Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **28** (2002), 233-240

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE STABILIZATION OF THE WAVE EQUATION BY THE BOUNDARY

Fernando Cardoso, Georgi Vodev

Communicated by V. Petkov

Abstract. We study the distribution of the (complex) eigenvalues for interior boundary value problems with dissipative boundary conditions in the case of C^1-smooth boundary under some natural assumption on the behaviour of the geodesics. As a consequence we obtain energy decay estimates of the solutions of the corresponding wave equation.

1. Introduction and statement of results. Let $\mathcal{O}^\sharp \subset \mathbf{R}^n$, $n \geq 2$, be a bounded, connected domain with a C^∞-smooth boundary $\partial\mathcal{O}^\sharp$, and let $g^\sharp = \sum_{i,j=1}^n g^\sharp_{ij}(x)dx_i dx_j$ be a Riemannian metric in $\mathcal{O}^\sharp$, $g^\sharp_{ij} \in C^\infty(\overline{\mathcal{O}^\sharp})$. Let $\mathcal{O} \subset \mathcal{O}^\sharp$, $\partial\mathcal{O} \cap \partial\mathcal{O}^\sharp = \emptyset$, be another bounded, connected domain with boundary of class C^1, equipped with the Riemannian metric $g = \sum_{i,j=1}^n g_{ij}(x)dx_i dx_j := g^\sharp|_{\mathcal{O}}$. Denote by Δ_g the (negative) Laplace-Beltrami operator on $(\mathcal{O}, g)$, i.e.

$$\Delta_g = p^{-1} \sum_{i,j=1}^n \partial_{x_i}\left(p g^{ij} \partial_{x_j}\right),$$

2000 *Mathematics Subject Classification:* 35B37, 35J15, 47F05
Key words: complex eigenvalues, energy decay.
Partially supported by CNPq (Brazil)

where $\left(g^{ij}\right)$ is the inverse matrix to (g_{ij}), and $p = (\det (g_{ij}))^{1/2} = \left(\det \left(g^{ij}\right)\right)^{-1/2}$. Our purpose is to study the energy decay of the solutions of the equation

$$
\left\{\begin{array}{c}
(\partial_t^2 - \Delta_g)u(t,x) = 0 \quad \text{in} \quad \mathcal{O} \times \mathbf{R}, \\
u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x) \quad \text{in} \quad \mathcal{O}, \\
-\partial_\nu u + a(x)\partial_t u = 0 \quad \text{on} \quad \partial\mathcal{O} \times \mathbf{R},
\end{array}\right. \tag{1.1}
$$

where ν is the unit inner normal to $\partial\mathcal{O}$ associated to the metric g, and $a(x) \in C(\partial\mathcal{O})$ is a non-identically zero real-valued function such that $a(x) \geq 0$, $\forall x \in \partial\mathcal{O}$. The energy of the solution $u(t,x)$ is given by

$$E(t) = \frac{1}{2}\int_{\mathcal{O}} \left(|\partial_t u(t,x)|^2 + |\nabla_g u(t,x)|^2\right) p dx.$$

When $\partial\mathcal{O}$ is of class C^∞, Bardos, Lebeau and Rauch [1] gave a necessary and sufficient condition which guarantees the exponential energy decay

$$E(t) \leq Ce^{-ct}E(0), \quad t \geq 1,\, C, c > 0. \tag{1.2}$$

Roughly speaking, this condition says that every generalized geodesic must meet the set $\Gamma := \{x \in \partial\mathcal{O} : a(x) > 0\}$ at a *nondiffractive* point at time $\leq T$ for some constant $T > 0$. We refer to [1] for more precise definitions and statements. Burq [2] extended their result to the case of C^3-smooth boundary and C^2-smooth metric g.

Consider in the Hilbert space $H = H_1(\mathcal{O}) \oplus L^2(\mathcal{O})$, where $L^2(\mathcal{O}) := L^2(\mathcal{O}, d\mathrm{Vol}_g)$, $H_1(\mathcal{O})$ is the closure of $C^\infty(\overline{\mathcal{O}})$ with respect to the norm $\int_{\mathcal{O}} |\nabla_g u|^2 d\mathrm{Vol}_g$, the operator

$$A = -i\begin{pmatrix} 0 & Id \\ \Delta_g & 0 \end{pmatrix}$$

with domain

$$D(A) = \{u = (u_1, u_2) \in H : Au \in H, -\partial_\nu u_1 + a u_2 = 0 \text{ on } \partial\mathcal{O}\}.$$

It is well known that iA is a generator of a semi-group, e^{itA}, and the solution of (1.1) is given by

$$\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{itA}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Moreover, the resolvent of A is a compact operator, so $\operatorname{spec} A$ is discrete, $0 \in \operatorname{spec} A$, with no other eigenvalues on $\operatorname{Im}\lambda = 0$. In other words, we have $\operatorname{spec} A \setminus \{0\} \subset \{\operatorname{Im}\lambda > 0\}$. It is easy to see that a $\lambda \in \mathbf{C}$ belongs to $\operatorname{spec} A$ iff the following problem has a non-trivial solution:

$$
\left\{\begin{array}{c}
(\Delta_g + \lambda^2)u = 0 \quad \text{in} \quad \mathcal{O}, \\
-\partial_\nu u + i\lambda a(x)u = 0 \quad \text{on} \quad \partial\mathcal{O}.
\end{array}\right. \tag{1.3}
$$

Clearly, the bound (1.2) would follow from

$$\|e^{itA}f\|_H \le C'e^{-ct/2}\|f\|_H, \quad \forall f \in H' := H \ominus \operatorname{Ker} A. \tag{1.4}$$

The bound (1.4) implies that $\operatorname{spec} A \setminus \{0\} \subset \{\operatorname{Im}\lambda \ge c/2\}$ and

$$\|(A-\lambda)^{-1}\|_{\mathcal{L}(H)} \le C_0 \quad \text{for} \quad \operatorname{Im}\lambda \le c_0,\ |\lambda| \ge 1, \tag{1.5}$$

for some constants $C_0, c_0 > 0$. Note that the inverse is also true, that is, (1.5) $\Rightarrow$ (1.4). In fact, to get (1.4) it suffices to have (1.5) for real λ, $|\lambda| \gg 1$, only.

Lebeau and Robbiano [4] proved without any conditions on the geodesics (still in the case of C^∞-smooth boundary, assuming only that $a \ge 0$ and $\Gamma \ne \emptyset$) that $\operatorname{spec} A \setminus \{0\} \subset \{\operatorname{Im}\lambda \ge C_1 e^{-C_2|\lambda|}\}$ for some constants $C_1, C_2 > 0$ and that

$$\|(A-\lambda)^{-1}\|_{\mathcal{L}(H)} \le \widetilde{C}_1 e^{\widetilde{C}_2|\lambda|} \quad \text{for} \quad \operatorname{Im}\lambda \le C_1' e^{-C_2'|\lambda|},\ |\lambda| \ge 1, \tag{1.6}$$

for some positive constants $C_1', \widetilde{C}_1, C_2', \widetilde{C}_2$. It is easy to see that (1.6) follows from (1.6) with $\operatorname{Im}\lambda = 0$. One can derive from (1.6) (e.g. see [3], Theorem 3) that for every integer $m \ge 1$,

$$E(t)^{1/2} \le C\|e^{itA}f\|_H \le C_m (\log t)^{-m} \|f\|_{D(A^m)},\ t \ge 2, \forall f \in D(A^m) \cap H', \tag{1.7}$$

where $\|f\|_{D(A^m)} := \|(A+1)^m f\|_H$.

The purpose of the present paper is to obtain an intermediate result between (1.2) and (1.7) for boundaries with little regularity. We make the following assumptions

$$a(x) \ge a_0 > 0 \quad \forall x \in \partial\mathcal{O}, \tag{1.8}$$

and

$\exists T > 0$ so that for every $g^\sharp$-geodesic $\gamma(t)$ with $\gamma(0) \in \mathcal{O}^\sharp$ there exists $t \in (0, T]$

$$\text{such that } \gamma(t) \in \partial\mathcal{O}^\sharp. \tag{1.9}$$

Note that (1.9) is trivially fulfilled for arbitrary $\mathcal{O}$ if $g^\sharp$ is the Euclidean metric $\sum_{j=1}^n dx_j^2$. It is worth noticing that the condition of [1] does not imply (1.9) as, to our best knowledge, it is not possible to define the generalized bicharacteristic flow when the boundary is only C^1-smooth. But such an implication is also hard to see (at least for the authors) even for C^∞-smooth boundary.

Our main result is the following

Theorem 1.1. *Under the assumptions* (1.8) *and* (1.9), *we have*

$$\operatorname{spec} A \setminus \{0\} \subset \{\operatorname{Im}\lambda \ge C(1+|\lambda|)^{-1}\}, \quad C > 0, \tag{1.10}$$

and

$$\|(A-\lambda)^{-1}\|_{\mathcal{L}(H)} \le C_1|\lambda| \quad \textit{for} \quad \operatorname{Im}\lambda \le C_2|\lambda|^{-1},\ |\lambda| \ge 1, \tag{1.11}$$

for some constants $C_1, C_2 > 0$.

In the same way as in [3], Theorem 3, (see also [5], Section 3) one can derive from the above theorem the following

Corollary 1.2. *Under the assumptions* (1.8) *and* (1.9), *for every integer* $m \geq 1$,

$$\|e^{itA}f\|_H \leq C_m \left(t^{-1}\log t\right)^m \|f\|_{D(A^m)}, \quad t \geq 2, \forall f \in D(A^m) \cap H'. \tag{1.12}$$

It is easy to see that to prove the above theorem it suffices to prove (1.11) for real λ, $|\lambda| \gg 1$, only. This in turn is done by proving suitable a priori estimates for the solutions of the equation (1.3) (with non zero RHS) with real $\lambda \gg 1$.

Acknowledgements. A part of this work was carried out while the second author was visiting Universidade Federal de Pernambuco, Recife, Brazil, in February 2002, and he would like to thank this institution for the hospitality. The authors were also partially supported by the agreement Brazil-France in Mathematics – Proc. 69.0014/01-5.

2. Uniform a priori estimates. Let $u \in H^2(\mathcal{O})$ satisfy the equation

$$\begin{cases} (\Delta_g + \lambda^2)u = v \quad \text{in} \quad \mathcal{O}, \\ u|_{\partial\mathcal{O}} = f, \, \partial_\nu u|_{\partial\mathcal{O}} = \lambda h, \end{cases} \tag{2.1}$$

where $\lambda \gg 1$ is real. In what follows, $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\|\cdot\|_\sharp$, $\langle\cdot,\cdot\rangle_\sharp$, $\|\cdot\|_0$, $\langle\cdot,\cdot\rangle_0$ will denote the norms and the scalar products in $L^2(\mathcal{O})$, $L^2(\mathcal{O}^\sharp)$, $L^2(\partial\mathcal{O})$, respectively. Here $L^2(\partial\mathcal{O}) := L^2(\partial\mathcal{O}, d\mathrm{Vol}_{\partial g})$, where ∂g denotes the Riemannian metric on $\partial\mathcal{O}$ induced by the metric g. We equipe the Sobolev space $H^s(\mathcal{O})$, $s \geq 0$, (and similarly $H^s(\mathcal{O}^\sharp)$) with the semi-classical norm

$$\|w\|_{H^s(\mathcal{O})} := \|(1 - \lambda^{-2}\Delta_g)^{s/2}w\|.$$

We will derive Theorem 1.1 from the following

Theorem 2.1. *Under the assumption* (1.9), *there exist constants* $C, \lambda_0 > 0$ *so that for* $\lambda \geq \lambda_0$ *we have*

$$\|u\| \leq C\lambda^{-1}\|v\| + C\lambda^{1/2}\|f\|_0 + C\lambda^{1/2}\|h\|_0. \tag{2.2}$$

Let $u \in H^2(\mathcal{O})$ satisfy the equation (with real $\lambda \gg 1$)

$$\begin{cases} (\Delta_g + \lambda^2)u = v \quad \text{in} \quad \mathcal{O}, \\ -\partial_\nu u + i\lambda a(x)u = 0 \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{2.3}$$

By Green's formula we have (with $f = u|_{\partial\mathcal{O}}$)

$$-\mathrm{Im}\,\langle \Delta_g u, u\rangle = \mathrm{Im}\,\langle \partial_\nu u|_{\partial\mathcal{O}}, f\rangle_0,$$

and hence, in view of (1.8),

$$-\mathrm{Im}\,\langle v, u\rangle = \lambda\langle af, f\rangle_0 \geq a_0\lambda\|f\|_0^2. \tag{2.4}$$

By (2.2) and (2.4),

$$\|u\|^2 \leq C_1\lambda^{-2}\|v\|^2 + C_1\lambda\|f\|_0^2$$

$$\leq C_1\lambda^{-2}\|v\|^2 + C_2\,|\langle v, u\rangle| \leq C_3\|v\|^2 + \frac{1}{2}\|u\|^2.$$

Hence,

$$\|u\| \leq C_4\|v\|, \quad C_4 > 0,$$

which yields

$$\|u\|_{H^1(\mathcal{O})} \leq C\|v\|, \quad C > 0. \tag{2.5}$$

It is easy to see that (2.5) implies (1.11) for real $\lambda \gg 1$, and hence the theorem itself.

P r o o f o f T h e o r e m 2.1. Recall first that bicharacteristic flow $\Phi(t) : T^*\mathcal{O}^\sharp \to T^*\mathcal{O}^\sharp$, $t \in \mathbf{R}$, associated to the metric $g^\sharp$ is defined by $(x(t), \xi(t)) = \Phi(t)(x^0, \xi^0)$, where

$$\begin{cases} \dot{x}(t) = \dfrac{\partial r^\sharp(x,\xi)}{\partial \xi}, \\ \dot{\xi}(t) = -\dfrac{\partial r^\sharp(x,\xi)}{\partial x}, \\ x(0) = x^0, \xi(0) = \xi^0, \end{cases} \tag{2.6}$$

$r^\sharp(x,\xi)$ being the principal symbol of $-\Delta_{g^\sharp}$. Fix $(x^0,\xi^0) \in T^*\mathcal{O}^\sharp$, $r^\sharp(x^0,\xi^0) = 1$, and choose a function $p(x,\xi) \in C_0^\infty(T^*\mathcal{O}^\sharp)$, $0 \leq p \leq 1$, $p = 1$ in a small neighbourhood of (x^0,ξ^0) and $p = 0$ outside another neighbourhood of (x^0,ξ^0) so that $\mathrm{supp}_x p \cap \partial\mathcal{O}^\sharp = \emptyset$. Let $t > 0$ be such that $x(\tau) \notin \partial\mathcal{O}^\sharp$, $\forall \tau \in [0,t]$, $x(0) \in \mathrm{supp}_x p$. For $\tau \in [0,t]$ denote $p_\tau(x,\xi) = p(\Phi(-\tau)(x,\xi)) \in C_0^\infty(T^*\mathcal{O}^\sharp)$. It is easy to see from (2.6) that we have

$$\partial_\tau p_\tau + \{r^\sharp, p_\tau\} = 0, \quad 0 \leq \tau \leq t, \tag{2.7}$$

where $\{\cdot,\cdot\}$ denotes the Poisson brackets. Denote by $p_\tau(x,\mathcal{D}_x)$, $\mathcal{D}_x := (i\lambda)^{-1}\partial_x$, the $\lambda - \Psi DO$ with symbol $p_\tau(x,\xi)$, i.e.

$$p_\tau(x,\mathcal{D}_x)u := \left(\frac{\lambda}{2\pi}\right)^n \int\!\!\int e^{i\lambda\langle x-y,\xi\rangle} p_\tau(x,\xi)u(y)d\xi dy.$$

It follows easily from (2.7) that, for $0 \le \tau \le t$, we have

$$Q := \lambda \partial_\tau p_\tau(x, \mathcal{D}_x) + i[\Delta_{g^\sharp}, p_\tau(x, \mathcal{D}_x)] = O(1) : L^2(\mathcal{O}^\sharp) \to L^2(\mathcal{O}^\sharp). \tag{2.8}$$

Given a function w defined in $\mathcal{O}$, $\widetilde{w}$ will denote its extension by zero outside $\mathcal{O}$. We have in sense of distributions

$$\Delta_{g^\sharp} \widetilde{w} = \widetilde{\Delta_g w} + (\partial_\nu w|_{\partial\mathcal{O}})\, \delta + (w|_{\partial\mathcal{O}})\, \delta', \tag{2.9}$$

where δ and δ' denote the delta density on $\partial\mathcal{O}$ and its first derivative defined by

$$\delta(\varphi) = \int_{\partial\mathcal{O}} \varphi d\mathrm{Vol}_{\partial g}, \quad \delta'(\varphi) = -\int_{\partial\mathcal{O}} \partial_\nu \varphi d\mathrm{Vol}_{\partial g}, \quad \varphi \in C_0^\infty(\mathbf{R}^n).$$

Suppose that $\mathrm{supp}_x p_\tau \cap \partial\mathcal{O} \neq \emptyset$. In view of (2.8) and (2.9), we have

$$\frac{1}{2}\frac{d}{d\tau} \|p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_\sharp^2 = \mathrm{Re}\, \langle \partial_\tau p_\tau(x, \mathcal{D}_x)\widetilde{u}, p_\tau(x, \mathcal{D}_x)\widetilde{u}\rangle_\sharp$$

$$= \lambda^{-1}\mathrm{Im}\, \langle [\Delta_{g^\sharp}, p_\tau(x, \mathcal{D}_x)]\widetilde{u}, p_\tau(x, \mathcal{D}_x)\widetilde{u}\rangle_\sharp + \lambda^{-1}\mathrm{Re}\, \langle Q\widetilde{u}, p_\tau(x, \mathcal{D}_x)\widetilde{u}\rangle_\sharp$$

$$= -\lambda^{-1}\mathrm{Im}\, \langle (\Delta_{g^\sharp} + \lambda^2)\widetilde{u}, p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u}\rangle_\sharp + \lambda^{-1}\mathrm{Re}\, \langle Q\widetilde{u}, p_\tau(x, \mathcal{D}_x)\widetilde{u}\rangle_\sharp$$

$$= -\lambda^{-1}\mathrm{Im}\, \langle \widetilde{v}, p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u}\rangle_\sharp + \mathrm{Im}\, \langle h, (p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\rangle_0$$

$$-\lambda^{-1}\mathrm{Im}\, \langle f, (\partial_\nu p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\rangle_0 + \lambda^{-1}\mathrm{Re}\, \langle Q\widetilde{u}, p_\tau(x, \mathcal{D}_x)\widetilde{u}\rangle_\sharp.$$

Hence

$$\left|\frac{d}{d\tau}\|p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_\sharp^2\right|$$

$$\le O(\lambda^{-1})\|v\|\|p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_\sharp + O(1)\|h\|_0\|(p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\|_0$$

$$+O(1)\|f\|_0\|(\mathcal{D}_\nu p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\|_0 + O(\lambda^{-1})\|u\|\|p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_\sharp.$$

On the other hand, by the trace theorem we have

$$\|(p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\|_0$$

$$\le O(\lambda^{1/2})\|p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_{H^{1/2}(\mathcal{O}^\sharp)} \le O(\lambda^{1/2})\|p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_\sharp,$$

$$\|(\mathcal{D}_\nu p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\|_0$$

$$\le O(\lambda^{1/2})\|p_\tau(x, \mathcal{D}_x)^* p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_{H^{3/2}(\mathcal{O}^\sharp)} \le O(\lambda^{1/2})\|p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_\sharp.$$

Hence

$$\left|\frac{d}{d\tau}\, \|p_\tau(x, \mathcal{D}_x)\widetilde{u}\|_\sharp\right| \le O(\lambda^{-1})\|v\| + O(\lambda^{1/2})(\|f\|_0 + \|h\|_0) + O(\lambda^{-1})\|u\|. \tag{2.10}$$

Clearly, if $\mathrm{supp}_x p_\tau \cap \partial\mathcal{O} = \emptyset$, (2.10) holds with $f = h = 0$. Thus we get

$$\|p(x,\mathcal{D}_x)\widetilde{u}\|_\sharp = \|p_t(x,\mathcal{D}_x)\widetilde{u}\|_\sharp - \int_0^t \frac{d}{d\tau}\,\|p_\tau(x,\mathcal{D}_x)\widetilde{u}\|_\sharp\,d\tau$$

$$\le \|p_t(x,\mathcal{D}_x)\widetilde{u}\|_\sharp + O(\lambda^{-1})\|v\| + O(\lambda^{1/2})(\|f\|_0 + \|h\|_0) + O(\lambda^{-1})\|u\|. \tag{2.11}$$

Clearly, there exist a domain $\mathcal{O}' \subset \mathcal{O}^\sharp$ and a constant $0 < \delta_0 \ll 1$ such that $\mathcal{O} \subset \mathcal{O}'$, $\partial\mathcal{O} \cap \partial\mathcal{O}' = \emptyset$, and $\mathrm{dist}(\mathcal{O}', \partial\mathcal{O}^\sharp) \ge \delta_0$. Fix now a $\zeta^0 = (x^0,\xi^0) \in T^*\mathcal{O}'$, $r^\sharp(x^0,\xi^0) = 1$. By (1.9), there exist a neighbourhood $U(\zeta^0) \subset T^*\mathcal{O}'$ of ζ^0 and $0 < t = t(\zeta^0) \le T$ so that

$$\pi_x\Phi(t)U(\zeta^0) \subset \{x \in \mathcal{O}^\sharp : \delta_0/4 \le \mathrm{dist}(x,\partial\mathcal{O}^\sharp) \le \delta_0/2\},$$

where $\pi_x(x,\xi) := x$. Choose a function $p(x,\xi) \in C_0^\infty(U(\zeta^0))$, $p = 1$ in a smaller neighbourhood of ζ^0. Let $p_t(x,\xi)$ be as above and choose a function $\eta(x) \in C_0^\infty(\mathcal{O}^\sharp)$ such that $\mathrm{supp}\,\eta \subset \{x \in \mathcal{O}^\sharp : \mathrm{dist}(x,\partial\mathcal{O}^\sharp) \le 2\delta_0/3\}$, $\eta = 1$ on $\mathrm{supp}_x p_t$. We have $\eta\widetilde{u} = 0$ and hence

$$p_t(x,\mathcal{D}_x)\widetilde{u} = \eta(x)p_t(x,\mathcal{D}_x)\widetilde{u} = [\eta(x), p_t(x,\mathcal{D}_x)]\widetilde{u},$$

so we obatin

$$\|p_t(x,\mathcal{D}_x)\widetilde{u}\|_\sharp \le O(\lambda^{-1})\|\widetilde{u}\|_\sharp. \tag{2.12}$$

By (2.11) and (2.12), we conclude

$$\|p(x,\mathcal{D}_x)\widetilde{u}\|_\sharp \le O(\lambda^{-1})\|v\| + O(\lambda^{1/2})(\|f\|_0 + \|h\|_0) + O(\lambda^{-1})\|u\|. \tag{2.13}$$

Fix now a $\zeta^0 = (x^0,\xi^0) \in T^*\mathcal{O}'$ such that $r^\sharp(x^0,\xi^0) \neq 1$. Suppose that $r^\sharp(x^0,\xi^0) > 1$ (the case $r^\sharp(x^0,\xi^0) < 1$ is treated similarly). Then there exists a (conic for $|\xi| \gg 1$) neighbourhood $W(\zeta^0) \subset T^*\mathcal{O}$ of ζ^0 such that $r^\sharp(x,\xi) > 1$ in $W(\zeta^0)$. Choose functions $q(x,\xi), q_1(x,\xi) \in C^\infty(W(\zeta^0))$, $q = 1$ in a smaller neighbourhood of ζ^0, $q_1 = 1$ on $\mathrm{supp}\,q$. Thus we have that the operator $-\lambda^{-2}\Delta_{g^\sharp} - 1$ considered as a semi-classical differential operator is elliptic on $\mathrm{supp}\,q_1$ with a strictly positive principal symbol. Therefore, by Gärding's inequality we have

$$\begin{aligned} &\mathrm{Re}\,\langle q_1(x,\mathcal{D}_x)(-\lambda^{-2}\Delta_{g^\sharp} - 1)\widetilde{u}, q(x,\mathcal{D}_x)\widetilde{u}\rangle_\sharp \\ &\ge C\|q(x,\mathcal{D}_x)\widetilde{u}\|_\sharp^2 - O(\lambda^{-2})\|\widetilde{u}\|_\sharp^2, \quad C > 0. \end{aligned} \tag{2.14}$$

On the other hand, as above we have

$$\mathrm{Re}\,\langle(-\lambda^{-2}\Delta_{g^\sharp} - 1)\widetilde{u}, q_1(x,\mathcal{D}_x)^*q(x,\mathcal{D}_x)\widetilde{u}\rangle_\sharp = -\lambda^{-2}\mathrm{Re}\,\langle\widetilde{v}, q_1(x,\mathcal{D}_x)^*q(x,\mathcal{D}_x)\widetilde{u}\rangle_\sharp$$

$$+\lambda^{-1}\mathrm{Re}\,\langle h, (q_1(x,\mathcal{D}_x)^*q(x,\mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\rangle_0 - \lambda^{-2}\mathrm{Re}\,\langle f, (\partial_\nu q_1(x,\mathcal{D}_x)^*q(x,\mathcal{D}_x)\widetilde{u})|_{\partial\mathcal{O}}\rangle_0$$

$$\le \Big(O(\lambda^{-2})\|v\| + O(\lambda^{-1/2})(\|f\|_0 + \|h\|_0)\Big)\,\|q(x,\mathcal{D}_x)\widetilde{u}\|_\sharp. \tag{2.15}$$

Combining (2.14) and (2.15) leads to the estimate

$$\|q(x, \mathcal{D}_x)\widetilde{u}\|_\sharp \leq O(\lambda^{-2})\|v\| + O(\lambda^{-1/2})(\|f\|_0 + \|h\|_0) + O(\lambda^{-1})\|u\|. \tag{2.16}$$

Now (2.2) follows from (2.13) and (2.16) by a microlocal partition of the unity on $T^*\mathcal{O}'$. □

REFERENCES

[1] C. Bardos, G. Lebeau, J. Rauch. Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30** (1992), 1024–1065.

[2] N. Burq. Controlabilité exacte des ondes dans des ouverts peu réguliers. *Asymptot. Anal.* **14** (1997), 157–191.

[3] N. Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.* **180** (1998), 1–29.

[4] G. Lebeau, L. Robbiano. Stabilization de l'équation des ondes par le bord. *Duke Math. J.* **86** (1997), 465–490.

[5] G. Popov, G. Vodev. Distribution of the resonances and local energy decay in the transmission problem. *Asymptot. Anal.* **19** (1999), 253–265.

Fernando Cardoso
Universidade Federal de Pernambuco
Departamento de Matemàtica
CEP. 50540-740 Recife-Pe, Brazil
e-mail: fernando@dmat.ufpe.br

Georgi Vodev
Université de Nantes
Département de Mathématiques
UMR 6629 du CNRS
2, rue de la Houssinière, BP 92208
44072 Nantes Cedex 03, France
e-mail: vodev@math.univ-nantes.fr

Received May 3, 2002
Revised June 26, 2002