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USING MONTE CARLO METHODS TO EVALUATE SUB-OPTIMAL EXERCISE POLICIES FOR AMERICAN OPTIONS

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Abstract. In this paper we use a Monte Carlo scheme to find the returns that an uninformed investor might expect from an American option if he followed one of several naïve exercise strategies rather than the optimal exercise strategy. We consider several such strategies that an ill-advised investor might follow. We also consider how the expected return is affected by how often the investor checks to see if his exercise criteria have been met.

1. Introduction. Options are derivative financial instruments giving the holder the right but not the obligation to buy (or sell) an underlying asset. They have numerous uses, such as speculation, hedging, generating income, and they contribute to market completeness. Although options have existed for much

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longer, their use has become much more widespread since 1973 when two of the most significant events in the history of options occured. The first of these was the publication the Black-Scholes option pricing formula, which enabled investors to price certain options, and the second important event was the opening of the Chicago Board Options Exchange (CBOE), which was really the first secondary market for options. Before the CBOE opened its doors, it was extremely difficult for an investor to sell any options that he might own, so that he was left with the choice of holding the option to expiry, or exercising early if that was permitted. With the advent of the CBOE, he had the additional choice of reselling the options to another investor.

There are various ways of categorizing options, one method being by the exercise characteristics. Options are usually either European, meaning they can be exercised only at expiry, which is a pre-determined date specified in the option contract, or American, meaning they can be exercised at or before expiry, at the holder's discretion. A third, less common, type is Bermudan, which can be exercised early, but only on a finite number of pre-specified occasions. European options are fairly easy to value. However, American options are much harder since because they can be exercised early, the holder must decide whether and when to exercise such an option, and this is one of the best-known problems in mathematical finance, leading to an optimal exercise boundary and an optimal exercise policy, which if followed will maximize the expected return. Ideally, an investor would be able to constantly calculate the expected return from continuing to hold the option, and if that is less than the return from immediate exercise, he should exercise the option. This process would tell the investor the location of the optimal exercise boundary. However, to date no closed form solutions are known for the location of the optimal exercise boundary, except for one or two very special cases such as the call with no dividends when early exercise is never optimal, and in general either numerical solutions or approximations must be used to locate the optimal exercise boundary. Both of these approaches are fairly well-developed, and we will mention some of the more important aspects of them below; for a complete review, the reader is referred to the monographs by Kwok [22] and Wilmott [30].

Amongst numerical methods, there is essentially a dichotomy amongst practitioners, with one approach being to formulate the problem as a stochastic differential equation (SDE) together with the appropriate boundary conditions and the other to formulate it as a partial differential equation (PDE) which can be derived by applying a no-arbitrage argument to the SDE. For the PDE approach, the finite-difference method is the standard approach [9, 31, 30], and this involves solving the PDE on a discrete grid. For the SDE approach, the more popular methods include binomial and trinomial trees [14, 6], which involve integrating the SDE backwards in time from expiry. Geske & Shastri [17] give an early comparison of finite-difference and binomial tree methods, although of course the state-of-the-art in both methods has come a long way since that study. More recently, several researchers have tried to price American options using Monte Carlo methods, which involve integrating forwards rather than backwards in time, and reviews of some of the more recent attempts are given in [7] and [27]. The use of Monte Carlo methods to value American options is still a nebulous problem. Several very promising approaches ranging from Malliavin calculus through the bundling algorithm of Tilley [29], the Grant-Vora-Weeks algorithm [18] which essentially treats the option as a Bermudan with exercise only at a series of discrete dates, and the Broadie & Glasserman algorithm [11] which produces a high and a low estimate for the option value, with the true value being between the two estimates. Other approaches include the work of Bossaert [5] who solved for the early exercise strategy, the paper of Ibanez & Zapatero [20] who used an optimization scheme to find the location of the optimal exercise boundary at a series of discrete points, and that of Mallier [25] who approximated the boundary using a series of basis functions. Although many of these Monte Carlo approaches are promising, many practioners feel that none of them is entirely satisfactory yet. We should mention that the difficulties in applying Monte Carlo methods to American options stem from the need to locate the optimal exercise boundary, and for the problem studied here, that is not an issue: rather, we are calculating what an option is worth if a pre-specified strategy is followed, so that location of our (sub-optimal) exercise boundary is already known.

Turning to approximate solutions, many different approaches have been taken over the years, and a review of some of them was given in the recent paper by Mallier [26]. That paper was primarily concerned with evaluating the accuracy of series solutions to the optimal exercise boundary [15, 4, 2, 3], but also contained a comparison between the series solutions and several other approximations, such as the quadratic approximation of MacMillan [24], which involves solving an approximate PDE for the early exercise premium, the LUBA (lower and upper bound approximation) of Broadie & Detemple [10], which involves finding very tight upper and lower bounds for the optimal exercise boundary, the Geske-Johnson formula [21, 16], which views an American option as a sequence of Bermudan options with the number of exercise dates increasing, and the method of lines [12]. The approximations mentioned above represent only a small sample of those in the literature, and more complete surveys are given in [22, 10, 26].

Although as we mentioned above, numerous studies have been done on the valuation of American options using both numerical solutions and approximations, Both of these approaches can be difficult and time-consuming, and whereas an institution can perform those calculations and thereby optimize their return, an individual may well be unable to do this, and instead have his own naïve exercise policy, choosing to exercise the option when certain conditions are met, for example when the value of the option reaches some multiple of the exercise price. We will refer to such an individual as an uninformed investor. The expected return from such sub-optimal strategies will be less than or equal to that when the optimal exercise policy is pursued.

2. Monte Carlo scheme. In this study, we use a Monte Carlo scheme to look at several such strategies that an ill-advised investor might follow, and calculate the expected return using these strategies. In terms of the stock price S and the initial stock price S_0, the 8 strategies we used for the call option to exercise the option when:

(1): Never (i.e. treat the option like a European).
(2): If S is 110% or more of S_0 (put: $S \leq 0.9\ S_0$).
(3): If S is 115% or more of S_0 and in money (put: $S \leq 0.85\ S_0$).
(4): If S is greater than S_0 and at or in money (put: $S < S_0$).
(5): If S goes down by 10% and still in money (put: $S \geq 1.1 S_0$).
(6): If S goes down by 5% (put: $S \geq 1.05 S_0$).
(7): If S goes down by 10% from its peak and in the money. (put: S up by 10% from trough).
(8): If S goes up on 5 successive time-steps and is in the money (put: down).

We should recall that for the call with no dividends, it is never optimal to exercise before the expiration date, so we would expect strategy 1 to be the best for the call. In addition to evaluating the expected return to an investor if he were to follow one of these naïve strategies, we will also look at how the expected return is affected by how often the investor checks to see if his exercise criteria have been met. As we mentioned above, we will tackle this problem with Monte Carlo simulation. This approach is well-suited for this particular problem, since the underlying stock price S is assumed to follow a random walk. The use of Monte Carlo methods for option pricing was pioneered by Boyle [8], and these methods have since become extremely popular because they are both powerful and extremely flexible. Although the use of Monte Carlo methods to value American options is still a nebulous problem, with for example several researchers pursuing Malliavin calculus while others are attempting different approaches, these difficulties stem from the need to locate the optimal exercise boundary, and for the problem studied here, that is not an issue: rather, we are calculating what an option is worth if one of several naïve strategies is followed, and so the location of our (sub-optimal) exercise boundary is fairly simple. Returning to option pricing in general, in this context, Monte Carlo methods involve the direct stochastic integration of the underlying Langevin equation for the stock price, which is assumed to follow a log-normal random walk or geometric Brownian motion. The heart of any Monte Carlo method is the random number generator, and our code employed the Netlib routine RANLIB, which produces random numbers which are uniformly distributed on the range $(0, 1)$ and which were then converted to normally distributed random numbers. This routine was itself based on the article by L'Ecuyer & Cote [23]. Antithetic variables were used to speed convergence, and a large number of realizations were performed to ensure accurate results. Our simulations, including other runs not presented here, required about a month's CPU time on a DEC Alpha and were performed on the Beowulf cluster at the University of Western Ontario.

The starting point of our analysis is the risk-neutral random walk for the price of the underlying S in the absence of dividends,

$$dS = rSdt + \sigma SdX, \tag{2.1}$$

where dX describes the random walk, dt is the step size, taken to be 0.01 in our simulations, r is the risk free rate and σ the volatility. If we assume that the simulation is started at time t_0 and ends at expiry T, then the other parameters which affect the simulations are the initial stock price $S_0 = S(t_0)$, the exercise price E and the time to expiry, $\tau = T - t_0$. For each value of the parameters, a separate set of runs was done for each of the exercise strategies. For each realization, at each time step, we first check to see if the exercise criteria has been satisfied, and either exercise at that step or continue to the next time step, and repeat this procedure either the option has been exercised or we reach expiry, at which time the option is either exercised or expires worthless. For each realization, we calculate the payoff, which is $\max\left[S\left(T_E\right) - E, 0\right]$ for the call and $\max\left[E - S\left(T_E\right), 0\right]$ for the put, where T_E is the time at which the option was exercised. We then

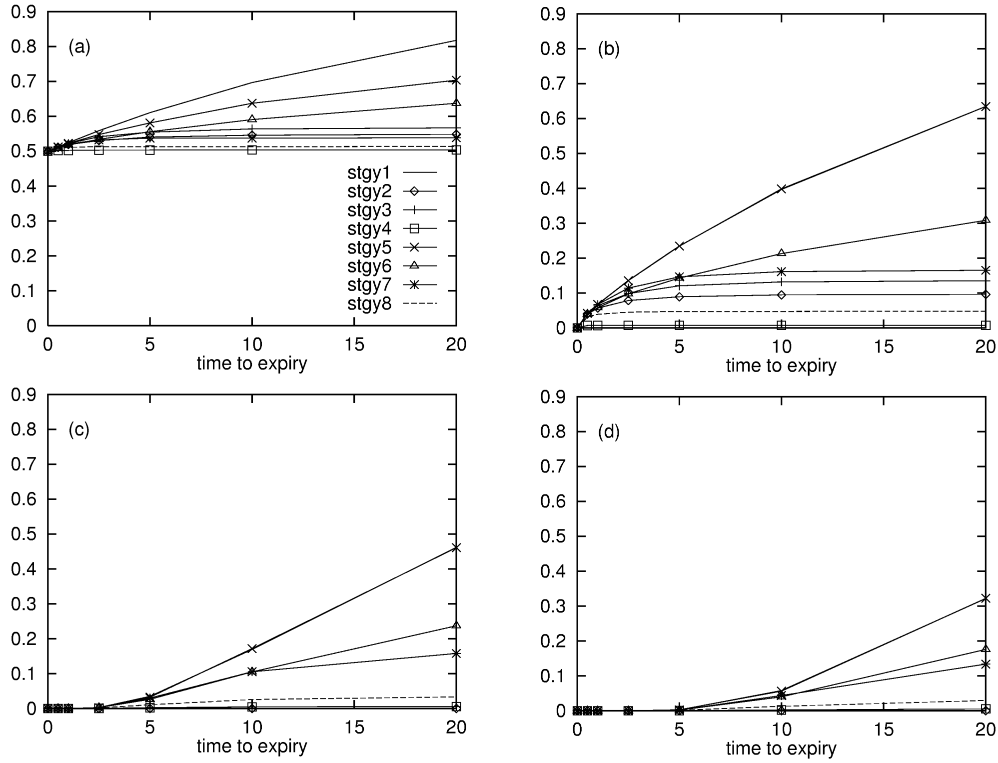

Fig. 1. Effect of E: call. $S_0 = 1$, $r = 0.05$, $\sigma = 0.1$. (a): E=0.5, (b) 1, (c) 1.5, (d) 2.

discount this value back to the starting time to find its present value. The value of the option is the average over all realizations of this present value.

3. Results. In this section, we present the results of some of our simulations, and in particular examine the effects of varying the various parameters. In figure 1, we look at the effect of varying the strike price E for the call while holding the other parameters constant; the corresponding runs for the put are in figure 2. For the call, strategy 1 (holding) is best, which is to be expected given that it is never optimal to exercise a call with no dividends. By contrast, for the put, no one strategy is best, and in actuality, they are all bad. Holding is no longer optimal and is sometimes the worst strategy amongst those studied. While for the call, the value always increased with time to expiry, for the put sometimes the value decreased and sometimes it increased. Presumably, this happens be-

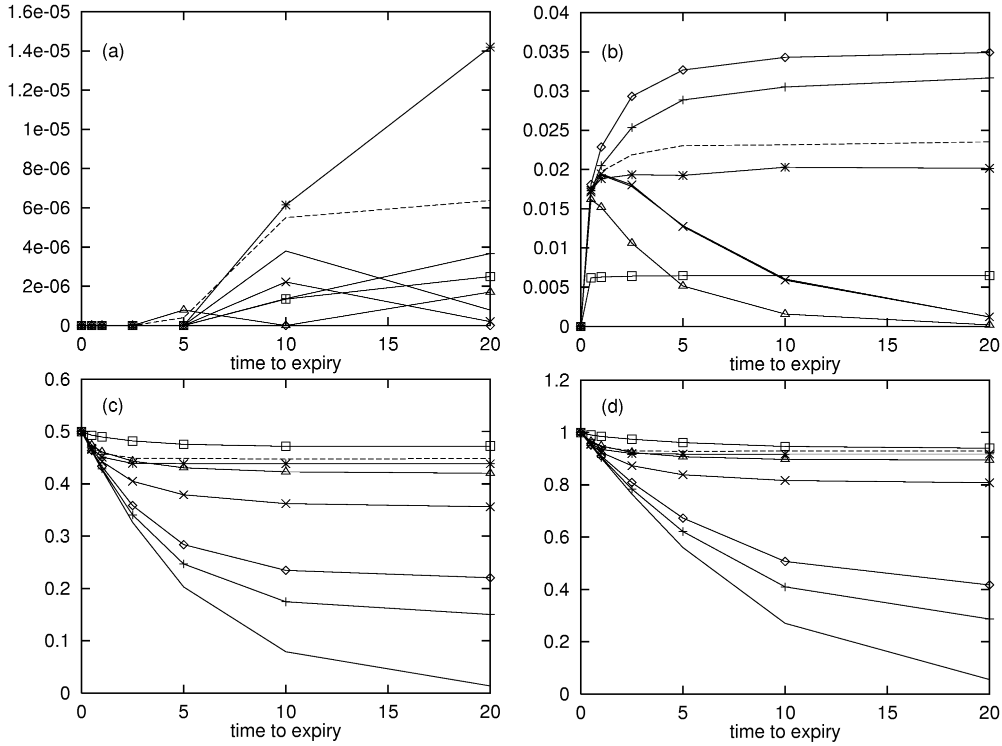

Fig. 2. As for figure 1 but for the put

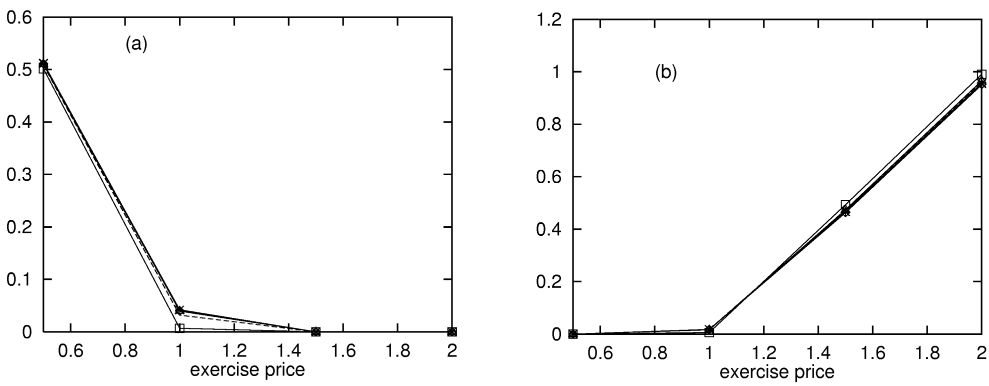

Fig. 3. Effect of E. $S_0 = 1$, $r = 0.05$, $\sigma = 0.1$, $\tau = 0.5$ (a) call, (b) put.

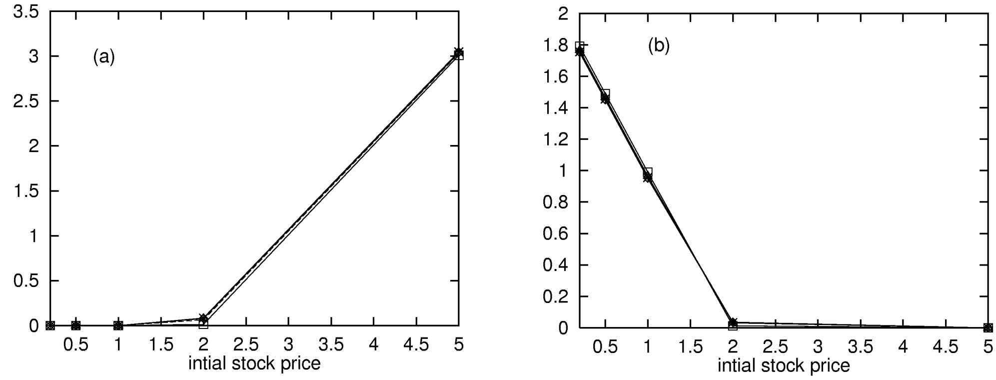

Fig. 4. Effect of S_0. $E = 2$, $r = 0.05$, $\sigma = 0.1$, $\tau = 0.5$. (a) call, (b) put.

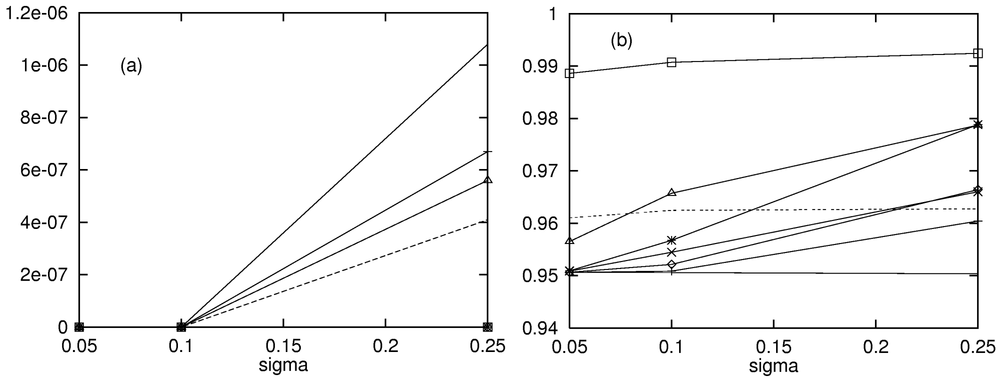

Fig. 5. Effect of σ. $E = 2$, $r = 0.05$, $S_0 = 1$, $\tau = 0.5$. (a) call, (b) put.

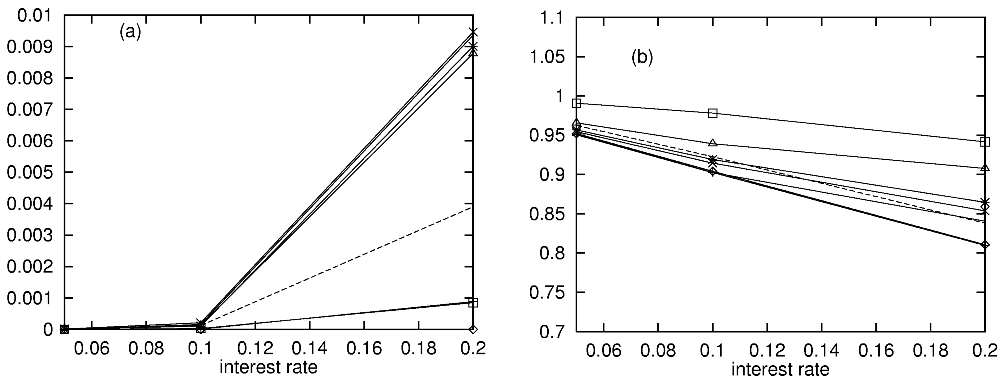

Fig. 6. Effect of r. $E = 2$, $\sigma = 0.1$, $S_0 = 1$. (a) call $\tau = 2.5$, (b) put $\tau = 0.5$.

cause some of the strategies for the put are especially bad, and increasing the tenor increases the possibility of inopportune exercise. In figure 3, we see that as we increased the exercise price, the value of the call decreased while that of the put increased. This dependence on exercise price is of course to be expected from our knowledge of the greeks. Similarly, we looked at the effect of varying the initial stock price, finding as expected that as we increased S_0 the option value increased for the call but decreased for the put. These results are summarized in figure 4. In figure 5, we examine the effects of varying the volatility, and find that for both the put and call, increasing σ leads to an increase in the value of the option, again as expected. In figure 6, we look at the effect of varying the risk-free rate r, and find that increasing r increases the option value for the call but decreases it for the put, once again as expected. We also studied the effect that the frequency of application of the strategy had on the expected returns from the option. Our results are shown in figure 7. The time-step used in our simulations

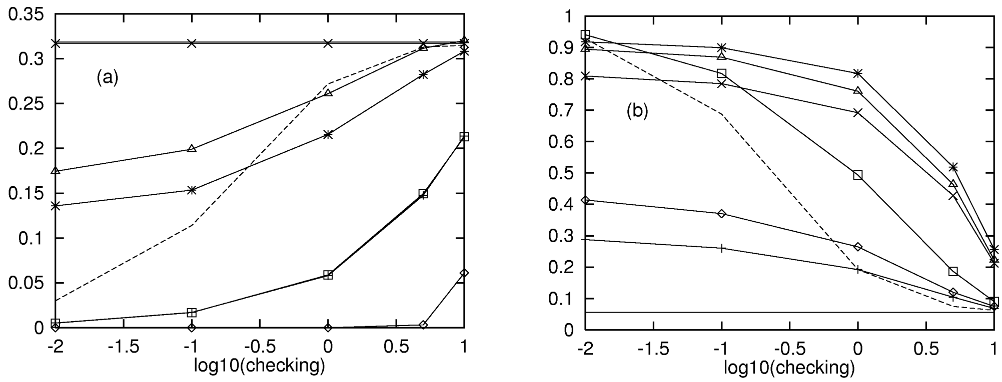

Fig. 7. Effect of checking. $E = 2$, $\sigma = 0.1$, $r = 0.05$, $S_0 = 1$, $\tau = 20$. (a) call, (b) put.

was $dt = 0.01$, and to examine the effects of frequency we applied the strategy initially every step or 0.01 time units, and then (in different runs) every 10 steps (0.1 units), 100 steps (1 units), 500 steps (5 units) and 1000 steps (10 units). The motivation for this was an attempt to model the real world behaviour of different classes of investor, ranging from institutions using computer trading through a day trader who is constantly checking prices, and an average investor who might check prices daily of weekly, to a pension fund investor gets report once a month. Here, we are essentially treating the option like a Bermudan, as indeed we have in this entire study since we are using a finite time-step. We see that for the call, strategy 1, which was holding, is unaffected by the frequency of checking and that strategy 5, which for these particular parameter values results in infrequent exercise, is little affected by the frequency, but that amongst the other strategies increasing the interval between checks leads to an increase in value. We should recall that it is never optimal to exercise the call without dividends, so that increasing the interval reduces the likelihood of inopportune exercise. For the put, strategy 1, which was holding, is again unaffected by the frequency, while for the other strategies, increasing the interval leads to a decrease in value. We should recall that it is sometimes optimal to exercise the put even without dividends, so that increasing the interval reduces exercise possibilities.

4. Conclusion. In this paper, we have looked at a number of naïve exercise strategies for American options, and used a Monte Carlo scheme to find the returns that an investor would expect if he followed one of these strategies, looking at the effects of varying the sundry parameters. The variation of the expected returns with these parameters was largely as expected from the greeks. As expected, for a call without dividends, holding was the best strategy. For the put, no single strategy amongst those studied was best, with different strategies being better in different areas of parameter space; in fact, all of the strategies for the put and all apart from holding for the call were fairly bad strategies from the point of view of the returns that an investor would expect if he pursued one of those strategies, and so our advice to an unsophisticated investor would be to steer clear of American options.

REFERENCES

[1] G. Alobaidi. American options and their strategies. Ph.D. Thesis, University of Western Ontario, London, Canada, 2000.

[2] G. Alobaidi, R. Mallier. Asymptotic analysis of American call options. *Int. J. Math. Math. Sci.* **27** (2001), 177–188.

[3] G. Alobaidi, R. Mallier. On the optimal exercise boundary for an American put option. *J. Appl. Math.* **1** (2001), 39–45.

[4] G. Barles, J. Burdeau, M. Romano, N. Samsoen. Critical stock price near expiration. *Math. Finance* **5** (1995), 77–95.

[5] P. Bossaert. Simulation estimators of optimal early exercise. Preprint, Carnegie Mellon University, 1989.

[6] P. P. Boyle. A lattice framework for option pricing with two state variables. *J. Financial and Quantitative Analysis* **23** (1988), 1–12.

[7] P. P. Boyle, A. W. Kolkiewicz, K. S. Tan. (2002). Pricing American derivatives using simulation: a biased low approach. In: Monte Carlo and Quasi-Monte Carlo Methods 2000 (Eds K.-T. Fang, F. J. Hickernell, H. Niederreiter) Springer, Berlin, 2002, 181–200.

[8] P. P. Boyle. Options: a Monte-Carlo approach. *J. Financial Economics* **4** (1979), 323–338.

[9] M. J. Brennan, E. S. Schwartz. The valuation of the American put option. *J. Finance* **32** (1977), 449–462.

[10] M. Broadie, J. Detemple. American options valuation: new bounds, approximations, and a comparison with existing methods. *Review of Financial Studies* **9** (1996), 1211–1250.

[11] M. Broadie, P. Glasserman. Pricing American-style securities using simulation. *J. Econom. Dynam. Control* **21** (1997) 1323–1352.

[12] P. Carr, P. Faquet. Fast evaluation of American options. Working Paper, Cornell University, 1994.

[13] Don M. Chance. An introduction to Derivatives, 3rd edition. The Dryden press, Orlando. 1995.

[14] J. Cox, S. Ross, M. Rubinstein. Option pricing: a simplified approach. *J. Financial Economics* **7** (1979), 229–264.

[15] J. N. Dewynne, S. D. Howison, I. Rupf, P. Wilmott. Some mathematical results in the pricing of American options. *European J. Appl. Math.* **4** (1993), 381–398.

[16] R. Geske, H. E. Johnson. The American put valued analytically. *J. Finance* **39** (1984), 1511–1524.

[17] R. Geske, K. Shastri. Valuation by approximation: a comparison of alternative option valuation techniques. *J. Financial and Quantitative Analysis* **20** (1985), 45–71.

[18] D. Grant, G. Vora, D. E. Weeks. (1996). Simulation and the early-exercise option problem. *J. Financial Engineering* **5** (1996), 211–227.

[19] J. C. Hull. Options, Futures and Other Derivatives, 4th edition. Prentice Hall, Upper Saddle River, NJ, 2000.

[20] A. Ibanez, F. Zapatero. Monte Carlo valuation of American options through computation of the optimal exercise frontier. Preprint **99-8**, Marshall School of Business, University of Southern California, 1999.

[21] H. E. Johnson. An analytical approximation to the American put price. *J. Financial and Quantitative Analysis* **18** (1983), 141–148.

[22] Y. K. Kwok. Mathematical Models of Financial Derivatives. Springer, Singapore, 1998.

[23] P. L'Ecuyer, S. Cote. Implementing a Random Number Package with Splitting Facilities. *ACM Trans. Math. Software* **17** (1991), 98–111.

[24] L. W. MacMillan. Analytic approximation for the American put option. *Advances in Futures and Options* **1A** (1986), 119–139.

[25] R. Mallier. Approximating the optimal exercise boundary for American options via Monte Carlo. In: Proceedings Computational Intelligence: Methods and Applications CIMA2001 (Eds L. I. Kuncheva et al.), ICSC Academic Press, Canada, 2001, 365–371.

[26] R. Mallier. Evaluating approximations to the optimal exercise boundary for American options. *J. Appl. Math.* **2** (2002), 71–92.

[27] J. A. Picazo. American option pricing: a classification-Monte-Carlo (CMC) approach. In: Monte Carlo and Quasi-Monte Carlo Methods 2000 (Eds K.-T. Fang, F. J. Hickernell, H. Niederreiter) Springer, Berlin, 422–433.

[28] V. Kerry Smith. Monte-Carlo methods. D. C. Heath and Company, 1973.

[29] J. Tilley. Valuing American options in a path simulation model. *Transactions of the Society of Actuaries* **45** (1983), 83–104.

[30] P. Wilmott. Paul Wilmott on Quantitative Finance. Wiley, Chichester, 2000.

[31] L. Wu, Y.-K. Kwok. A front-fixing finite difference method for the valuation of American options. *J. Financial Engineering* **6** (1997), 83–97.

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