First we recall one construction from [5]. Let $p$ and $q$ be distinct primes and let $\mathbb{F}_q$ be the field with $q$ elements. We denote by $\mathbb{F}_q(\alpha)$ the subfield of the algebraic closure of $\mathbb{F}_q$ generated by $\alpha$. If $\varepsilon_i$ is a primitive $p^i$-th root of 1 ($i = 0, 1, 2, \ldots$), put $F_i = \mathbb{F}_q(\varepsilon_i)$ and $F = \bigcup_{i=0}^{\infty} F_i$. Let $A$ be the additive group of $\mathbb{F}$, and let $B$ be the multiplicative group which contains the $p^i$-th roots of 1, where $i = 0, 1, 2, \ldots$. The rule

$$bab^{-1} = b^{p^m} \cdot a,$$

where $a \in A$, $b \in B$ and $b^{p^m} \cdot a$ is the product of the elements $b^{p^m}$ and $a$ in the field $F$, $m$ is some non-negative integer, defines an action of $B$ on $A$. The group $G_m = A \times B$ constructed in this manner is called a Čarín group [5]. The groups $G_0$ were first considered by Čarín [10].

Since $A = G'_m$, $G_m$ is an $HM^*$-group. Moreover it is a minimal non-“nilpotent-by-finite” group.

(ii) Let $F$ be a field defined as in (i) and $J$ a non-trivial $F$-$F$-bialgebra (it is well known that such a bialgebra $J$ exists). We denote by $A$ and $I$ the subalgebra

$$
\begin{pmatrix}
F & J & \cdots & J \\
0 & F & \cdots & J \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & F
\end{pmatrix}
$$

of the algebra of $n \times n$ matrices with the identity element 1 ($n \geq 2$) and the
nilpotent ideal
\[
\begin{pmatrix}
0 & J & \ldots & J \\
0 & 0 & \ldots & J \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{pmatrix}
\]
of nilpotency index \(n\), respectively. Then the ring \(A/I \cong F \oplus \cdots \oplus F\) is a direct sum of \(n\) copies of \(F\). It is known that a nilpotent ring \(I\) with two operations "+" and "," forms a group under the operation \(a \circ b = a + b + a \cdot b\) for all \(a, b \in I\).

This group is called the adjoint group of \(I\) and denoted by \(I^\circ\). By Lemma 2.4 of [2] \(I^\circ\) is a \(q\)-group. Moreover, by the lemma from [21, p. 27], see also [1], \(I^\circ\) is a nilpotent group of class \(n\). Since the unipotent subgroup \(1 + I\) is isomorphic to \(I^\circ\) and the multiplicative group \(F^*\) of \(F\) is a \(q'\)-group, we see that the unit group

\[U(A) = (1 + I) \rtimes (F^* \times \cdots \times F^*)\]
of \(A\) is a semidirect product of the normal subgroup \(1 + I\) and the direct product of \(n\) copies of \(F^*\). It is clear that \(U(A)\) contains the subgroup

\[G = (1 + I) \rtimes (B \times \cdots \times B)\]
of finite index, where \(B\) is a quasicyclic \(p\)-subgroup of \(F^*\).

We want to prove that \(G\) is an \(HM^*\)-group. First we consider the case \(n = 2\). The commutator

\[\left[ \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \right] = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^{-1} & 0 \\ 0 & x_2^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x_1ix_2^{-1} - i \\ 0 & 1 \end{pmatrix} \in 1 + I\]
for all elements \(i \in J\) and \(x_1, x_2 \in B\) and, as a consequence,

\[\left[1 + I, B \times B\right] = 1 + I.\]

Now we turn to the general case and, by using a similar argument, obtain that

\[\left[1 + I, B \times \cdots \times B\right] = 1 + I,\]

\(n\) times

i.e. \(G\) is an \(HM^*\)-group with nilpotent commutator subgroup \(G' = 1 + I\).
(iii) Let \( J \) be a left \( T \)-nilpotent \( F \)-\( F \)-bialgebra with \( F \) and \( B \) as before. An example of a left \( T \)-nilpotent bialgebra is contained, for instance, in [19, Example 1]. We denote by \( J^1 \) the bialgebra obtained by adjoining an identity element to \( J \). Writing \( A \) for the subalgebra

\[
\begin{pmatrix}
J^1 & J \\
0 & F
\end{pmatrix}
\]

of the \( 2 \times 2 \) matrix algebra with the identity element 1 and \( I \) for the left \( T \)-nilpotent ideal

\[
\begin{pmatrix}
J & J \\
0 & 0
\end{pmatrix},
\]

we have that \( 1 + I \) is a hypercentral \( q \)-subgroup by Lemma 2.4 of [2] and the lemma from [21, p. 27]. By the same argument, as in (ii), we can prove that the condition \((*)\) holds, i.e. \( G = (1 + I) \rtimes (B \times B) \) is an \( HM^* \)-group.

2. Proof of Theorem. \((\Leftarrow)\) This direction of the proof is immediate. 
\((\Rightarrow)\) Assume that \( G \) is a non-“nilpotent-by-finite” group. Then \( G \) has a strictly descending subnormal series

\[
G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_m = S,
\]

where \( S \) is a non-“nilpotent-by-finite” group with all proper normal subgroups nilpotent-by-finite. By Lemma 1, \( G_j/G_{j+1} \) is a \( Černikov \) group \((j = 0, 1, \ldots, m - 1)\).

Let \( z \in G_{m-1} \). Then \( S^z \triangleleft G_{m-1} \) and hence \( S' \triangleleft G_{n-1} \). We denote by \( D_{m-1} \) the subgroup of finite index in \( G_{m-1} \) such that \( D_{m-1}/S' \) is the divisible part of \( G_{m-1}/S' \). Then \( D'_{m-1} = S' \). Since \( D_{m-1} D^{y}_{m-1}/S' \) is a \( Černikov \) group for every \( y \in G_{m-2} \) and \( S' \) does not contain a proper \( S \)-invariant subgroup of finite index, we conclude that \( D_{m-1} \triangleleft G_{m-2} \). Continuing this process, after a finite number steps we obtain that \( G \) has a normal subgroup \( D \) of finite index with divisible \( Černikov \) group \( D/D' \) and \( D' = S' \). Consequently

\[
D = D_1 \cdot \ldots \cdot D_n (n \geq 1),
\]

where \( D_s/D' \) is a divisible \( Černikov \) Sylow \( p_s \)-subgroup of \( D/D' \) \((s = 1, \ldots, n)\) and \( p_s \neq p_l \) if \( s \neq l \) \((1 \leq s, l \leq n)\). If \( D_k \) is a nilpotent-by-finite subgroup for some integer \( k \) \((1 \leq k \leq n)\), then it is nilpotent. This yields that \( D_m \) is not nilpotent-by-finite for some integer \( m \), where \( 1 \leq m \leq n \). Then it contains a
subnormal subgroup $T$ with all proper normal subgroups nilpotent-by-finite. As above we can prove that $T' = D'_m = S'$. Hence $D_m$ is an $HM^*$-group with nilpotent commutator subgroup $D'_m$. The theorem is proved. □

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