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COMPOSITIONS, GENERATED BY SPECIAL NETS IN AFFINELY CONNECTED SPACES

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Abstract. Let a net $(\underset{1}{v},\underset{2}{v},\dots \underset{N}{v})$ be given in the space A_N with an affine connectedness $\Gamma^\sigma_{\alpha\beta}$, without a torsion . If the covectors $\overset{\alpha}{v}_\sigma$ are defined such that $\underset{\alpha}{v}^\sigma \overset{\beta}{v}_\sigma = \delta^\beta_\alpha$, then the affinor $a^\beta_\alpha = \underset{1}{v}^\beta \overset{1}{v}_\alpha + \underset{2}{v}^\beta \overset{2}{v}_\alpha + \dots + \underset{n}{v}^\beta \overset{n}{v}_\alpha - \underset{n+1}{v}^\beta \overset{n+1}{v}_\alpha - \dots - \underset{N}{v}^\beta \overset{N}{v}_\alpha$ is uniquely determinate by the net. Since $a^\beta_\alpha a^\sigma_\beta = \delta^\sigma_\alpha$, then a^β_α defines a composition $(X_n \times X_m)$ in A_N, i.e. the net $(\underset{1}{v},\underset{2}{v},\dots \underset{N}{v})$ defines a composition.
Special nets which characterize Cartesian, geodesic, Chebyshevian, geodesic-Chebyshevian and Chebyshevian-geodesic compositions are introduced. Conditions for the coefficients of the connectedness in the parameters of these special nets are found.
The following three affinors are considered : a^β_α, $b^\beta_\alpha = \underset{1}{v}^\beta \overset{1}{v}_\alpha + \underset{2}{v}^\beta \overset{2}{v}_\alpha + \cdots + \underset{k}{v}^\beta \overset{k}{v}_\alpha - \underset{k+1}{v}^\beta \overset{k+1}{v}_\alpha - \dots - \underset{N}{v}^\beta \overset{N}{v}_\alpha$, $c^\beta_\alpha = \underset{1}{v}^\beta \overset{1}{v}_\alpha + \underset{2}{v}^\beta \overset{2}{v}_\alpha + \dots + \underset{k}{v}^\beta \overset{k}{v}_\alpha - \underset{k+1}{v}^\beta \overset{k+1}{v}_\alpha - \dots - \underset{n}{v}^\beta \overset{n}{v}_\alpha + \underset{n+1}{v}^\beta \overset{n+1}{v}_\alpha + \dots + \underset{N}{v}^\beta \overset{N}{v}_\alpha$.
These affinors define a three interrelated compositions and satisfy $a^\beta_\alpha b^\sigma_\beta = c^\sigma_\alpha$, $b^\beta_\alpha c^\sigma_\beta = a^\sigma_\alpha$, $c^\beta_\alpha a^\sigma_\beta = b^\sigma_\alpha$. It is proved that if two of the three interrelated compositions are Cartesian (Chebyshevian), then the third one is Cartesian (Chebyshevian) too.

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1. Preliminary. Let A_N be an affinely connected space without a torsion, with coefficients of the connectedness $\Gamma^{\sigma}_{\alpha\beta}$. The space A_N assumes a composition $X_n \times X_m$ of two base manyfolds X_n and X_m $(n+m=N)$ if and only if there exists an affinor a^{β}_{α}, such that $a^{\beta}_{\alpha}a^{\sigma}_{\beta}=\delta^{\sigma}_{\alpha}$ [1]. This space will be denoted $A_N(X_n \times X_m)$. Two positions $P(X_n), P(X_m)$ of the base manyfolds pass through any point of $A_N(X_n \times X_m)$.
Let accept: $\alpha,\beta,\gamma,\sigma,\nu,\ldots \in \{1,2,\ldots,N\}$; $i,j,k,p,q,r,s,\ldots \in \{1,2,\ldots,n\}$; $\overline{i},\overline{j},\overline{k},\overline{p},\overline{q},\overline{r},\overline{s},\ldots \in \{n+1,n+2,\ldots,N\}$.
We shall consider an affinely connected spaces $A_N(X_n \times X_m)$ with integrable structure of the compositions. According to [4] the integrability condition of the structure is characterized with the equality

$$a^{\sigma}_{\beta}\nabla_{[\alpha}a^{\nu}_{\sigma]} - a^{\sigma}_{\alpha}\nabla_{[\beta}a^{\nu}_{\sigma]} = 0. \tag{1}$$

For the projecting affinors $\overset{n}{a}{}^{\beta}_{\alpha}$, $\overset{m}{a}{}^{\beta}_{\alpha}$, defined by the conditions [5]

$$\overset{n}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^{\beta}_{\alpha} + a^{\beta}_{\alpha}) , \quad \overset{m}{a}{}^{\beta}_{\alpha} = \frac{1}{2}(\delta^{\beta}_{\alpha} - a^{\beta}_{\alpha}) , \tag{2}$$

the following equalities are fulfilled: $\overset{n}{a}{}^{\beta}_{\alpha}\overset{n}{a}{}^{\sigma}_{\beta} = \overset{n}{a}{}^{\sigma}_{\alpha}$, $\overset{m}{a}{}^{\beta}_{\alpha}\overset{m}{a}{}^{\sigma}_{\beta} = \overset{m}{a}{}^{\sigma}_{\alpha}$, $\overset{n}{a}{}^{\beta}_{\alpha}\overset{m}{a}{}^{\sigma}_{\beta} = \overset{m}{a}{}^{\beta}_{\alpha}\overset{n}{a}{}^{\sigma}_{\beta} = 0$. From [2] and [3] it is known:
The composition $X_n \times X_m$ is called Cartesian if the positions $P(X_m)$ and $P(X_n)$ are parallelly translated along any line in the space.
The composition $X_n \times X_m$ is called geodesic if the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along $P(X_n)$ and $P(X_m)$, respectively.
The composition $X_n \times X_m$ is called Chebyshevian if the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along $P(X_m)$ and $P(X_n)$, respectively.
The composition $X_n \times X_m$ is called g, Ch-composition if the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along $P(X_n)$.
The composition $X_n \times X_m$ is called Ch, g-composition if the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along $P(X_m)$.
The following propositions are proved in the paper [2]:
The composition $X_n \times X_m$ is Cartesian if and only if

$$\nabla_{\alpha}a^{\sigma}_{\beta} = 0. \tag{3}$$

The composition $X_n \times X_m$ is geodesic if and only if

$$a^{\sigma}_{\alpha}\ \nabla_{\beta} a^{\nu}_{\sigma} + a^{\sigma}_{\beta}\ \nabla_{\sigma} a^{\nu}_{\alpha} = 0. \tag{4}$$

The composition $X_n \times X_m$ is Chebyshevian if and only if

$$\nabla_{[\alpha} a^{\sigma}_{\beta]} = 0. \tag{5}$$

The composition $X_n \times X_m$ is g, Ch-composition if and only if

$$\overset{n}{a}{}^{\sigma}_{\alpha}\ \nabla_{\sigma} \overset{n}{a}{}^{\nu}_{\beta} = 0. \tag{6}$$

The composition $X_n \times X_m$ is Ch, g-composition if and only if

$$\overset{m}{a}{}^{\sigma}_{\alpha}\ \nabla_{\sigma} \overset{m}{a}{}^{\nu}_{\beta} = 0. \tag{7}$$

According to [4] for an arbitrary vector $v^{\alpha} \in A_N$ we have

$$v^{\alpha} = \overset{n}{a}{}^{\alpha}_{\sigma}\ v^{\sigma} + \overset{m}{a}{}^{\alpha}_{\sigma}\ v^{\sigma} = \overset{n}{V}{}^{\alpha} + \overset{m}{V}{}^{\alpha},$$

where

$$\overset{n}{V}{}^{\alpha} = \overset{n}{a}{}^{\alpha}_{\sigma}\ v^{\sigma} \in P(X_n),\ \overset{m}{V}{}^{\alpha} = \overset{m}{a}{}^{\alpha}_{\sigma}\ v^{\sigma} \in P(X_m). \tag{8}$$

Let N independent fields of directions $\underset{1}{v}{}^{\sigma}, \underset{2}{v}{}^{\sigma}, \ldots, \underset{N}{v}{}^{\sigma}$ be given in A_N.They define the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{N}{v})$. The reciprocal covectors $\overset{\alpha}{v}_{\sigma} (\alpha = 1, 2, \ldots, N)$ are defined by the equalities

$$\underset{\alpha}{v}{}^{\sigma} \overset{\beta}{v}_{\sigma} = \delta^{\beta}_{\alpha} \quad iff \quad \underset{\alpha}{v}{}^{\sigma} \overset{\alpha}{v}_{\nu} = \delta^{\sigma}_{\nu}. \tag{9}$$

As in the paper [6] can be written the following derivative equations

$$\nabla_{\alpha}\ \underset{\beta}{v}{}^{\sigma} = \overset{\nu}{\underset{\beta}{T}}_{\alpha}\ \underset{\nu}{v}{}^{\sigma} \quad , \quad \nabla_{\alpha}\ \overset{\beta}{v}_{\sigma} = -\overset{\beta}{\underset{\nu}{T}}_{\alpha}\ \overset{\nu}{v}_{\sigma} \quad . \tag{10}$$

2. Special nets and compositions in A_N. Introduce the following affinor

$$a^{\beta}_{\alpha} = \underset{1}{v}{}^{\beta} \overset{1}{v}_{\alpha} + \underset{2}{v}{}^{\beta} \overset{2}{v}_{\alpha} + \cdots + \underset{n}{v}{}^{\beta} \overset{n}{v}_{\alpha} - \underset{n+1}{v}{}^{\beta} \overset{n+1}{v}_{\alpha} - \cdots - \underset{N}{v}{}^{\beta} \overset{N}{v}_{\alpha}, \tag{11}$$

are uniquely determinated from the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$. According to (9), (11) we obtain $a^{\beta}_{\alpha} a^{\sigma}_{\beta} = \delta^{\sigma}_{\alpha}$, from where it follows that the affinor (11) defines a composition $X_n \times X_m$ in A_N. If the affinor (11) satisfies (1) the structure of the space $A_N(X_n \times X_m)$ it will be integrable.

Definition 1. *Any composition* $(X_n \times X_m)$ *generated by the affinor* (11), *which satisfies the condition* (1), *will be called associated with the net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$.

From (2), (9), (11) we find

$$\overset{n}{a}{}^{\alpha}_{\sigma} \underset{s}{v}^{\sigma} = \underset{s}{v}^{\alpha}, \quad \overset{n}{a}{}^{\alpha}_{\sigma} \underset{\overline{s}}{v}^{\sigma} = 0, \quad \overset{m}{a}{}^{\alpha}_{\sigma} \underset{\overline{s}}{v}^{\sigma} = \underset{\overline{s}}{v}^{\alpha}, \quad \overset{m}{a}{}^{\alpha}_{\sigma} \underset{s}{v}^{\sigma} = 0, \tag{12}$$

from where it follows $\underset{s}{v}^{\alpha} \in P(X_n)$, $\underset{\overline{s}}{v}^{\alpha} \in P(X_m)$.
Taking into account (11), (12) we establish

$$\overset{n}{a}{}^{\alpha}_{\beta} = \underset{s}{v}^{\alpha} \overset{s}{v}_{\beta}, \quad \overset{m}{a}{}^{\alpha}_{\beta} = \underset{\overline{s}}{v}^{\alpha} \overset{\overline{s}}{v}_{\beta}, \quad a^{\alpha}_{\beta} = \underset{s}{v}^{\alpha} \overset{s}{v}_{\beta} - \underset{\overline{s}}{v}^{\alpha} \overset{\overline{s}}{v}_{\beta} \ . \tag{13}$$

The derivative equations (10) are equivalent to the following four equations

$$\begin{aligned} &\nabla_{\alpha} \underset{s}{v}^{\sigma} = \underset{s}{\overset{k}{T}}_{\alpha} \underset{k}{v}^{\sigma} + \underset{s}{\overset{\overline{k}}{T}}_{\alpha} \underset{\overline{k}}{v}^{\sigma}, \quad \nabla_{\alpha} \overset{s}{v}_{\sigma} = -\underset{k}{\overset{s}{T}}_{\alpha} \overset{k}{v}_{\sigma} - \underset{\overline{k}}{\overset{s}{T}}_{\alpha} \overset{\overline{k}}{v}_{\sigma}, \\ &\nabla_{\alpha} \underset{\overline{s}}{v}^{\sigma} = \underset{\overline{s}}{\overset{k}{T}}_{\alpha} \underset{k}{v}^{\sigma} + \underset{\overline{s}}{\overset{\overline{k}}{T}}_{\alpha} \underset{\overline{k}}{v}^{\sigma}, \quad \nabla_{\alpha} \overset{\overline{s}}{v}_{\sigma} = -\underset{k}{\overset{\overline{s}}{T}}_{\alpha} \overset{k}{v}_{\sigma} - \underset{\overline{k}}{\overset{\overline{s}}{T}}_{\alpha} \overset{\overline{k}}{v}_{\sigma}. \end{aligned} \tag{14}$$

Definition 2. *A net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N$ *will be called* C*-net if for any* $k, s, \overline{p}, \overline{r}$ *the coefficients from the derivative equations* (14) *satisfy the equalities* $\underset{\overline{p}}{\overset{k}{T}}_{\alpha} = \underset{s}{\overset{\overline{r}}{T}}_{\alpha}$.

Theorem 1. *The composition* $(X_n \times X_m) \in A_N$ *is Cartesian if and only if it is associated with a* C*-net.*

P r o o f. the composition $(X_n \times X_m) \in A_N$ be associated with a C-net. According to (13), (14), Definition 2 we find

$$\nabla_{\alpha} \overset{n}{a}{}^{\sigma}_{\beta} = \nabla_{\alpha}(\underset{s}{v}^{\sigma} \overset{s}{v}_{\beta}) = \underset{s}{\overset{k}{T}}_{\alpha} \underset{k}{v}^{\sigma} \overset{s}{v}_{\beta} - \underset{k}{\overset{s}{T}}_{\alpha} \underset{s}{v}^{\sigma} \overset{k}{v}_{\beta} = 0,$$

$$\nabla_\alpha \overset{m}{a}{}^{\sigma}_{\beta} = \nabla_\alpha(\underset{\bar{s}}{v}{}^{\sigma}\overset{\bar{s}}{v}_\beta) = \overset{\bar{k}}{\underset{\bar{s}}{T}}{}_\alpha \underset{\bar{k}}{v}{}^{\sigma}\overset{\bar{s}}{v}_\beta - \overset{\bar{s}}{\underset{\bar{k}}{T}}{}_\alpha \underset{\bar{s}}{v}{}^{\sigma}\overset{\bar{k}}{v}_\beta.$$

Thus taking into account again (13) we get $\nabla_\alpha\, a{}^{\sigma}_{\beta} = 0$. Now from (3) it follows that the composition $(X_n \times X_m) \in A_N$ is Cartesian.

Let the composition $(X_n \times X_m) \in A_N$, associated with the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ be Cartesian. Then (3) will be fulfilled. According to (2) the equality (3) is equivalent to $\nabla_\alpha \overset{n}{a}{}^{\sigma}_{\beta} = 0$. Applying (13), (14) we obtain

$$\nabla_\alpha \overset{n}{a}{}^{\sigma}_{\beta} = \nabla_\alpha(\underset{s}{v}{}^{\sigma}\overset{s}{v}_\beta) = \overset{\bar{k}}{\underset{s}{T}}{}_\alpha \underset{\bar{k}}{v}{}^{\sigma}\overset{s}{v}_\beta - \overset{s}{\underset{\bar{k}}{T}}{}_\alpha \underset{s}{v}{}^{\sigma}\overset{\bar{k}}{v}_\beta\ . \tag{15}$$

Because of the independence of the vectors $\underset{\alpha}{v}{}^{\sigma}$ and covectors $\overset{\alpha}{v}_\sigma$ from (15) we obtain consecutively $\overset{\bar{k}}{\underset{s}{T}}{}_\alpha \overset{s}{v}_\beta = 0$, $\overset{s}{\underset{\bar{k}}{T}}{}_\alpha \overset{\bar{k}}{v}_\beta = 0$ and $\overset{k}{\underset{\bar{p}}{T}}{}_\alpha = \overset{\bar{r}}{\underset{s}{T}}{}_\alpha = 0$, i.e. the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ is Cartesian. □

Theorem 2. *If the coordinate net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N(X_n \times X_m)$ *is a* C*-net, then the coefficients of the connectedness satisfy the conditions*

$$\Gamma^{s}_{\alpha\bar{k}} = \Gamma^{\bar{p}}_{\alpha m} = 0.$$

Proof. Let the C-net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N(X_n \times X_m)$ be chosen as a coordinate one. After contraction of the first equality in (10) by $\overset{\nu}{v}_\sigma$ and taking into account (9) we obtain

$$\overset{\nu}{v}_\sigma \nabla_\alpha \underset{\beta}{v}{}^{\sigma} = \overset{\nu}{\underset{\beta}{T}}{}_\alpha. \tag{16}$$

Now according to (16), Definition 2 we find $\overset{k}{v}_\sigma \nabla_\alpha \underset{\bar{p}}{v}{}^{\sigma} = \overset{k}{\underset{\bar{p}}{T}}{}_\alpha = 0$, $\overset{\bar{r}}{v}_\sigma \nabla_\alpha \underset{s}{v}{}^{\sigma} = \overset{\bar{r}}{\underset{s}{T}}{}_\alpha = 0$, from where follow $\overset{k}{v}_\sigma(\partial_\alpha \underset{\bar{p}}{v}{}^{\sigma} + \Gamma^{\sigma}_{\alpha\nu}\underset{\bar{p}}{v}{}^{\nu}) = 0$, $\overset{\bar{r}}{v}_\sigma(\partial_\alpha \underset{s}{v}{}^{\sigma} + \Gamma^{\sigma}_{\alpha\nu}\underset{s}{v}{}^{\nu}) = 0$. Finally, taking into account that the C-net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N(X_n \times X_m)$ is coordinate, we get $\Gamma^{s}_{\alpha\bar{k}} = \Gamma^{\bar{p}}_{\alpha m} = 0$.

It is easy to prove that if the conditions $\Gamma^{s}_{\alpha\bar{k}} = \Gamma^{\bar{p}}_{\alpha m} = 0$ are fulfilled in the parameters of the coordinate net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$, then this net is C-net. $\square$

Definition 3. *A net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N$ *will be called g-net if for any* $k,s,\overline{k},\overline{s}$ *the coefficients from the derivative equations* (14) *and the affinor* a_α^β *satisfy the equalities*

$$(17) \qquad \overset{\overline{k}}{\underset{s}{T}}{}_\alpha \; a_\sigma^\alpha + \overset{\overline{k}}{\underset{s}{T}}{}_\sigma = 0, \quad \overset{k}{\underset{\overline{s}}{T}}{}_\alpha \; a_\sigma^\alpha - \overset{k}{\underset{\overline{s}}{T}}{}_\sigma = 0.$$

Theorem 3. *The composition* $(X_n \times X_m) \in A_N$ *is a g-composition if and only if it is associated with a g-net.*

P r o o f. According to (2) we can write

$$a_\beta^\sigma \; \nabla_\alpha a_\sigma^\nu + a_\alpha^\sigma \; \nabla_\sigma a_\beta^\nu = (\overset{n}{a}{}_\beta^\sigma - \overset{m}{a}{}_\beta^\sigma)\nabla_\alpha(\overset{n}{a}{}_\sigma^\nu - \overset{m}{a}{}_\sigma^\nu) + (\overset{n}{a}{}_\alpha^\sigma - \overset{m}{a}{}_\alpha^\sigma)\nabla_\sigma(\overset{n}{a}{}_\beta^\nu - \overset{m}{a}{}_\beta^\nu).$$

Then because of (9), (13), (14) we get

$$(18) \qquad a_\beta^\alpha \; \nabla_\sigma a_\alpha^\nu + a_\sigma^\alpha \; \nabla_\alpha a_\beta^\nu = 2(\overset{\overline{k}}{\underset{s}{T}}{}_\alpha \; a_\sigma^\alpha + \overset{\overline{k}}{\underset{s}{T}}{}_\sigma)\underset{\overline{k}}{v}{}^\nu \; \overset{s}{v}_\beta + 2(\overset{k}{\underset{\overline{s}}{T}}{}_\sigma - \overset{k}{\underset{\overline{s}}{T}}{}_\alpha a_\sigma^\alpha)\overset{\overline{s}}{v}_\beta \; \underset{k}{v}{}^\nu.$$

Let the composition $(X_n \times X_m) \in A_N$ be associated with a g-net. Then taking into account (17), (18) we obtain (4), i.e. $(X_n \times X_m) \in A_N$ is a g-composition.

Let the composition $(X_n \times X_m) \in A_N$, associated with the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ be g-composition. According to (4), (18) and the independence of $\underset{\alpha}{v}{}^\sigma$ and $\overset{\alpha}{v}_\sigma$ we obtain (17) which means that $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ is a g-net. $\square$

Theorem 4. *If the coordinate net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$ *is a g-net, then the coefficients of the connectedness satisfy the conditions* $\Gamma_{rs}^{\overline{k}} = \Gamma_{\overline{rs}}^{k} = 0$.

P r o o f. Let the g-net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$ be chosen as a coordinate one. According to (16) the equalities (17) accept the form

$$\overset{\overline{k}}{v}_\sigma(\partial_\alpha \underset{p}{v}{}^\sigma + \Gamma_{\alpha\nu}^\sigma \underset{p}{v}{}^\nu)a_\beta^\alpha = -\overset{\overline{k}}{v}_\sigma(\partial_\beta \underset{p}{v}{}^\sigma + \Gamma_{\beta\nu}^\sigma \underset{p}{v}{}^\nu), \; \overset{k}{v}_\sigma(\partial_\alpha \underset{\overline{p}}{v}{}^\sigma + \Gamma_{\alpha\nu}^\sigma \underset{\overline{p}}{v}{}^\nu)a_\beta^\alpha = \overset{k}{v}_\sigma(\partial_\beta \underset{\overline{p}}{v}{}^\sigma + \Gamma_{\beta\nu}^\sigma \underset{\overline{p}}{v}{}^\nu).$$

Now taking into account that the g-net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})\in A_N(X_n\times X_m)$ is coordinate, we get $\Gamma^{\overline{k}}_{sp}\overset{s}{v}_\beta-\Gamma^{\overline{k}}_{\overline{s}p}\overset{\overline{s}}{v}_\beta=-\Gamma^{\overline{k}}_{\beta p}$, $\Gamma^{k}_{s\overline{p}}\overset{s}{v}_\beta-\Gamma^{k}_{\overline{s}\,\overline{p}}\overset{\overline{s}}{v}_\beta=\Gamma^{k}_{\beta\overline{p}}$, from where we obtain $\Gamma^{\overline{k}}_{rs}=\Gamma^{k}_{\overline{rs}}=0$.

It is easy to prove that if the conditions $\Gamma^{\overline{k}}_{rs}=\Gamma^{k}_{\overline{r}\,\overline{s}}=0$ are fulfilled in the parameters of the coordinate net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})\in A_N(X_n\times X_m)$, then this net is a g-net. $\square$

Definition 4. *A net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})\in A_N$ *will be called Ch-net if for any* $k,s,\overline{p},\overline{r}$ *the coefficients from the derivative equations* (14) *satisfy the equalities* $\underset{\overline{p}}{\overset{k}{T}}{}_{[\alpha}\overset{\overline{p}}{v}_{\beta]}=\underset{s}{\overset{\overline{r}}{T}}{}_{[\alpha}\overset{s}{v}_{\beta]}=0$.

Theorem 5. *The composition* $(X_n\times X_m)\in A_N$ *is Chebyshevian if and only if it is associated with a Ch-net.*

Proof. According to (13), (14) we find

$$\nabla_{[\alpha}\overset{n}{a}{}^{\sigma}_{\beta]}=\underset{s}{\overset{\overline{r}}{T}}{}_{[\alpha}\overset{s}{v}_{\beta]}\underset{\overline{r}}{v}^{\sigma}-\underset{\overline{p}}{\overset{k}{T}}{}_{[\alpha}\overset{\overline{p}}{v}_{\beta]}\underset{k}{v}^{\sigma}=0. \tag{19}$$

Let the composition $(X_n\times X_m)\in A_N$ be associated with a Ch-net. From (19) taking into account (2) and Definition 3 we get $\nabla_{[\alpha}\,a^{\sigma}_{\beta]}=0$, i.e. the composition $(X_n\times X_m)\in A_N$ is Chebyshevian.

Let the composition $(X_n\times X_m)\in A_N$, associated with the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ be Chebyshevian. Then (5) will be fulfilled. According to (2) the equality (5) is equivalent to $\nabla_{[\alpha}\overset{n}{a}{}^{\sigma}_{\beta]}=0$. From (19) because of the independence of the vectors $\underset{\alpha}{v}^{\sigma}$ we obtain $\underset{\overline{p}}{\overset{k}{T}}{}_{[\alpha}\overset{\overline{p}}{v}_{\beta]}=\underset{s}{\overset{\overline{r}}{T}}{}_{[\alpha}\overset{s}{v}_{\beta]}=0$, which means that $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ is a Ch-net. $\square$

Theorem 6. *If the coordinate net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})\in A_N(X_n\times X_m)$ *is a Ch-net, then the coefficients of the connectedness satisfy the conditions* $\Gamma^{\overline{r}}_{\overline{p}s}=\Gamma^{k}_{s\overline{p}}=0$.

Proof. Let the Ch-net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})\in A_N(X_n\times X_m)$ be chosen as a coordinate one. According to (16) and Definition 3 we find $\overset{\overline{r}}{v}_\sigma\nabla_{[\alpha}\underset{s}{v}^{\sigma}\overset{s}{v}_{\beta]}=0$, $\overset{k}{v}_\sigma\nabla_{[\alpha}\underset{\overline{p}}{v}^{\sigma}\overset{\overline{p}}{v}_{\beta]}=0$, from where follow $\overset{k}{v}_\sigma(\partial_\alpha\underset{\overline{p}}{v}^{\sigma}+\Gamma^{\sigma}_{\alpha\nu}\underset{\overline{p}}{v}^{\nu})\overset{\overline{p}}{v}_\beta-\overset{k}{v}_\sigma(\partial_\beta\underset{\overline{p}}{v}^{\sigma}+\Gamma^{\sigma}_{\beta\nu}\underset{\overline{p}}{v}^{\nu})\overset{\overline{p}}{v}_\alpha=0$,

$\overset{\overline{r}}{v}_\sigma(\partial_\alpha \underset{s}{v}^\sigma + \Gamma^\sigma_{\alpha\nu}\underset{s}{v}^\nu)\overset{s}{v}_\beta - \overset{\overline{r}}{v}_\sigma(\partial_\beta \underset{s}{v}^\sigma + \Gamma^\sigma_{\beta\nu}\underset{s}{v}^\nu)\overset{s}{v}_\alpha = 0$. Taking into account that the Ch-net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N(X_n \times X_m)$ is coordinate, we get

$$\Gamma^k_{\alpha\overline{p}}\overset{\overline{p}}{v}_\beta - \Gamma^k_{\beta\overline{p}}\overset{\overline{p}}{v}_\alpha = 0, \quad \Gamma^{\overline{r}}_{\alpha s}\overset{s}{v}_\beta - \Gamma^{\overline{r}}_{\beta s}\overset{s}{v}_\alpha = 0. \tag{20}$$

Supposing in (20) $\alpha = s, \alpha = \overline{p}$ consecutively and granting the independence of $\overset{\sigma}{v}_\beta$, we find $\Gamma^{\overline{r}}_{\overline{p}s} = \Gamma^k_{s\overline{p}} = 0$.
It is easy to prove that if the conditions $\Gamma^{\overline{r}}_{\overline{p}s} = \Gamma^k_{s\overline{p}} = 0$ are fulfilled in the parameters of the coordinate net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N(X_n \times X_m)$, then this net is a Ch-net. □

Definition 5. *A net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N$ *will be called g, Ch-net if for any* $s, \overline{k}$ *the coefficients from the derivative equations (14) satisfy the equalities* $\underset{\overline{k}}{\overset{s}{T}}_\alpha \overset{n}{a}{}^\alpha_\beta = \underset{s}{\overset{\overline{k}}{T}}_\alpha \overset{n}{a}{}^\alpha_\beta = 0$.

Theorem 7. *The composition* $(X_n \times X_m) \in A_N$ *is a* g, Ch*-composition if and only if it is associated with a* g, Ch*-net.*

P r o o f. According to (6), (13), (14) we find

$$\overset{n}{a}{}^\sigma_\alpha \nabla_\sigma \overset{n}{a}{}^\nu_\beta = \underset{s}{\overset{\overline{k}}{T}}_\sigma \overset{n}{a}{}^\sigma_\alpha \overset{s}{v}_\beta \underset{\overline{k}}{v}^\nu - \underset{\overline{k}}{\overset{s}{T}}_\sigma \overset{n}{a}{}^\sigma_\alpha \overset{\overline{k}}{v}_\beta \underset{s}{v}^\nu. \tag{21}$$

Let the composition $(X_n \times X_m) \in A_N$ be associated with a g, Ch-net. From (21) because of (2) and Definition 4 we get $\overset{n}{a}{}^\sigma_\alpha \nabla_\sigma \overset{n}{a}{}^\nu_\beta = 0$, i.e. $(X_n \times X_m) \in A_N$ is a g, Ch-composition.

Let the composition $(X_n \times X_m) \in A_N$, associated with the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ be g, Ch-composition. According to (6), (21) and the independence of $\underset{\alpha}{v}^\sigma$ and $\overset{\alpha}{v}_\sigma$ we obtain $\overset{n}{a}{}^\sigma_\alpha \nabla_\sigma \overset{n}{a}{}^\nu_\beta = 0$, which means that $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ is a g, Ch-net. □

Theorem 8. *If the coordinate net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v}) \in A_N(X_n \times X_m)$ *is a g, Ch-net, then the coefficients of the connectedness satisfy the conditions* $\Gamma^{\overline{k}}_{rs} = \Gamma^k_{r\overline{s}} = 0$.

P r o o f. Let the g, Ch-net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$ be chosen as a coordinate one. According to (13) the equalities in Definition 4 accept the form $\underset{s}{\overset{\overline{k}}{T}}_\alpha \overset{p}{v}_\beta \underset{p}{v}^\alpha = \underset{\overline{k}}{\overset{s}{T}}_\alpha \overset{p}{v}_\beta \underset{p}{v}^\alpha = 0$. Now using (16) we establish

$$\overset{s}{v}_\sigma(\partial_\alpha \underset{\overline{k}}{v}^\sigma + \Gamma^\sigma_{\alpha\nu}\underset{\overline{k}}{v}^\nu)\overset{p}{v}_\beta \underset{p}{v}^\alpha = \overset{\overline{k}}{v}_\sigma(\partial_\alpha\underset{s}{v}^\sigma+\Gamma^\sigma_{\alpha\nu}\underset{s}{v}^\nu)\overset{p}{v}_\beta\underset{p}{v}^\alpha=0.$$

Taking into account that the g, Ch-net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$ is coordinate, we get $\Gamma^{\overline{k}}_{rs} = \Gamma^{k}_{r\overline{s}} = 0$.
It is easy to prove that if the conditions $\Gamma^{\overline{k}}_{rs} = \Gamma^{k}_{r\overline{s}} = 0$ are fulfilled in the parameters of the coordinate net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$, then this net is a g, Ch-net. □

Definition 6. *A net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N$ *will be called Ch,g-net if for any* $s,\overline{k}$ *the coefficients from the derivative equations* (14) *satisfy the equalities* $\underset{\overline{k}}{\overset{s}{T}}_\alpha \overset{m}{a}{}^{\alpha}_{\beta} = \underset{s}{\overset{\overline{k}}{T}}_\alpha \overset{m}{a}{}^{\alpha}_{\beta} = 0$.

The proof of the next two theorems is essentially the same as the proof of the Theorems 7, 8.

Theorem 9. *The composition* $(X_n \times X_m) \in A_N$ *is a* Ch, g*-composition if and only if it is associated with a* Ch, g*-net.*

Theorem 10. *If the coordinate net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$ *is a Ch,g-net, then the coefficients of the connectedness satisfy the conditions* $\Gamma^{\overline{k}}_{s\overline{p}} = \Gamma^{k}_{\overline{s}\,\overline{p}} = 0$.

Of cause if in the parameters of the coordinate net

$$(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v}) \in A_N(X_n \times X_m)$$

the equalities $\Gamma^{\overline{k}}_{s\overline{p}} = \Gamma^{k}_{\overline{s}\,\overline{p}} = 0$. are fulfilled, then this net is a Ch, g-net.

The characteristics of the spaces in the parameters of special coordinate nets of the spaces $A_N(X_n \times X_m)$ which contain special compositions (Theorems 2, 4, 6, 8) coincide with the characteristics in the adapted with these compositions coordinate systems found in [2].

3. Three interrelated compositions generated by a net in $\boldsymbol{A_N}$. Let the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ be given in A_N . Consider the affinors (11) and

$$\begin{aligned} b_\alpha^\beta &= \underset{1}{v}^\beta \overset{1}{v}_\alpha + \underset{2}{v}^\beta \overset{2}{v}_\alpha + \dots + \underset{k}{v}^\beta \overset{k}{v}_\alpha - \underset{k+1}{v}{}^\beta \overset{k+1}{v}{}_\alpha - \dots - \underset{N}{v}^\beta \overset{N}{v}_\alpha, \quad k < n, \\ c_\alpha^\beta &= \underset{1}{v}^\beta \overset{1}{v}_\alpha + \underset{2}{v}^\beta \overset{2}{v}_\alpha + \dots + \underset{k}{v}^\beta \overset{k}{v}_\alpha - \underset{k+1}{v}{}^\beta \overset{k+1}{v}{}_\alpha - \dots - \underset{n}{v}^\beta \overset{n}{v}_\alpha + \\ &\quad + \underset{n+1}{v}{}^\beta \overset{n+1}{v}{}_\alpha + \dots + \underset{N}{v}^\beta \overset{N}{v}_\alpha, \end{aligned} \tag{22}$$

uniquely determinate from the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$. From (9), (11), (22) follow

$$a_\alpha^\beta b_\beta^\sigma = c_\alpha^\sigma, \quad b_\alpha^\beta c_\beta^\sigma = a_\alpha^\sigma, \quad c_\alpha^\beta a_\beta^\sigma = b_\alpha^\sigma; \tag{23}$$

$$a_\alpha^\beta a_\beta^\sigma = \delta_\alpha^\sigma, \quad b_\alpha^\beta b_\beta^\sigma = \delta_\alpha^\sigma, \quad c_\alpha^\beta c_\beta^\sigma = \delta_\alpha^\sigma. \tag{24}$$

According to (24) the affinors $a_\alpha^\beta,\ b_\alpha^\beta,\ c_\alpha^\beta$ define compositions which we will denote $(X_n \times X_m),\ (Y_n \times Y_m),\ (Z_n \times Z_m)$, respectively . These three compositions will be called three interrelated compositions. The projecting affinors of $(Y_n \times Y_m)$ and $(Z_n \times Z_m)$ will be denoted by $\overset{k}{b}{}_\alpha^\beta,\ \overset{p}{b}{}_\alpha^\beta$ and $\overset{r}{c}{}_\alpha^\beta,\ \overset{s}{c}{}_\alpha^\beta$, where $k+p=r+s=n+m=N$. Because of (22) we can write

$$\overset{k}{b}{}_\alpha^\beta = \underset{i}{v}^\beta \overset{i}{v}_\alpha \ , \quad \overset{p}{b}{}_\alpha^\beta = \underset{\bar{i}}{v}^\beta \overset{\bar{i}}{v}_\alpha \ , \quad \overset{r}{c}{}_\alpha^\beta = \underset{j}{v}^\beta \overset{j}{v}_\alpha \ , \quad \overset{s}{c}{}_\alpha^\beta = \underset{\bar{j}}{v}^\beta \overset{\bar{j}}{v}_\alpha \ , \tag{25}$$

where $i = 1,2,\dots,k$; $\bar{i} = k+1, k+2, \dots, N$; $j = 1,2,\dots,k,n+1,n+2,\dots,N$; $\bar{j} = k+1, k+2, \dots, n$.
According to (13), (25) we obtain $\overset{n}{a}{}_\alpha^\beta = \overset{k}{b}{}_\alpha^\beta + \overset{s}{c}{}_\alpha^\beta,\ \overset{r}{c}{}_\alpha^\beta = \overset{k}{b}{}_\alpha^\beta + \overset{m}{a}{}_\alpha^\beta,\ \overset{p}{b}{}_\alpha^\beta = \overset{s}{c}{}_\alpha^\beta + \overset{m}{a}{}_\alpha^\beta$.
Hence

$$\nabla_\nu \overset{p}{b}{}_\alpha^\beta = \nabla_\nu \overset{s}{c}{}_\alpha^\beta + \nabla_\nu \overset{m}{a}{}_\alpha^\beta. \tag{26}$$

Now from (26) follow

Proposition 1. *If two of the three interrelated compositions are Cartesian, then the third one is Cartesian, too.*

Proposition 2. *If two of the three interrelated compositions are Chebyshevian, then the third one is Chebyshevian, too.*

Definition 7. *A net* $(\underset{1}{v},\underset{2}{v},\ldots,\underset{N}{v}) \in A_N$ *will be called C3-net if the compositions, generated by the affinors* (11) *and* (22) *are Cartesian.*

According to Theorem 1 the derivative equations (10) for the $C3$-net accept the form

$$\begin{aligned}
&\nabla_\alpha \underset{p}{v}^\sigma = \overset{i}{\underset{p}{T}}{}_\alpha \underset{i}{v}^\sigma\ ,\ p=1,2,\ldots,k;\ i=1,2,\ldots,k\ ,\\
(27)\qquad &\nabla_\alpha \underset{p}{v}^\sigma = \overset{i}{\underset{p}{T}}{}_\alpha \underset{i}{v}^\sigma\ ,\ p=k+1,k+2,\ldots,n;\ i=k+1,k+2,\ldots,n\ ,\\
&\nabla_\alpha \underset{p}{v}^\sigma = \overset{i}{\underset{p}{T}}{}_\alpha \underset{i}{v}^\sigma\ ,\ p=n+1,n+2,\ldots,N;\ i=n+1,n+2,\ldots,N\ .
\end{aligned}$$

From Theorem 2 it follows

Corollary 1. *If the coordinate net* $(\underset{1}{v},\underset{2}{v},\ldots,\underset{N}{v}) \in A_N$ *is a C3-net, then the coefficients of the connectedness satisfy the conditions* $\Gamma^\beta_{\alpha\sigma}=0$ *for any* σ *and* $(\beta=1,2,\ldots,k;\ \alpha=k{+}1,k{+}2,\ldots,N)$, $(\beta=n{+}1,n{+}2,\ldots,N;\ \alpha=1,2,\ldots,n)$, $(\beta=k+1,k+2,\ldots,n;\ \alpha=1,2,\ldots,k,n+1,n+2,\ldots,N)$.

Definition 8. *A net* $(\underset{1}{v},\underset{2}{v},\ldots,\underset{N}{v}) \in A_N$ *will be called Ch3-net if the compositions, generated by the affinors* (11) *and* (22) *are Chebyshevian.*

From Theorem 6 it follows

Corollary 2. *If the coordinate net* $(\underset{1}{v},\underset{2}{v},\ldots,\underset{N}{v}) \in A_N$ *is a Ch3-net, then the coefficients of the connectedness satisfy the conditions* $\Gamma^\beta_{\alpha\sigma}=0$ *for* $(\alpha,\beta=1,2,\ldots,n;\ \sigma=n+1,n+2,\ldots,N)$, $(\alpha,\beta=k+1,k+2,\ldots,N;\ \sigma=1,2,\ldots,k)$, $(\alpha,\beta=1,2,\ldots,k,n+1,n+2,\ldots,N,\ \sigma=k+1,k+2,\ldots,n)$.

Example. Let the net $(\underset{1}{v},\underset{2}{v},\underset{3}{v},\underset{4}{v})$ be given in the space A_4 without torsion. Obviously the affinors $a^\beta_\alpha=\underset{1}{v}^\beta\overset{1}{v}_\alpha+\underset{2}{v}^\beta\overset{2}{v}_\alpha+\underset{3}{v}^\beta\overset{3}{v}_\alpha-\underset{4}{v}^\beta\overset{4}{v}_\alpha$, $b^\beta_\alpha=\underset{1}{v}^\beta\overset{1}{v}_\alpha+\underset{2}{v}^\beta\overset{2}{v}_\alpha-\underset{3}{v}^\beta\overset{3}{v}_\alpha-\underset{4}{v}^\beta\overset{4}{v}_\alpha$, $c^\beta_\alpha=\underset{1}{v}^\beta\overset{1}{v}_\alpha+\underset{2}{v}^\beta\overset{2}{v}_\alpha-\underset{3}{v}^\beta\overset{3}{v}_\alpha+\underset{4}{v}^\beta\overset{4}{v}_\alpha$, satisfy (23), (24). If the net $(\underset{1}{v},\underset{2}{v},\underset{3}{v},\underset{4}{v})$ is a $C3$-net, then the derivative equations (27) accept the form $\nabla_\alpha\underset{1}{v}^\sigma=\overset{1}{\underset{1}{T}}{}_\alpha\underset{1}{v}^\sigma+\overset{2}{\underset{1}{T}}{}_\alpha\underset{2}{v}^\sigma$, $\nabla_\alpha\underset{2}{v}^\sigma=\overset{1}{\underset{2}{T}}{}_\alpha\underset{1}{v}^\sigma+\overset{2}{\underset{2}{T}}{}_\alpha\underset{2}{v}^\sigma$, $\nabla_\alpha\underset{3}{v}^\sigma=\overset{3}{\underset{3}{T}}{}_\alpha\underset{3}{v}^\sigma$, $\nabla_\alpha\underset{4}{v}^\sigma=\overset{4}{\underset{4}{T}}{}_\alpha\underset{4}{v}^\sigma$, from where it follows that the fields of directions $\underset{3}{v}^\sigma$, and $\underset{4}{v}^\sigma$, are absolutely parallel.

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