Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **28** (2002), 361-378

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

NEARLY COCONVEX APPROXIMATION

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Communicated by K. G. Ivanov

Dedicated to the memory of our colleague Vasil Popov
January 14, 1942 – May 31, 1990

Abstract. Let $f \in \mathbb{C}[-1,1]$ change its convexity finitely many times, in the interval. We are interested in estimating the degree of approximation of f by polynomials, and by piecewise polynomials, which are nearly coconvex with it, namely, polynomials and piecewise polynomials that preserve the convexity of f except perhaps in some small neighborhoods of the points where f changes its convexity. We obtain Jackson type estimates and summarize the positive and negative results in a truth-table as we have previously done for nearly comonotone approximation.

1. Introduction.

Let $f \in \mathbb{C}[-1,1]$ change its convexity finitely many times, say $s \geq 0$ times, in the interval. We are interested in estimating the degree of approximation of f by polynomials which are nearly coconvex with it,

2000 *Mathematics Subject Classification:* 41A10, 41A17, 41A25, 41A29.

Key words: Nearly coconvex approximation, polynomial approximation, piecewise polynomial approximation, Jackson estimates.

*Part of this work was done while the second author was on a visit at Tel Aviv University in March 2001 namely, polynomials that preserve the convexity of f except perhaps in some small neighborhoods of the points where f changes its convexity.

Recently [8], we have investigated the degree of coconvex approximation by polynomials, that is, approximation by polynomials that change convexity exactly where f does. We have been able to obtain Jackson-type estimates involving the various moduli of smoothness of f and its derivatives (whenever they exist). However, the estimates are usually valid only for polynomials of sufficiently high degree $n \ge N$, where N depends on the location of the changes in convexity, or even on f itself. Also, if there is at least one change of convexity, then the estimates involving the moduli of smoothness of f, are valid only up to $\omega_3(f, \cdot)$, those involving the moduli of smoothness of f', are valid only up to $\omega_2(f', \cdot)$, and those involving the moduli of smoothness of f'', are valid only up to $\omega_3(f'', \cdot)$.

Our aim here is to relax somewhat the constraints in the expectation to obtain better estimates on the approximation. We will show that this is indeed the case, however the improvement is limited. A similar phenomenon is known from nearly comonotone approximation of a function which changes its monotonicity finally many times in the interval. (See our recent survey [5] where we have collected all known positive and negative results on nearly comonotone approximation on a finite interval, by algebraic polynomials in the uniform norm (see also [3, 4]).) We intend here to obtain the analogous results for nearly coconvex approximation.

Let $I := [-1, 1]$ and denote by $\mathbb{C} = \mathbb{C}^0$ and $\mathbb{C}^r$, respectively the space of continuous functions, and that of r-times continuously differentiable functions on I, equipped with the uniform norm

$$\|f\| := \max_{x \in I} |f(x)|.$$

Denote by $\mathbb{Y}_\sigma$, $\sigma \in \mathbb{N}$, the set of all collections $Y_\sigma := \{y_i\}_{i=1}^{\sigma}$, such that $-1 < y_\sigma < \ldots < y_1 < 1$, and for $\sigma = 0$, we write $\mathbb{Y}_0 := \{\emptyset\}$. For later reference set $y_0 := 1$ and $y_{\sigma+1} := -1$.

For $s \in \mathbb{N}$, we denote by $\Delta^2(Y_s)$, the collection of all functions $f \in \mathbb{C}$ that change convexity at the set Y_s, and are convex in $[y_1, 1]$, that is, f is convex in $[y_{2i+1}, y_{2i}]$, $0 \le i \le [s/2]$, and it is concave in $[y_{2i}, y_{2i-1}]$, $1 \le i \le [(s+1)/2]$. In particular $\Delta^2 := \Delta^2(Y_0)$ is the set of convex functions on I. Also, let

$$\rho_n(x) := \frac{1}{n^2} + \frac{1}{n}\varphi(x) := \frac{1}{n^2} + \frac{1}{n}\sqrt{1 - x^2},$$

and for $r \ge 0$, $k \ge 0$, and a constant $c > 0$, denote

$$O(r, k, c, Y_s) := \begin{cases} \bigcup_{i=0}^{s+1} (y_i - c\rho_n(y_i), y_i + c\rho_n(y_i)) & (r, k) = (0, 4) \text{ or } (1, 3) \\ \bigcup_{i=1}^{s} (y_i - c\rho_n(y_i), y_i + c\rho_n(y_i)) & \text{otherwise.} \end{cases}$$

We wish to approximate a general function $f \in \Delta^2(Y_s)$, by means of polynomials which are nearly coconvex with f, that is, we require that the polynomials be coconvex with f except perhaps in some neighborhood of Y_s. We will show that for the appropriate (r,k), we can find polynomials which are coconvex with f in $[-1,1] \setminus O(r,k,c,Y_s)$, for some $c > 0$. Thus, in particular, these polynomials belong to $\Delta^2(Y_\sigma)$ for some $\sigma \geq 0$, so that $Y_\sigma \subseteq O(r,k,c,Y_s)$. (We know by experience from other shape preserving approximation, that it is in these neighborhoods of the points of change of convexity, where it is the hardest to fulfill the requirements.)

There are two main ingredients in the proofs of positive results. First one establishes the existence of piecewise polynomials which are both nearly coconvex with f and sufficiently close to it, and second, one should show that such piecewise polynomials may be well approximated by polynomials which are coconvex with them, that is, polynomials that change their convexity exactly where the piecewise polynomial does. The latter was the main contents of our recent paper [7]. Thus we concentrate here on establishing the former and on drawing the final conclusions from having obtained the two needed ingredients.

We will first construct continuous piecewise polynomials on the Chebyshev partition, that are nearly coconvex with $f \in \Delta^2(Y_s)$, and approximate it well. Namely, given $n \in \mathbb{N}$, $n > 1$, we set $x_j := x_{j,n} := \cos(j\pi/n)$, $j = 0, \dots, n$, the Chebyshev partition of $[-1,1]$, and we denote $I_j := I_{j,n} := [x_j, x_{j-1}]$, $j = 1, \dots, n$. Let $\Sigma_{k,n}$ be the collection of all continuous piecewise polynomials of degree $k-1$, on the Chebyshev partition, that is, if $S \in \Sigma_{k,n}$, then

$$S|_{I_j} = p_j, \quad j = 1, \dots, n,$$

where $p_j \in \Pi_{k-1}$, the space of polynomials of degree $\leq k-1$, and

$$p_j(x_j) = p_{j+1}(x_j), \quad j = 1, \dots, n-1.$$

As alluded to above, we will construct an $S \in \Delta^2(Y_\sigma)$, so that $Y_\sigma \subseteq O(r,k,c,Y_s)$. In order to state the following result, recently proved by the authors [7], we need more notation.

Given $Y_\sigma \in \bigcup_{\mu=0}^{\infty} \mathbb{Y}_\mu$, let

$$O_i := O_{i,n}(Y_\sigma) := (x_{j+1}, x_{j-2}), \quad \text{if} \quad y_i \in [x_j, x_{j-1}), \quad 1 \leq i \leq \sigma,$$

where $x_{n+1} := -1$, $x_{-1} := 1$, and denote

$$O := O(n, Y_\sigma) := \bigcup_{i=1}^{\sigma} O_i, \quad O(n, \emptyset) := \emptyset.$$

Finally, we write $j \in H = H(n, Y_\sigma)$, if $I_j \cap O = \emptyset$. We denote by $\Sigma_{k,n}(Y_\sigma) \subseteq \Sigma_{k,n}$, the subset of those piecewise polynomials for which

$$p_j \equiv p_{j+1}, \quad \text{whenever both} \quad j, (j+1) \notin H.$$

Theorem LS. *For every $k \in \mathbb{N}$ and $\sigma \in \mathbb{N}_0$ there are constants $c = c(k,\sigma)$ and $c_* = c_*(k,\sigma)$, such that if $n \in \mathbb{N}$ and $Y_\sigma \in \mathbb{Y}_\sigma$, and $S \in \Sigma_{k,n}(Y_\sigma) \cap \Delta^2(Y_\sigma)$, then there is a polynomial $P_n \in \Delta^2(Y_\sigma)$ of degree $\leq c_* n$, satisfying*

$$\|S - P_n\| \leq c\omega_k^\varphi(S, 1/n). \tag{1.1}$$

(For the definition of $\omega_k^\varphi(f,t)$, see Section 2.)

Thus, if we are able to construct a good piecewise polynomial approximation, of the above type, to $f \in \Delta^2(Y_s)$, (with an appropriate Y_σ), then we will have a good polynomial approximation to f.

We state the main results in Section 2, and after some auxiliary lemmas in Section 3, we have the proof in Section 4.

In the sequel we will have absolute positive constants C, and we will have positive constants c that depend only on s, k and r that are going to be indicated. We will use the notation C and c for such constants which are of no significance to us and may differ on different occurrences, even in the same line. However, sometimes we need to keep track of the constants and then they will have indices like $C_0, C_1, \ldots$ and $c_0, c_1, \ldots$.

2. Nearly coconvex approximation.

In addition to the spaces of continuously differentiable functions we need two additional spaces. We will use the norm

$$\|f\| := \operatorname*{esssup}_{x \in I} |f(x)|,$$

also for a function that is essentially bounded on I, and with this notation, let the space W^r, be the set of functions $f \in \mathbb{C}$ which possess an absolutely continuous $(r-1)$st derivative in I, such that $\|f^{(r)}\| < \infty$. Also let the space B^r, be the set of functions $f \in \mathbb{C}$ which possess a locally absolutely continuous $(r-1)$st derivative in $(-1,1)$, such that $\|\varphi^r f^{(r)}\| < \infty$, where $\varphi(x) := \sqrt{1-x^2}$.
We sometimes wish to restrict ourselves to a subinterval $[a,b] \subseteq I$ in which case we will use the notation $\|\cdot\|_{[a,b]}$ for the above norms on the interval $[a,b]$. Given $f \in \mathbb{C}$, and $k \in \mathbb{N}$, let

$$\Delta_h^k f(x) := \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f\left(x - \frac{k}{2}h + ih\right),$$

be the symmetric difference of order k, defined for all x and $h \ge 0$, such that $x \pm \frac{k}{2}h \in [a,b]$. The ordinary moduli of smoothness of f in $[a,b]$, $\omega_k(f,t;[a,b])$, are defined by

$$\omega_k(f,t;[a,b]) := \sup_{0\le h\le t}\sup_x |\Delta_h^k f(x)|, \quad t \ge 0,$$

where the inner supremum is taken over all x such that $x \pm \frac{k}{2}h \in [a,b]$. In particular when $[a,b] = I$, we write $\omega_k(f,t) := \omega_k(f,t;I)$. We also need the Ditzian-Totik (DT-)moduli of smoothness [1] defined by

$$\omega_k^\varphi(f,t) := \sup_{0\le h\le t}\sup_x |\Delta_{h\varphi(x)}^k f(x)|, \quad t \ge 0,$$

where the inner supremum is taken over all x such that $x \pm \frac{k}{2}h\varphi(x) \in [-1,1]$. It is well known that

$$\omega_k^\varphi(f,t) \le c(k)\omega_k(f,t), \quad t > 0,$$

and that

$$\omega_k^\varphi(f,t) \le c(k)\omega_{k-1}^\varphi(f,t), \quad t > 0, \quad k > 1. \tag{2.1}$$

If $f \in \mathbb{C}^r$, then

$$\omega_k(f,t) \le c(k,r)t^r\omega_{k-r}(f^{(r)},t), \quad t > 0, \quad k > r, \tag{2.2}$$

and

$$\omega_k^\varphi(f,t) \le c(k,r)t^r\omega_{k-r}^\varphi(f^{(r)},t), \quad t > 0, \quad k > r, \tag{2.3}$$

Also if $f \in W^r$, then

$$\omega_r(f,t) \le c(r)t^r\|f^{(r)}\|, \quad t > 0,$$

and if $f \in B^r$, then

$$\omega_r^\varphi(f,t) \le c(r)t^r\|\varphi^r f^{(r)}\|, \quad t > 0. \tag{2.4}$$

The main result of this paper is the approximation of $f \in \Delta^2(Y_s)$, $s \ge 0$, by polynomials which are nearly coconvex with it. We emphasize that these results are new even for convex functions, namely, also for $s = 0$. We prove

Theorem 2.1. *Let $r = 0$ and $k \le 4$, or $r = 1$ and $k \le 3$, or $r \ge 2$ and any $k \ge 1$. For every $f \in \Delta^2(Y_s) \cap \mathbb{C}^r$ and each $n \ge k + r - 1$, there exists a polynomial $p_n \in \Pi_n$ such that*

$$p_n''(x)\prod_{i=1}^{s}(x - y_i) \ge 0, \quad x \in I \setminus O(r,k,c^*,Y_s), \tag{2.5}$$

and

$$\|f - p_n\| \le \frac{c}{n^r}\omega_k^{\varphi}(f^{(r)}, 1/n) \le \frac{c}{n^r}\omega_k(f^{(r)}, 1/n), \tag{2.6}$$

where $c^* = c^*(k, r, s)$ *and* $c = c(k, r, s)$. *Furthermore, if* $f \in \Delta^2(Y_s) \cap W^r$, $r \ge 1$, *then there exists a* $p_n \in \Pi_n$ *satisfying*

$$p_n''(x)\prod_{i=1}^{s}(x - y_i) \ge 0, \quad x \in I \setminus O(r, 0, c^*, Y_s), \tag{2.7}$$

with $c^* = c^*(r, s)$, *such that*

$$\|f - p_n\| \le \frac{c}{n^r}\|f^{(r)}\|, \tag{2.8}$$

where $c = c(r, s)$.

Recently, Nissim and Yushchenko [9] proved that (2.6) is invalid for $r = 1$ and $k > 3$, and (which by (2.2) is implied by it), for $r = 0$ and $k > 4$, even if we allow the constant c to depend on f, and we assume that $p_n''(x)\prod_{i=1}^{s}(x - y_i) \ge 0$ except on some arbitrary sets of measures $\to 0$, as $n \to \infty$. Thus, in particular there exists no c^* even dependent on f such that we may exclude $O(r, k, c^*, Y_s)$ and still have (2.6).

Thus, we can summarize the validity of nearly coconvex polynomial approximation in the following truth table.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
2	+	+	+	+	+	+	$\cdots$
1	+	+	+	+	−	−	$\cdots$
0		+	+	+	+	−	$\cdots$
	0	1	2	3	4	5	k

Fig. 1

where the symbol "+" in columns $k \ge 1$ stands for the validity of (2.6) and in column $k = 0$ it stands for the validity of (2.8) with constants above, and the symbol "−" indicates that the above inequalities cannot be had even with constants that depend on f.

3. Auxiliary lemmas. We begin with two lemmas that we need for the case where f is merely continuous.

Lemma 3.1. *Let $h > 0$ be arbitrary, and let P, P_1, and P_2, be three cubic polynomials such that*

$$P(0) = P_2(0), \quad P'(0) = P_2'(0), \quad P(h) = P_1(h), \quad \text{and} \quad P'(h) = P_1'(h). \tag{3.1}$$

Then

$$\|P - P_1\|_{[0,h]} \le 4\|P_1 - P_2\|_{[0,h]}. \tag{3.2}$$

P r o o f. Straightforward computations show that (3.1) implies that

$$\begin{aligned} P(x) - P_1(x) &= (P_2(0) - P_1(0))\left(2\frac{x^3}{h^3} - 3\frac{x^2}{h^2} + 1\right) \\ &\quad + h(P_2'(0) - P_1'(0))\left(\frac{x^3}{h^3} - 2\frac{x^2}{h^2} + \frac{x}{h}\right). \end{aligned} \tag{3.3}$$

By Markov's inequality

$$|P_1'(0) - P_2'(0)| \le \frac{2}{h}3^2\|P_1 - P_2\|_{[0,h]},$$

and (3.2) readily follows by (3.3). □

Next we have

Lemma 3.2. *Let the interval $[a, b]$ be partitioned into*

$$T: \ a =: a_{13} < a_{12} < \cdots < a_1 < a_0 := b.$$

Denote $J_i := [a_i, a_{i-1}]$, $1 \le i \le 13$, and $J := [a_{12}, a_1]$, and set

$$\Lambda := \frac{b - a}{\min\limits_{1 \le i \le 13} |J_i|}. \tag{3.4}$$

Let S be a continuous piecewise cubic on the above partition, set $S_{|J_i} =: P_i$ and assume that

$$\|P_i - P_j\|_{[a,b]} \le 1, \quad 1 \le i, j \le 13. \tag{3.5}$$

Suppose that S does not change convexity in $[a_3, a_0]$, and in $[a_{13}, a_{10}]$. Then there is a collection $Y^* \subset [a_{11}, a_3]$, of at most three points (including the possibility of none), and a continuous piecewise cubic polynomial $S^* \in \Sigma_{4,13}([a, b], T, Y^*)$ (where the latter is the set of piecewise cubic polynomials defined as in the introduction, but for the partition T of the interval $[a, b]$), satisfying

$$S^*(x) = P_i(x), \quad x \in J_i, \quad i = 1, 13,$$

such that

$$\|S^* - S\|_{[a,b]} \le C\Lambda^4. \tag{3.6}$$

Proof. Without loss of generality assume that S is convex in $[a_3, a_0]$. Let P be the cubic polynomial satisfying

$$P(a_1) = P_1(a_1),\ P'(a_1) = P_1'(a_1),\ P(a_{12}) = P_{13}(a_{12}),\ \text{and } P'(a_{12}) = P_{13}'(a_{12}).$$

Clearly, P'' is a linear function. Suppose first that S is convex in $[a_{13}, a_{10}]$. If $P'' \ge 0$ in $[a_{12}, a_1]$, then we put $Y^* = \emptyset$, and define

$$S^*(x) := \begin{cases} P_{13}(x), & a_{13} \le x \le a_{12} \\ P(x), & a_{12} < x < a_1 \\ P_1(x), & a_1 \le x \le a_0. \end{cases} \tag{3.7}$$

Then (3.6) follows from (3.2) and (3.5).

Otherwise, either $P''(x)P_1''(x) \le 0$ for some $x \in [a_2, a_1]$, or $P''(x)P_{12}''(x) \le 0$ for some $x \in [a_{12}, a_{11}]$. In any case we conclude that either $P''(x)P_1''(x) \le 0$ in $[a_1, a_0]$, or $P''(x)P_{13}''(x) \le 0$ in $[a_{13}, a_{12}]$. Set $i_0 = 1$ in the former case, and $i_0 = 13$ in the latter. Then by Markov's inequality,

$$\begin{aligned} \|P''\|_{J_{i_0}} &\le \|P'' - P_{i_0}''\|_{J_{i_0}} \\ &\le 2\Lambda\|P'' - P_{i_0}''\|_J \\ &\le \frac{C\Lambda}{|J|^2}\|P - P_{i_0}\|_J \\ &\le \frac{C\Lambda}{|J|^2}, \end{aligned} \tag{3.8}$$

where for the last inequality we have applied Lemma 3.1 and (3.5). Finally, (3.8) implies

$$\|P''\|_J \le \frac{C\Lambda^2}{|J|^2}. \tag{3.9}$$

Let

$$s_2(x) = \begin{cases} \alpha, & x \in [a_9, a_8], \\ \beta, & x \in [a_5, a_4], \\ 0, & \text{elsewhere in } [a_{12}, a_1], \end{cases} \tag{3.10}$$

where α and β are selected so that the piecewise polynomial

$$S_2(x) := P(a_{12}) + P'(a_{12})(x - a_{12}) + \int_{a_{12}}^{x} (x-u)s_2(u)du, \tag{3.11}$$

satisfies $S_2(a_1) = P(a_1)$ and $S_2'(a_1) = P'(a_1)$. It follows by (3.9) that

$$\left|\int_{a_{12}}^{a_1} (a_1 - u)s_2(u)\,du\right| = \left|\int_{a_{12}}^{a_1} (a_1 - u)P''(u)\,du\right| \le C\Lambda^2, \tag{3.12}$$

and

$$\left|\int_{a_{12}}^{a_1} s_2(u)\,du\right| = \left|\int_{a_{12}}^{a_1} P''(u)\,du\right| \le \frac{C\Lambda^2}{|J|}. \tag{3.13}$$

Now, (3.12) and (3.13) imply that

$$|\alpha|, |\beta| \le \frac{C\Lambda^4}{|J|^2},$$

which in turn yields

$$\|S_2 - P\|_J \le C\Lambda^4. \tag{3.14}$$

By virtue of Lemma 3.1 and (3.5)

$$\|P - S\|_J \le C,$$

which combined with (3.14) yields

$$\|S_2 - S\|_J \le C\Lambda^4. \tag{3.15}$$

Hence, letting

$$S^*(x) := \begin{cases} P_{13}(x), & a_{13} \le x \le a_{12} \\ S_2(x), & a_{12} < x < a_1 \\ P_1(x), & a_1 \le x \le a_0, \end{cases} \tag{3.16}$$

we obtain (3.6) from (3.15). Finally, depending on the signs of α and β, we take $Y^* \subseteq \{a_{11}, a_7, a_3\}$ (either two or none), and observe that $S^* \in \Sigma_{4,13}([a,b], T, Y^*)$, namely, that S^* is a single polynomial in the required intervals about the points which are candidates for Y^*, that is, in $[a_{12}, a_9]$, in $[a_8, a_5]$, and in $[a_4, a_1]$.

If, on the other hand, S is concave in $[a_{13}, a_{10}]$, then suppose first that $P''(a_{11}) < 0$ and $P''(a_2) > 0$. We recall that P'' is linear, thus we may take S^* as in (3.7) and $Y^* := \{y^*\}$, y^* being the unique point where $P''(y^*) = 0$. Again, evidently, $S^* \in \Sigma_{4,13}([a,b], T, Y^*)$, and (3.6) follows from (3.2).

Otherwise, either $P''(x)S''(x) \le 0$, for x in $[a_3, a_2]$ (this happens when $P''(a_{11}) < 0$ and $P''(a_2) \le 0$), or $P''(x)S''(x) \le 0$, for x in $[a_{11}, a_{10}]$ (which occurs when $P''(a_{11}) \ge 0$ and $P''(a_2) > 0$), or both are valid in $[a_2, a_1]$ and in $[a_{12}, a_{11}]$, respectively (as is the case when $P''(a_{11}) \ge 0$ and $P''(a_2) \le 0$). We conclude as above, that (3.8) is valid for at least one i_0 among $\{2, 3, 11, 12\}$. Hence we proceed with the definition of s_2 and S_2 as in (3.10) and (3.11), respectively. We define S^* by (3.16) and conclude that (3.6) holds, and that $S^* \in \Sigma_{4,13}([a,b], T, Y^*)$, where $Y^* \subseteq \{a_{11}, a_7, a_3\}$ is taken depending on the signs of α and β, this time we need either all three points or one. This completes the proof. $\square$

If f is twice differentiable we prove

Lemma 3.3. *Let $k \ge 4$ and $0 < h_1, h_2 \le h$, and let $f \in C^2[-5h, 5h]$, be such that f is convex in $[5h - h_1, 5h]$, and f is either convex or concave in $[-5h, -5h+h_2]$. Then there is a polynomial P_{k+1} coconvex with f in $[-5h, -5h+h_2]$ and in $[5h - h_1, 5h]$, such that*

$$\begin{aligned} P_{k+1}(-5h) &= f(-5h), \quad P_{k+1}(5h) = f(5h), \\ P'_{k+1}(-5h) &= f'(-5h), \quad P'_{k+1}(5h) = f'(5h), \end{aligned} \tag{3.17}$$

and

$$\|f - P_{k+1}\|_{[-5h,5h]} \le ch^2 \omega_k(f'', h; [-5h, 5h]), \tag{3.18}$$

where $c = c(k)$.

Proof. Denote by L_{k-1} the Lagrange polynomial, that interpolates f'' at k equidistant points in $[-5h, 5h]$, including the endpoints. Then we have by Whitney's theorem

$$\|f'' - L_{k-1}\|_{[-5h,5h]} \le c\omega_k(f'', h; [-5h, 5h]) =: c^* \omega. \tag{3.19}$$

Hence, the polynomial

$$p_{k+1}(x) := f(-5h) + f'(-5h)(x + 5h) + \int_{-5h}^{x} (x - t) L_{k-1}(t)\, dt,$$

satisfies

$$|f(x) - p_{k+1}(x)| \le 10h \int_{-5h}^{5h} |f''(t) - L_{k-1}(t)|\, dt \le 100c^* h^2 \omega, \tag{3.20}$$

and

$$|f'(x) - p'_{k+1}(x)| \le \int_{-5h}^{5h} |f''(t) - L_{k-1}(t)|\, dt \le 10c^* h\omega. \tag{3.21}$$

Thus, we set

$$\begin{aligned} P^*_{k+1}(x) := p_{k+1}(x) + \Bigg[&\frac{f'(5h) - p'_{k+1}(5h)}{(10h)^2}(x - 5h) \\ &-2\frac{f(5h) - p_{k+1}(5h)}{(10h)^3}(x - 10h) \Bigg] (x + 5h)^2, \end{aligned}$$

and readily see that (3.17) is satisfied with P_{k+1} replaced by P^*_{k+1}. Also (3.18) with P_{k+1} replaced by P^*_{k+1}, follows by virtue of (3.20) and (3.21). Finally let

$$q(x) := \begin{cases} \dfrac{c^*\omega}{90h^2}(x^2-(5h)^2)^2, & \text{if } f \text{ is convex in } [-5h,-5h+h_2] \\ \dfrac{c^*\omega}{80h^3}x(x^2-(5h)^2)^2, & \text{otherwise.} \end{cases}$$

Then at worst q is a quintic polynomial (accounting for the restriction $k \geq 4$), and it is readily seen that

$$\|q\|_{[-5h,5h]} \leq ch^2\omega,$$

and

$$q(-5h) = q(5h) = q'(-5h) = q'(5h) = 0.$$

Moreover, straightforward computations show that

$$|q''(x)| > c^*\omega, \quad \text{if} \quad 4h < x < 5h, \quad \text{and if} \quad -5h < x < -4h,$$

while q'' is always positive in $(4h,5h)$, and it is positive or negative in $(-5h,-4h)$, respectively, if f is convex or concave in $[-5h,-5h+h_2]$. Therefore, if we define

$$P_{k+1}(x) := P^*_{k+1}(x) + q(x),$$

then it readily follows that P_{k+1} satisfies (3.17) and (3.18), and it is coconvex with f in $[-5h,-5h+h_2]$ and in $[5h-h_1,5h]$. □

Finally we need a result from [8, Corollary 2.4], namely,

Lemma LS. *Let $k \geq 1$ and let $f \in \mathbb{C}^2[a,a+h]$, $h > 0$, be convex. Then there exists a convex polynomial $P \in \Pi_{k+1}$ satisfying $P(a) = f(a)$ and $P(a+h) = f(a+h)$, and either $P'(a) = f'(a)$ and $P'(a+h) \leq f'(a+h)$, or $P'(a) \geq f'(a)$ and $P'(a+h) = f'(a+h)$, such that*

$$\|f-P\|_{[a,a+h]} \leq ch^2\omega_k(f'',h;[a,a+h]),$$

where $c = c(k)$.

4. Proof of the main results. The proof of Theorem 2.1 will be divided into two parts. First we will prove it for $r = 0$ and $k = 4$, which in turn implies, by virtue of (2.3), the validity of (2.6) for all cases $r = 0$ and $k \leq 4$, and $r = 1$ and $k \leq 3$, and by (2.4), also the validity of (2.8) for $r = 1$ and $r = 2$, except for the justification for the smaller excluded set $O(2,0,c^*,Y_s)$. We will explain this improvement at the end of the first part of the proof. Then we will prove the theorem for $r=2$ and all $k\ge 4$, which again by virtue of (2.3), implies the validity of (2.6) for all $r\ge 2$ and all $k\ge 1$, and the validity of (2.8) for all $r>2$.

For a finite number of n's, $n\le c$, the proof readily follows by taking the Lagrange polynomial, interpolating f at $k+r$ equidistant points in $[-1,1]$, including the endpoints ± 1. So we fix $n>100(k+r)(s+1)$. For any integer $\tau\ge 0$ denote

$$O_i^{(\tau)} := O_i^{(\tau)}(Y_s) := (x_{j+\tau}, x_{j-1-\tau}), \quad \text{if} \quad y_i\in[x_j, x_{j-1}),$$

(where $x_j:=1$, if $j<0$, $x_j:=-1$, if $j>n$), and let

$$O^{(\tau)} := \bigcup_{i=1}^{s} O_i^{(\tau)}.$$

Denote by

$$G_\nu^{(\tau)} =: (x_{M_{\tau,\nu}}, x_{m_{\tau,\nu}}), \quad \nu=1,\dots,l_\tau\le s,$$

the connected components of $O^{(\tau)}$, enumerated so that $m_{\tau,\nu+1}\ge M_{\tau,\nu}$. Note that $G_\nu^{(\tau)}$ $1<\nu<l_\tau$, contains at least $2\tau+1$ consecutive intervals (x_j,x_{j-1}); and the same holds for $G_1^{(\tau)}$ and $G_{l_\tau}^{(\tau)}$, if $x_1\notin G_1^{(\tau)}$ and if $x_{n-1}\notin G_{l_\tau}^{(\tau)}$, respectively. On the other hand the total number of intervals in $G_\nu^{(\tau)}$, $1\le\nu\le l_\tau$ is less than $(2\tau+1)s$. Hence,

$$|G_\nu^{(\tau)}| \le c \min_{m_{\tau,\nu}+1\le j\le M_{\tau,\nu}} |I_j| < c\rho_n(x), \quad x\in G_\nu^{(\tau)}. \tag{4.1}$$

P r o o f o f T h e o r e m 2.1. (the case $r=0$, $k=4$) We are given $Y_s\in\mathbb{Y}_s$ and $f\in\Delta^2(Y_s)$. In order to apply Theorem LS it suffices to find a $Y_\sigma\in\mathbb{Y}_\sigma$, $\sigma\le c(s+1)$, such that $Y_\sigma\subset O^{(6)}\cup[-1,x_{n-2}]\cup[x_2,1]$, and an $S^*\in\Sigma_{4,n}(Y_\sigma)\cap\Delta^2(Y_\sigma)$, satisfying

$$\|f-S^*\|\le c\omega_4^\varphi\left(f,\frac{1}{n}\right).$$

A closer look at the proof of Theorem 2 in [10] reveals that there is an $S\in\Sigma_{4,n}$, which is coconvex with f in $[-1,1]\setminus(O^{(2)}\cup[-1,x_{n-2}]\cup[x_2,1])$, and such that

$$\|f-S\|\le c\omega_4^\varphi\left(f,\frac{1}{n}\right). \tag{4.2}$$

Note that this in turn implies

$$\omega_4^\varphi\left(S,\frac{1}{n}\right)\le c\omega_4^\varphi\left(f,\frac{1}{n}\right). \tag{4.3}$$

We define

$$S^*(x) := S(x), \quad x \in [-1,1] \setminus (O^{(6)} \cup [-1, x_{n-2}] \cup [x_2, 1]).$$

Denote

$$p_j := S_{|I_j}, \quad j = 1, \dots n.$$

By [6, Lemma 9]

$$\|p_i - p_j\|_{[x_i,x_j]} \le c\omega_4^{\varphi}\left(S, \frac{1}{n}\right) \le c\omega_4^{\varphi}\left(f, \frac{1}{n}\right) \quad 1 \le i < j \le i + 13s. \tag{4.4}$$

Let $1 < \nu < l_6$. In order to apply Lemma 3.2 for constructing S^* in $G_\nu^{(6)}$, we write $a_{13,\nu} := x_{M_{6,\nu}}$, $a_{12,\nu} := x_{M_{6,\nu}-1}, \dots, a_{8,\nu} := x_{M_{6,\nu}-5}$, $a_{0,\nu} := x_{m_{6,\nu}}$, $a_{1,\nu} := x_{m_{6,\nu}+1}$, $\dots$, $a_{5,\nu} := x_{m_{6,\nu}+5}$, and put $a_{\mu,\nu}$, $\mu = 6, 7$ to be any pair of x_j's $m_{6,\nu} + 5 < j < M_{6,\nu} - 5$, in the correct order. This is a subset of the Chebyshev nodes so that we may regard the piecewise polynomial that we construct by Lemma 3.2 as a proper one for our needs. Therefore we conclude by Lemma 3.2 that for each $1 < \nu < l_6$, we have an $S_\nu^* \in \Sigma_{4,13}(G_\nu^{(6)}, T_\nu, Y_\nu^*)$, where $T_\nu := \{a_{\mu,\nu}\}_{\mu=0}^{13}$, and $Y_\nu^* \subseteq [x_{M_{6,\nu}-2}, x_{m_{6,\nu}+3}]$, containing at most three points, such that $S^*_{\nu|I_{m_{6,\nu}+1}} = p_{m_{6,\nu}+1}$, and $S^*_{\nu|I_{M_{6,\nu}}} = p_{M_{6,\nu}}$. By virtue of (4.1) the constant Λ in (3.4), is bounded by a constant $c = c(s)$, and combining with (4.3) we obtain,

$$\|S^* - S\|_{G_\nu^{(6)}} \le c\omega_4^{\varphi}\left(f, \frac{1}{n}\right). \tag{4.5}$$

For the interval $G_1^{(6)}$ we have two possibilities. If $x_2 \notin G_1^{(6)}$, then we define S^* on $G_1^{(6)}$ by the previous construction, and we are left with the need to define S^* in the interval $[x_2, 1]$. Observe that in this case S is convex in I_3 (since in that interval f is convex and they are coconvex), that is, p_3 is convex in I_3, and being a cubic polynomial it has at most one inflection point in $[x_2, 1]$. So we put

$$S^* := p_3, \quad x \in [x_2, 1].$$

Then (4.2) through (4.4) imply

$$\|f - S^*\|_{[x_2,1]} \le c\omega_4^{\varphi}\left(f, \frac{1}{n}\right). \tag{4.6}$$

Otherwise, $x_2 \in G_1^{(6)}$, whence $M_{6,1} \le 13s + 1$. Then S is coconvex with f in $I_{M_{6,1}}$. Again, $p_{M_{6,1}}$ has at most one inflection point in $[x_{M_{6,1}-1}, 1]$. As above we put

$$S^* := p_{M_{6,1}}, \quad x \in [x_{M_{6,1}}, 1],$$

and get

$$\|f - S^*\|_{[x_{M_{6,1}},1]} \le c\omega_4^\varphi\left(f, \frac{1}{n}\right). \tag{4.7}$$

The component $G_{l_6}^{(6)}$ is treated similarly.

Altogether we have $Y_\sigma \subset O^{(6)} \cup [-1, x_{n-2}] \cup [x_2, 1] \subset O^{(7)} \cup [-1, x_{n-2}] \cup [x_2, 1]$, with $\sigma \le 3l_6 + 2 \le 3s + 2$ points, and by our construction $S^* \in \Sigma_{4,n}(Y_\sigma) \cap \Delta^2(Y_\sigma)$. Hence, by Theorem LS there exists a polynomial of degree $\le c_* n$ for which (2.6) with $r = 0$ and $k = 4$, holds.

What remains in order to complete this part of the proof is to justify the smaller excluded set in the case $r = 0$, $k \le 3$, and in turn, when $f \in W^r$, $1 \le r \le 3$. In these cases we show that if $x_1 \notin G_1^{(7)}$, then we can guarantee the correct approximation order and no inflection points in $[x_2, 1]$. (The same argument holds at the other endpoint.) We observe that in this situation we are guaranteed that S^* constructed above, is convex in I_3, (evidently if $x_2 \notin G_1^{(7)}$, and by Lemma 3.2 if $x_2 \in G_1^{(7)}$). We are going to redefine S^* in $[x_2, 1]$ as the quadratic Taylor polynomial T_2 of S^* about x_2-. It follows that the resulting piecewise polynomial is coconvex with f in $[x_3, 1]$, and has no inflection points in that interval. We have

$$|T_2(x) - S^*(x)| \le c\omega_3(S^*, |I_3|; I_3), \quad x \in I_3,$$

which together with (4.3) through (4.5) yields

$$|T_2(x) - f(x)| \le c\omega_3^\varphi\left(f, \frac{1}{n}\right), \quad x \in [x_3, 1].$$

Hence, (2.6) is satisfied for $r = 0$ and $k \le 3$, and (2.8) is satisfied for $r = 1, 2, 3$. □

Proof of Theorem 2.1. (the case $r = 2$, $k \ge 4$) We need the notion of the length of an interval $J := [a, b] \subseteq I$, relative to its position in I, namely,

$$/J/ := \frac{|J|}{\varphi((a+b)/2)}, \tag{4.8}$$

where $|J| := b - a$(see [6]). It follows from [6, (2.21)] that

$$\omega_k(f, |J|; J) \le \omega_k^\varphi(f, /J/). \tag{4.9}$$

In particular,

$$\omega_k(f'', |I_j|; I_j) \le c\omega_k^\varphi(f'', 1/n).$$

By Lemma LS there is an $S \in \Sigma_{k+2,n}$, coconvex with f in $(-1,1) \setminus O^{(0)}$, and such that

$$\|f - S\| \le cn^{-2}\omega_k^{\varphi}\left(f'', \frac{1}{n}\right). \tag{4.10}$$

(In the connected components of $O^{(0)}$, we merely take the Lagrange interpolating polynomial on k equidistant points including the endpoints of this component.) As in the previous proof we denote

$$p_{j,k+1} := S_{|I_j}, \quad j = 1, \dots, n,$$

and we put

$$S^*(x) := S(x), \quad x \in (-1,1) \setminus O^{(10)}.$$

Now we observe, that if $x_1 \notin G_\nu^{(10)}$ and $x_{n-1} \notin G_\nu^{(10)}$, then $M_{10,\nu} - m_{10,\nu} \ge 21$, whence

$$10|I_{M_{10,\nu}}| \le |G_\nu^{(10)}|, \quad \text{and} \quad 10|I_{m_{10,\nu}+1}| \le |G_\nu^{(10)}|.$$

Thus, if we denote $h := \frac{1}{10}|G_\nu^{(10)}|$, then

$$h_2 := x_{M_{10,\nu}-1} - x_{M_{10,\nu}} \quad \text{and} \quad h_1 := x_{m_{10,\nu}} - x_{m_{10,\nu}+1} \le h,$$

so that we may apply Lemma 3.3 with these h, h_1 and h_2. We obtain a polynomial $P_{\nu,k+1}$ which is coconvex with f in both $(x_{M_{10,\nu}}, x_{M_{10,\nu}-1})$ and $(x_{m_{10,\nu}+1}, x_{m_{10,\nu}})$, and such that

$$\begin{aligned} P_{\nu,k+1}(x_{M_{10,\nu}}) &= f(x_{M_{10,\nu}}), \quad P_{\nu,k+1}(x_{m_{10,\nu}}) = f(x_{m_{10,\nu}}), \\ P'_{\nu,k+1}(x_{M_{10,\nu}}) &= f'(x_{M_{10,\nu}}), \quad P'_{\nu,k+1}(x_{m_{10,\nu}}) = f'(x_{m_{10,\nu}}), \end{aligned} \tag{4.11}$$

and

$$\|f - P_{\nu,k+1}\|_{G_\nu^{(10)}} \le c|G_\nu^{(10)}|^2\omega_k(f'', |G_\nu^{(10)}|; G_\nu^{(10)}), \tag{4.12}$$

By virtue of (4.1) and (4.9), (4.12) yields

$$\|f - P_{\nu,k+1}\|_{G_\nu^{(10)}} \le cn^{-2}\omega_k^{\varphi}(f, 1/n). \tag{4.13}$$

If $x_{m_{10,1}} = 1$, then we have $M_{10,1} \le 21s$. Hence, we take $P_{1,k+1}(x) := p_{M_{10,1}}(x)$, $x \in [x_{M_{10,1}}, 1]$, and obtain (4.12) and whence (4.13). The case $x_{M_{10,l}} = -1$ is similar.

Thus we define

$$S^*_{|G_\nu} := P_{\nu,k+1}, \quad 1 \le \nu \le l_{10}.$$

Altogether we have $Y_\sigma \subset O^{(10)}$ with $\sigma \le (k+1)l_{10} \le (k+1)s$ points. By (4.10), (4.11) and (4.13) $S^* \in \Sigma_{k+2,n}(Y_\sigma) \cap \Delta^2(Y_\sigma)$, and is such that

$$\|f - S^*\| \le cn^{-2}\omega_k^\varphi(f'', 1/n). \tag{4.14}$$

Hence

$$\omega_{k+2}^\varphi(S^*, 1/n) \le cn^{-2}\omega_k^\varphi(f'', 1/n).$$

We now apply Theorem LS together with (4.14) to obtain a polynomial P_n which is coconvex with S and therefore nearly coconvex with f, such that

$$\|f - P_n\| \le \|f - S^*\| + \|S^* - P_n\| \le cn^{-2}\omega_k^\varphi(f'', 1/n), \quad k \ge 4. \tag{4.15}$$

This completes the proof. □

5. Concluding remarks. We can extend the results of Theorem 2.1 to the space B^r (recall the definition from the beginning of Section 2).

Theorem 5.1. *If $f \in \Delta^2(Y_s) \cap B^r$, then for each $n \ge r-1$ there exists a $p_n \in \Pi_n$ satisfying*

$$p_n''(x)\prod_{i=1}^{s}(x - y_i) \ge 0, \quad x \in I \setminus \begin{cases} O(r, 0, c^*, Y_s), & \text{if } r \ne 4 \\ O(0, 4, c^*, Y_s), & \text{if } r = 4, \end{cases} \tag{5.1}$$

such that

$$\|f - p_n\| \le \frac{c}{n^r}\|\varphi^r f^{(r)}\|, \tag{5.2}$$

where $c = c(r, s)$ and $c^* = c^*(r, s)$.

Proof. The proof for $r \le 4$ follows immediately by Theorem 2.1 cases $r = 0$, $k \le 4$. Note that the case $r = 0$, $k = 4$ accounts for the bigger excluded set in (5.1). For $r > 4$ Theorem 5.1 follows from the following result. □

Theorem 5.2. *Let $r \ge 2$ and $k \ge 1$. For every $f \in \Delta^2(Y_s) \cap \mathbb{C}^r(-1, 1) \cap B^r$ and each $n \ge k + r - 1$, there exists a polynomial $p_n \in \Pi_n$ such that*

$$p_n''(x)\prod_{i=1}^{s}(x - y_i) \ge 0, \quad x \in I \setminus \begin{cases} O(0, 4, c^*, Y_s), & \text{if } 2 \le r \le 4 \\ O(r, k, c^*, Y_s), & \text{if } r > 4 \end{cases} \tag{5.3}$$

and

$$\|f - p_n\| \le \frac{c}{n^r}\omega_{k,r}^\varphi(f^{(r)}, 1/n), \tag{5.4}$$

where $c^* = c^*(k, r, s)$, $c = c(k, r, s)$.

See [6] for the definition of the moduli of smoothness $\omega^{\varphi}_{k,r}(f^{(r)},t)$. They provide an extra fine-tuning of the D–T moduli near the endpoints ±1. In particular $\omega^{\varphi}_{k,0}(f,t)=\omega^{\varphi}_{k}(f,t)$, and for a function $f\in B^r$ we have the inequality (see [6, (2.5)]),

$$\omega^{\varphi}_{k,r-k}(f^{(r-k)},t)\le ct^r\|\varphi^r f^{(r)}\|,\quad k<r. \tag{5.5}$$

Clearly Theorem 5.1 for $r>4$ follows immediately from Theorem 5.2 and (5.5).

Proof of Theorem 5.2. One proves this Theorem 5.2 exactly in the same manner, as Theorem 2.1 for $r\ge 2$. So we will not give details, except that we have to take care separately of the intervals I_1 and I_n when $r\le 4$. This is due to the fact that if $r\le 4$, then f'' may be unbounded on these two intervals, and we may not apply Lemma LS there. Thus we have included these two intervals in the excluded set. □

Remark. Actually, Theorem 5.2 holds with the smaller excluded set $O(r,k,c^*,Y_s)$ also when $r=2$, $k=1$. However, in all other cases, $2\le r\le 4$, it is in general impossible to remove the end intervals from the excluded set. The same is true also for Theorem 5.1, $r=4$, and for Theorem 2.1, $r=0$, $k=4$, and $r=1$, $k=3$. One can prove that following the arguments by Kopotun [2].

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Received October 1, 2002