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CHARACTERIZATIONS OF THE SOLUTION SETS OF GENERALIZED CONVEX MINIMIZATION PROBLEMS

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Abstract. In this paper we obtain some simple characterizations of the solution sets of a pseudoconvex program and a variational inequality. Similar characterizations of the solution set of a quasiconvex quadratic program are derived. Applications of these characterizations are given.

1. Introduction. Throughout this work $\mathbb{R}^n$ is the real Euclidean vector space, $X \subset \mathbb{R}^n$ is an open set, and $S \subset X$ is convex. The purpose of the paper is to give characterizations of the solution set of the nonlinear programming problem

$$\text{(P)} \qquad \min\ f(x) \quad \text{subject to} \quad x \in S.$$

Most of them are extensions of the characterizations of the solution set of a pseudolinear differentiable program due to Jeyakumar and Yang [5]. Similar characterizations of convex programs are derived in the earlier work of Mangasarian [9].

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Characterizations of the solutions sets are useful for understanding the behavior of solution methods for programs that have multiple optimal solutions.

The paper is organized as follows. In Section 3 we consider the case when f is pseudoconvex. The pseudoconvex functions don't satisfy the properties of pseudolinear functions that Jeyakumar and Yang use, however, similar characterizations can be derived. We show that in the common case these characterizations cannot be extended to quasiconvex program. In Section 4 we consider a quasiconvex quadratic program. We derive characterizations of the solution set of this program and give two applications. In Section 5 we apply the obtained characterizations to study the solution set of a variational inequality.

2. Preliminaries. We denote the set of the reals by $\mathbb{R}$, and $\overline{\mathbb{R}}$ is the extended real line. Consider a given function $f : X \to \mathbb{R}$. Suppose that $h(x,u)$ is a generalized directional derivative of f at the point x in the direction u. The function $h(x,u)$ may be considered as a bifunction $h : X \times \mathbb{R}^n \to \overline{\mathbb{R}}$.

Recall the following well-known concepts.

The point $x \in X$ is said to be *stationary* with respect to h if $h(x,u) \geq 0$ for all $u \in \mathbb{R}^n$.

The function $f : X \to \mathbb{R}$ is said to be *quasiconvex* on S if

$$f(x + t(y - x)) \leq \max\{f(x), f(y)\}, \quad \text{whenever} \quad x, y \in S \quad \text{and} \quad 0 \leq t \leq 1.$$

The following cone is connected to the quasiconvex function f at each fixed point $x \in S$:

$$\mathcal{N}(x) = \{\xi \in \mathbb{R}^n \mid y \in S,\ f(y) \leq f(x) \quad \text{imply} \quad \langle \xi, y - x\rangle \leq 0\}.$$

Indeed, this is the normal cone to the sublevel set of $f(x)$. Here by $\langle \cdot, \cdot \rangle$ we have denoted the usual scalar product in $\mathbb{R}^n$.

The function f is said to be *pseudoconvex* with respect to $h(x,u)$ on S if

$$x,\ y \in S,\ f(y) < f(x) \quad \text{imply} \quad h(x, y - x) < 0.$$

The standard notion of convex subdifferential may also be applied to our directional derivative h. Each continuous linear functional ξ over $\mathbb{R}^n$ satisfying $\langle \xi, u\rangle \leq h(x,u)$ for all $u \in \mathbb{R}^n$ is called a *subgradient* of f with respect to h at x. The set of all subgradients $\partial f(x)$ at x is called the *subdifferential* of f at x. $\partial f(x)$ is (possibly empty) closed convex subset of $\mathbb{R}^n$.

The function f is said to be *radially lower semicontinuous* on the convex set $X \subset \mathbb{R}^n$, if the function $\varphi(t) = f(a + t(b - a))$ is lower semicontinuous on $[0, 1]$ for every $a,\ b \in X$.

Throughout we assume that for each $x,\ y \in S$ and $u \in \mathbb{R}^n$ the generalized directional derivative satisfies some of the following properties:

1. $h(x,u) < \infty$.

2. If f is quasiconvex on S, then $f(y) \le f(x)$ implies $h(x, y-x) \le 0$.

3 (Fermat rule). If $f(\overline{x}) = \min\,\{f(x) \mid x \in S\}$, then $h(\overline{x}, x-\overline{x}) \ge 0$.

4. The set $\partial f(x)$ is nonempty.

5. $h(x,u)$ considered as a function of u is the support function of $\partial f(x)$, and $h(x,u) = \max\{\langle \xi, u\rangle \mid \xi \in \partial f(x)\}$.

6. If f is quasiconvex, then $\partial f(x) \subset \mathcal{N}(x)$.

Some examples of generalized directional derivatives h that satisfy the considered properties are given in author's work [4].

3. Characterizations of the solution set of a pseudoconvex program. Consider the global minimization problem (P). We give characterizations of the solution set of the program (P) in terms of its minimizer $\overline{x}$.

Denote by $\overline{S}$ the solution set $\arg\min\,\{f(x) \mid x \in S\}$, and let it be nonempty. Suppose that $\overline{x}$ is any fixed element of this set. Consider the following notations of sets:

$$\tilde{S} := \{z \in S \mid h(z, \overline{x} - z) = 0\}, \quad \tilde{S}_1 := \{z \in S \mid h(z, \overline{x} - z) \ge 0\},$$

$$\hat{S} := \{z \in S \mid h(\overline{x}, z - \overline{x}) = 0\}, \quad S^* := \{z \in S \mid h(z, \overline{x} - z) = h(\overline{x}, z - \overline{x})\},$$

$$S^{\#} := \{z \in S \mid h(z\lambda\overline{x}, \overline{x} - z) = 0 \quad \text{for all} \quad \lambda \in (0,1]\},$$

$$\text{and} \quad S^0 := \{z \in S \mid h(z\lambda\overline{x}, \overline{x} - z) = h(\overline{x}, z - \overline{x}) \text{ for all } \lambda \in (0,1]\}.$$

Here we have denoted by $z\lambda\overline{x}$ the sum $z\lambda\overline{x} = \overline{x} + \lambda(z - \overline{x})$.

Theorem 3.1. *Assume that $f : X \to \mathbb{R}$ is pseudoconvex on the convex set S, and Properties* 1, 2, 3, 4, 5 *are satisfied. Let $\overline{x}$ be any fixed point of $\overline{S}$. Then*

$$\overline{S} = S^{\#} \cap \hat{S} = \tilde{S} \cap \hat{S} = \tilde{S} = \tilde{S}_1 = S^* = S^0.$$

P r o o f. It is obvious that

$$S^{\#} \cap \hat{S} \subset \tilde{S} \cap \hat{S} \subset \tilde{S} \subset \tilde{S}_1, \quad S^{\#} \cap \hat{S} \subset \tilde{S} \cap \hat{S} \subset S^*, \quad S^{\#} \cap \hat{S} \subset S^0 \subset S^*.$$

We shall prove that $\overline{S} \subset S^{\#} \cap \hat{S}$. Suppose that z is an arbitrary point of $\overline{S}$. By Property 4 the function f is quasiconvex on S [4, Proposition 5.1], and consequently the solution set $\overline{S}$ is convex. Assume that λ is any number of the interval $(0, 1]$. Then $z\lambda\overline{x} \in \overline{S}$. It follows from Property 3 that $h(z\lambda\overline{x}, x - z\lambda\overline{x}) \geq 0$ for all $x \in S$. Taking in the last inequality $x = \overline{x}$, we get $h(z\lambda\overline{x}, \overline{x} - z) \geq 0$, since $h(x, u)$ is positively homogeneous function of u as a consequence of Property 5. It follows from the equality $f(\overline{x}) = f(z\lambda\overline{x})$, by Property 2, that

$$h(z\lambda\overline{x}, \overline{x} - z\lambda\overline{x}) \leq 0. \tag{1}$$

Hence, $h(z\lambda\overline{x}, \overline{x} - z) = 0$. According to Property 3, $h(\overline{x}, z - \overline{x}) \geq 0$. By Property 2, we obtain from the equality $f(z) = f(\overline{x})$ that $h(\overline{x}, z - \overline{x}) \leq 0$. Hence, $h(\overline{x}, z - \overline{x}) = 0$. Thus, $z \in S^{\#} \cap \hat{S}$.

To show the inclusion $\tilde{S}_1 \subset \overline{S}$, assume that $z \in \tilde{S}_1$. Consequently, $h(z, \overline{x} - z) \geq 0$. By pseudoconvexity, $f(\overline{x}) \geq f(z)$, which implies that $z \in \overline{S}$.

At last, we shall establish the inclusion $S^* \subset \tilde{S}_1$. Let z be arbitrary point of S^*. Using Property 3, it follows from $\overline{x} \in \overline{S}$ that $h(\overline{x}, z - \overline{x}) \geq 0$. By $h(z, \overline{x} - z) = h(\overline{x}, z - \overline{x})$, we obtain that $z \in \tilde{S}_1$. The proof is complete. □

Corollary 3.1. *Suppose additionally that the function f is twice Frèchet differentiable everywhere on X. Then*

$$\overline{S} = S^{\#} \cap \hat{S} \cap \{z \in S \mid \langle \overline{x} - z, \nabla^2 f(z\lambda\overline{x})(\overline{x} - z)\rangle = 0 \quad \forall\, \lambda \in [0, 1]\}.$$

P r o o f. Let $z \in \overline{S}$. By Theorem 3.1, $\langle \nabla f(\overline{x} + \lambda(z - \overline{x})), \overline{x} - z\rangle = 0$, and $\langle \nabla f(\overline{x} + (\lambda + \mu)(z - \overline{x})), \overline{x} - z\rangle = 0$ for all λ, μ such that $\lambda \in [0, 1]$ and $\lambda + \mu \in [0, 1]$, because of $\overline{S} = S^{\#} \cap \hat{S}$. Thus, we conclude that

$$\langle \frac{\nabla f(\overline{x} + \lambda(z - \overline{x}) + \mu(z - \overline{x})) - \nabla f(\overline{x} + \lambda(z - \overline{x}))}{\mu}, \overline{x} - z\rangle = 0.$$

By taking the limits, as $\mu \to 0$, we get that $\langle \overline{x} - z, \nabla^2 f(z\lambda\overline{x})(\overline{x} - z)\rangle = 0$ for all $\lambda \in [0, 1]$.

The remaining part of the proof is obvious. □

Theorem 3.2. *Consider a radially lower semicontinuous function $f : X \to \mathbb{R}$ that is pseudoconvex on the convex set S, and Properties 1, 2, 4 are satisfied. Let $\overline{x}$ be any fixed point of $\overline{S}$. Then $\overline{S} = S_1^{\#}$, where*

$$S_1^{\#} = \{z \in S \mid \langle \xi, \overline{x} - z\rangle = 0,\ \forall\, \xi \in \partial f(z\lambda\overline{x}),\ \forall\, \lambda \in (0, 1)\}.$$

P r o o f. We shall prove the inclusion $\overline{S} \subset S_1^{\#}$. Assume that $z \in \overline{S}$ and $\lambda \in (0, 1)$. Since $f(\overline{x}) = f(z\lambda\overline{x})$, by Properties 4, 2, inequality (1) holds.

Hence, $\langle \xi, \overline{x} - z \rangle \leq 0$ for all $\xi \in \partial f(z\lambda\overline{x})$. Since $f(z) = f(z\lambda\overline{x})$, by Property 2, $h(z\lambda\overline{x}, z - z\lambda\overline{x}) \leq 0$. Therefore, $\langle \xi, \overline{x} - z \rangle \geq 0$ for all $\xi \in \partial f(z\lambda\overline{x})$. Thus, $z \in S_1^{\#}$.

To show the reverse inclusion assume that there exists $z \in S_1^{\#} \setminus \overline{S}$. As a consequence we have that $f(\overline{x}) < f(z)$. Using that f is radially lower semicontinuous, we obtain that there exists $\lambda \in (0, 1)$ such that $f(\overline{x}) < f(z\lambda\overline{x})$. By pseudoconvexity, $\langle \xi, \overline{x} - z \rangle < 0$ for all $\xi \in \partial f(z\lambda\overline{x})$, which is a contradiction. □

It is well-known that each pseudoconvex function, which satisfies Property 4, is quasiconvex. The following lemma from author's work [4, Theorem 5.1] is a necessary and sufficient condition for pseudoconvexity of a quasiconvex function. It is a generalization of the well-known property of the differentiable pseudoconvex functions due to Crouzeix and Ferland [3].

Lemma 3.1. *Let f be a quasiconvex and upper semicontinuous function defined on the open convex set $S \subset \mathbb{R}^n$. Assume that the derivative $h(x, u)$ satisfies Properties 3, 4, 5 and 6. Then f is pseudoconvex on S if and only if the set of global minimizers coincides with the set of stationary points.*

The following question arises from Theorem 3.1. Are there some class of functions, which include the pseudoconvex ones, and such that they satisfy considered characterizations? The following two theorems are connected to this question.

Theorem 3.3. *Let $S \subset \mathbb{R}^n$ be convex and open. Consider the upper semicontinuous quasiconvex function f, which is defined on S, and satisfies Properties 1, 2, 3, 4, 5, and 6. Assume that $\overline{x}$ is any fixed element from $\overline{S}$. Then the following claims are equivalent:*

a) f *is pseudoconvex on* S;

b) $\overline{S} = \tilde{S}$;

c) $\overline{S} = \tilde{S}_1$.

P r o o f. The implication $a) \Rightarrow b)$ follows from Theorem 3.1.

Let's prove the implication $b) \Rightarrow a)$. Assume that $\overline{S} = \tilde{S}$ and $z \in S$ is any stationary point. Therefore, $h(z, u) \geq 0$ for all $u \in \mathbb{R}^n$. In particular, $h(z, \overline{x} - z) \geq 0$. Since $\overline{x} \in \overline{S}$, then $f(\overline{x}) \leq f(z)$. Therefore, by Property 2, $h(z, \overline{x} - z) \leq 0$. Hence, $z \in \tilde{S}$. According to assumption $b)$, z is a global minimizer. Property 3 implies that each global minimizer is a stationary point. Thus, by Lemma 3.1, f is pseudoconvex.

The proof of the claim $c) \iff a)$ is similar. □

Theorem 3.4. *Let $S \subset \mathbb{R}^n$ be convex and open. Consider the Frèchet differentiable quasiconvex function f, which is defined on S. Assume that $\overline{x}$ is any fixed element from $\overline{S}$. Then the following claims are equivalent:*

a) f *is pseudoconvex on* S*;*
b) $\overline{S} = \tilde{S} \cap \hat{S}$*;*
c) $\overline{S} = S^*$.

P r o o f. The implication $a) \Rightarrow b)$ follows from Theorem 3.1. To show the implication $b) \Rightarrow a)$, suppose that $\nabla f(z) = 0$. It follows from inclusion $\overline{x} \in \overline{S}$ that $\nabla f(\overline{x}) = 0$. Therefore, $z \in \tilde{S} \cap \hat{S}$. Then, by Lemma 3.1, f is pseudoconvex.

The proof of the claim $a) \iff c)$ is similar. □

4. Characterizations of the solution set of a quasiconvex quadratic program. Consider the special case when f is a quadratic function, that is

$$f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle,$$

where Q is a constant symmetric $n \times n$ matrix, and c a constant n-dimensional vector. If $S \equiv \mathbb{R}^n$, then the quadratic function is quasiconvex if and only if it is convex (see, for example [10, Theorem 9.2.23]), but when $S \not\equiv \mathbb{R}^n$, Martos has shown in his earlier papers [11, 12] that the quasiconvex or pseudoconvex quadratic functions may not be convex. When S coincide with the nonnegative orthant he has given necessary and sufficient conditions for pseudoconvexity and quasiconvexity of quadratic nonconvex functions.

Example 4.1. *The following example of Arrow and Enthoven [1] shows that a quasiconvex quadratic function may not be pseudoconvex. Consider the function of two variables* $f(x_1, x_2) = -x_1 \cdot x_2$*. It is quasiconvex, but fails to be pseudoconvex on the set* $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0\}$*, since it is not pseudoconvex at the origin.*

The considered characterizations may be extended to the quasiconvex quadratic case.

Consider the sets

$$\hat{S} = \{z \in S \mid \langle \nabla f(\overline{x}), z - \overline{x}\rangle = 0\} = \{z \in S \mid \langle Q\overline{x} + c, z - \overline{x}\rangle = 0\},$$

$$\tilde{S} = \{z \in S \mid \langle \nabla f(z), \overline{x} - z\rangle = 0\} = \{z \in S \mid \langle Qz + c, \overline{x} - z\rangle = 0\}.$$

Theorem 4.1. *Let* $f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle$ *be a quasiconvex quadratic function, defined on the open set* X *with a symmetric matrix* Q*, and* $S \subset X$ *be convex. Suppose that* $\overline{x}$ *is an arbitrary fixed element of* $\overline{S}$*. Then*

$$\overline{S} = S^{\#} \cap \hat{S} = \tilde{S} \cap \hat{S} = S^* = S^0.$$

P r o o f. It is obvious that $S^{\#} \cap \hat{S} \subset \tilde{S} \cap \hat{S} \subset S^*$.

We shall prove that $\overline{S} \subset S^{\#} \cap \hat{S}$. Let $z \in \overline{S}$ be arbitrary. Since the solution set of a quasiconvex program is convex, then $f(z) = f(\overline{x}) = f(z\lambda\overline{x})$ for all $\lambda \in (0,1]$. By quasiconvexity, $\langle \nabla f(\overline{x}), z - \overline{x} \rangle \leq 0$, and $\langle \nabla f(z\lambda\overline{x}), \overline{x} - z\lambda\overline{x} \rangle \leq 0$ for all $\lambda \in (0,1]$. Using the convexity of S we get by the Fermat rule that $\langle \nabla f(\overline{x}), z - \overline{x} \rangle \geq 0$, and $\langle \nabla f(z\lambda\overline{x}), \overline{x} - z\lambda\overline{x} \rangle \geq 0$ for all $\lambda \in (0,1]$. Thus, $z \in S^{\#} \cap \hat{S}$.

To establish the inclusion $S^* \subset \overline{S}$, assume that $z \in S^*$, i.e.

$$\langle Qz + c, \overline{x} - z \rangle = \langle Q\overline{x} + c, z - \overline{x} \rangle. \tag{2}$$

Since Q is a symmetric matrix, then $\langle Q\overline{x}, z \rangle = \langle Qz, \overline{x} \rangle$. Therefore, equality (2) may be rewritten as $f(\overline{x}) = f(z)$. Hence, $z \in \overline{S}$.

The equality $\overline{S} = S^0$ follows from the inclusions $S^{\#} \cap \hat{S} \subset S^0 \subset S^*$. The proof is complete. □

Corollary 4.1. *Let Q be a constant symmetric $n \times n$ matrix, and c a constant n-dimensional vector. Consider the quasiconvex quadratic function $f(x) = \frac{1}{2}\langle x, Qx \rangle + \langle c, x \rangle$, which is defined on the open convex set $S \subset \mathbb{R}^n$. Suppose that $\overline{S} \neq \emptyset$. Then f is pseudoconvex on S.*

P r o o f. Let $\overline{x}$ be any fixed element from $\overline{S}$. According to Theorem 4.1, $\overline{S} = \tilde{S} \cap \hat{S}$. By Theorem 3.4, f is pseudoconvex. □

Remark 4.1. It follows from the characterization $\overline{S} = \tilde{S} \cap \hat{S}$ that

$$\langle z - \overline{x}, Q(z - \overline{x}) \rangle = 0 \quad \text{for all} \quad z \in \overline{S},$$

and therefore, if $\overline{x}$ is a known point of $\overline{S}$, then the set $\overline{S} \setminus \{\overline{x}\}$ is independent of c. It is obvious that $\overline{S}$ depends of c.

The following definition is well-known [13]: The set $M \subset \mathbb{R}^n$ is called *affine* if

$$x + \alpha(y - x) \in M \quad \text{for every} \quad x,\ y \in M \quad \text{and} \quad \alpha \in \mathbb{R}.$$

As a consequence of our characterizations we shall give a shorter proof of the following result of Benson, Smith, Schochetman and Bean [2, Theorem 2.2].

Corollary 4.2. *Assume that the quadratic function $f(x) = \frac{1}{2}\langle x, Qx \rangle + \langle c, x \rangle$, with a symmetric matrix Q, is convex, and S is affine. Then, $\overline{S}$ is an affine set, too.*

P r o o f. By Theorem 4.1, $\overline{S} = \tilde{S} \cap \hat{S}$. Let $z_1,\ z_2 \in \overline{S}$, $\alpha \in \mathbb{R}$. Since S is affine, $z_1 + \alpha(z_2 - z_1) \in S$. It is obvious that $z_1 + \alpha(z_2 - z_1) \in \hat{S}$.

Since $\overline{S}$ is convex, then $\frac{1}{2}(z_1+z_2) \in \overline{S}$. Therefore,

$$\langle Q\frac{z_1+z_2}{2}+c, \overline{x}-\frac{z_1+z_2}{2}\rangle = 0.$$

The last equality may be rewritten as

$$\langle Qz_1+c,\overline{x}-z_1\rangle + \langle Qz_2+c,\overline{x}-z_2\rangle + \langle Qz_1+c,\overline{x}-z_2\rangle + \langle Qz_2+c,\overline{x}-z_1\rangle = 0.$$

By $z_1,\ z_2 \in \tilde{S}$, it follows from this equality that

$$\langle Qz_1+c,\overline{x}-z_2\rangle + \langle Qz_2+c,\overline{x}-z_1\rangle = 0 \tag{3}$$

We shall show that $z_1+\alpha(z_2-z_1) \in \tilde{S}$. Using equality (3), we have

$$\langle Q(z_1+\alpha(z_2-z_1))+c,\overline{x}-z_1-\alpha(z_2-z_1)\rangle =$$

$$\alpha^2\langle Qz_2+c,\overline{x}-z_2\rangle + (1-\alpha)^2\langle Qz_1+c,\overline{x}-z_1\rangle +$$

$$\alpha(1-\alpha)(\langle Qz_1+c,\overline{x}-z_2\rangle + \langle Qz_2+c,\overline{x}-z_1\rangle) = 0.$$

Consequently, $\overline{S}$ is affine. □

Remark 4.2. Since a quasiconvex quadratic function $f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle$ with a symmetric matrix Q, defined on the whole space $\mathbb{R}^n$, is always convex [10, Theorem 9.2.23], then a quasiconvex quadratic function $f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle$ with a symmetric matrix Q, defined on affine set, is always convex, too. The proof of this fact repeats the proof of Theorem 9.2.23 in Reference [10].

5. Characterizations of the solution set of a variational inequality. It is well-known that when the function f is pseudoconvex on the convex set $S \subset \mathbb{R}^n$ and Frèchet differentiable, $z \in \overline{S}$ if and only if $\langle \nabla f(z), x - z\rangle \geq 0$ for all $x \in S$. When the function f is Frèchet differentiable, ∇f is pseudomonotone map if and only if f is pseudoconvex [6].

Let $V \subset \mathbb{R}^n$ and an operator $F : V \to \mathbb{R}^n$ be given. The *Standard Variational Inequality Problem* consists in finding $y \in V$ such that

$$\langle F(y), x-y\rangle \geq 0 \quad \text{for all} \quad x \in V. \tag{SVI}$$

Recall the following well-known concept [6]. The operator $F : V \to \mathbb{R}^n$ is called *pseudomonotone*, if

$$x,\ y \in V,\ \langle F(x), y-x\rangle \geq 0 \quad \text{imply} \quad \langle F(y), y-x\rangle \geq 0. \tag{4}$$

Denote the solution set of (SVI) by $\overline{V}$, and let $\overline{x}$ be any fixed element of $\overline{V}$. In consistency with our notations denote

$$\hat{V} := \{z \in V \mid \langle F(\overline{x}), z-\overline{x}\rangle = 0\}, \quad \tilde{V} := \{z \in V \mid \langle F(z), \overline{x}-z\rangle = 0\},$$

$$V^* := \{z \in V \mid \langle F(z), \overline{x} - z\rangle = \langle F(\overline{x}), z - \overline{x}\rangle\},$$

$$V^{\#} := \{z \in V \mid \langle F(z\lambda\overline{x}), \overline{x} - z\rangle = 0 \quad \text{for all} \quad \lambda \in [0,1]\}.$$

Theorem 5.1. *If the operator $F : V \to \mathbb{R}^n$ is pseudomonotone, and $\overline{x} \in V$ is any fixed element of $\overline{V}$, then*

$$\overline{V} \subset \tilde{V} \cap \hat{V} = V^* = \tilde{V} \subset \hat{V}.$$

If in addition F is a continuous map and the set V is closed and convex, then $\overline{V} \subset V^{\#}$.

Proof. We shall prove that $\overline{V} \subset \tilde{V} \cap \hat{V}$. Let $z \in \overline{V}$. Therefore,

$$\langle F(z), \overline{x} - z\rangle \geq 0. \tag{5}$$

Using the pseudomonotonicity of F, we get that $\langle F(\overline{x}), \overline{x} - z\rangle \geq 0$. Since $\overline{x} \in \overline{V}$, we have

$$\langle F(\overline{x}), z - \overline{x}\rangle \geq 0, \tag{6}$$

which implies that $\langle F(\overline{x}), z - \overline{x}\rangle = 0$. By the pseudomonotonicity, from (6) we obtain the inequality

$$\langle F(z), z - \overline{x}\rangle \geq 0. \tag{7}$$

By (5), we conclude that $\langle F(z), \overline{x} - z\rangle = 0$. Thus, $z \in \tilde{V} \cap \hat{V}$.

Obviously $\tilde{V} \cap \hat{V} \subset V^*$.

The inclusion $V^* \subset \tilde{V}$ is a consequence of (6), (7), and the definition of V^*.

To show the inclusion $\tilde{V} \subset \hat{V}$ we suppose that $z \in \tilde{V}$. By the pseudomonotonicity we conclude that $\langle F(\overline{x}), \overline{x} - z\rangle \geq 0$. According to inequality (6), $z \in \hat{V}$.

If F is continuous and pseudomonotone, V is closed and convex, then the solution set is convex [7]. Hence, $z\lambda\overline{x} \in \overline{V}$ for all $\lambda \in [0,1]$. Consequently,

$$\langle F(z\lambda\overline{x}), \overline{x} - z\lambda\overline{x}\rangle \geq 0 \quad \text{for all} \quad \lambda \in [0,1]. \tag{8}$$

Using $\overline{x} \in \overline{V}$, we get that $\langle F(\overline{x}), z\lambda\overline{x} - \overline{x}\rangle \geq 0$ for all $\lambda \in [0,1]$. By the pseudomonotonicity, $\langle F(z\lambda\overline{x}), z\lambda\overline{x} - \overline{x}\rangle \geq 0$ for all $\lambda \in [0,1]$. We obtain from inequality (8) that $z \in V^{\#}$. (The equality $\langle F(\overline{x}), \overline{x} - z\rangle = 0$ is already shown.) □

Remark 5.1. The converse inclusions of Theorem 5.1 do not hold. Indeed, assume that $V \equiv \mathbb{R}^n$. Then the solution set coincides with the solution set of the equality $F(x) = 0$. Suppose that $\overline{x} \in \overline{V}$. Therefore, $\hat{V} \equiv \mathbb{R}^n$, and $V^* = \tilde{V} = \tilde{V} \cap \hat{V}$. If $z \in \tilde{V}$, then $\langle F(z), \overline{x} - z\rangle = 0$. This does not imply that $F(z) = 0$.

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