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ON THE TRANSFORMATIONS OF SYMPLECTIC EXPANSIONS AND THE RESPECTIVE BÄCKLUND TRANSFORMATION FOR THE KDV EQUATION

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Abstract. By using the Deift–Trubowitz transformations for adding or removing bound states to the spectrum of the Schrödinger operator on the line we construct a simple algorithm allowing one to reduce the problem of deriving symplectic expansions to its simplest case when the spectrum is purely continuous, and vice versa. We also obtain the transformation formulas for the correponding recursion operator. As an illustration of this approach, the Bäcklund transformations for the KdV equation are constructed.

1. Introduction. Let us consider the Schrödinger operator

$$l(v)y \equiv -y'' + v(x)y = k^2 y, \quad -\infty < x < \infty, (\,' = D = d/dx) \tag{1.1}$$

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with a spectral parameter $\lambda = k^2$, where the real potential $v(x)$ is in the space

$$L_2^1 = \{f(x) : \int_{-\infty}^{\infty} (1 + |x|^2)|f(x)|dx < \infty\}.$$

Let $f^{(\pm)}(x, k)$ be the Jost solutions of equation (1.1), defined as usual by the asymptotics

$$\lim_{x\to\infty} f^{(+)}(x, k)e^{-ikx} = 1, \quad \lim_{x\to-\infty} f^{(-)}(x, k)e^{ikx} = 1. \tag{1.2}$$

In the limit $x \to \mp\infty$, the relations

$$f^{(\pm)}(x, k) \sim \pm b(\mp k)e^{\mp ikx} + a(k)e^{\pm ikx}, \quad x \to \mp\infty, \tag{1.3}$$

take place. The operator $l(v)$, considered in the Hilbert space $L_2 = L_2(\mathbb{R})$ with a scalar product $(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g}(x)\, dx$, $\|f\| = (f, f)^{1/2}$, has a double and absolutely continuous spectrum for $k^2 \in [0, \infty)$ and a finite number of bound states $\lambda_j = k_j^2$, $k_j = i\tau_j, \tau_j > 0$, $j = 1, \ldots, N$ where

$$f^{(+)}(x, k_j) = C_j f^{(-)}(x, k_j). \tag{1.4}$$

Here and below, some well-known results concerning the Schrödinger operator considered are given and will be used without reference. They can be found in [1, 2, 3, 4].

The functions $a(k)$ and $b(k)$ determine the scattering matrix for $k \in (-\infty, \infty)$, and the numbers k_j are the zeros of $a(k)$ for $\operatorname{Im} k > 0$. We denote by $\sigma(v)$ the set

$$\sigma(v) = \{k_j : \ k_j = i\tau_j, \ \tau_j > 0, \ a(k_j) = 0, \ j = 1, \ldots, N\}, \tag{1.5}$$

and the corresponding set of potentials by $\Omega(N)$. It is well known, that the scattering data

$$\mathbf{s}(v) : p(v; k) = \frac{2k}{\pi} \ln|a(k)|, \quad q(v; k) = \arg b(k), \quad k \in \mathbf{R}^+,$$

$$p_j(v) = \lambda_j = k_j^2, \quad q_j(v) = \ln|C_j|, \quad k_j \in \sigma(v), \tag{1.6}$$

determines uniquely the potential $v(x)$ in (1.1). The map $v \to \mathbf{s}(v)$ defines [5] the action-angle variables for the Korteweg-de Vries equation

$$v_t(x, t) = 6v(x, t)v(x, t)_x - v(x, t)_{xxx} \equiv 4DL(v)v, \quad v(x, 0) \in \mathcal{S}(\mathbf{R}), \tag{1.7}$$

where $\mathcal{S}$ is the Schwartz space of smooth functions that are rapidly decreasing at infinity. The representation of the KdV equation in terms of the recursion operator

$$L(v) = -\frac{1}{4}D^2 + v(x) + \frac{1}{4}\left(\int_x^{\infty} - \int_{-\infty}^{x}\right) dy v'(y) \tag{1.8}$$

is proposed in [6]. The inverse scattering method, discovered in the classical work of GGKM [7], yields for $\mathbf{s}(v(t))$ the following evolution in time:

$$p(t;k) = p(0;k),\ q(t;k) = q(0;k) + 8k^3 t,\ k \in \mathbf{R}^+;$$

$$p_j(t) = p_j(0),\ q_j(t) = q_j(0) + i8k_j^3 t,\ k_j \in \sigma(v). \tag{1.9}$$

One of the simplest ways to show that $\mathbf{s}(v)$ are canonical variables for the KdV is based on the symplectic expansions associated with $L(v)$, where the variational derivatives of the scattering data in $\mathbf{s}(v)$ with respect to v form a complete system of eigenfunctions (see, e.g. Sec. 2 for more details.) As mentioned above, this idea goes back to the pioneering work of Zakharov and Faddeev [5] and was investigated in [8, 9, 10]. The properties of the operator L related to the KdV Equation, in particular its connection to the Bäcklund transformations, is studied in detail in [11].

From the work of Levinson [12] up to [3, 4] it is known that it is easier to investigate the map $v \leftrightarrow \mathbf{s}(v)$ starting from Eq.(1.1) in the case of purely continuous spectrum, i.e. $\sigma(v) = \emptyset$. The procedure of adding and removing bound states is based on a suitable modification of the Darboux – Crum transform [13] (see, also Sec.3). As part of the detailed investigation in [4] on the inverse problem for the operator $l(v)$, the transformation

$$r(x) = \mathcal{G}(v) \overset{\text{def}}{=} v(x) - 2\frac{d^2}{dx^2}\log g(v;x), \quad v \in \Omega(N) \tag{1.10}$$

where

$$g(v;x) = f^{(+)}(v;x,k_0) + \alpha f^{(-)}(v;x,k_0), \quad \alpha > 0. \tag{1.11}$$

has been proposed for adding a bound state $\lambda_0 = -\tau_0^2 : \tau_0 > \max\{\tau_j\}_{j=1}^N$ to the discrete spectrum of the operator $l(v)$. Then the transformation which removes the bound state λ_0, (considered as inverse to $\mathcal{G}$) is given by the operator

$$v(x) = \mathcal{F}(r) \overset{\text{def}}{=} r(x) - 2\frac{d^2}{dx^2}\log h(r;x),\ h(r;x) = \frac{1}{g(v;x)},\ r \in \Omega(N+1), \tag{1.12}$$

where $h(r;x)$ is the eigenfunction, corresponding to the lowest eigenvalue λ_0 : $l(r)h = \lambda_0 h$ (see, e.g. Sec. 3).

The main purpose of this work is a construction of transformations similar to those in [14, 15], which allows one to obtain symplectic expansions corresponding to the potential $r \in \Omega(N+1)$ starting from the ones for $v \in \Omega(N)$, and vise versa. We also obtain the transformation for the operators $L(v)$ and $L(r)$ (sec. 5). In Sec. 3 we derive the main transformations for the eigenfunctions of the operator L. In Sec. 4 we show that the transforming operators from Sec. 3 naturally arise as functional derivatives $\partial\mathcal{G}/\partial v$ and $\partial\mathcal{F}/\partial r$. In the last Sec. 6, as a simple application of those results, we obtain that the transformations $\mathcal{G}$ and $\mathcal{F}$ preserve the KdV equation. This fact is known as Bäcklund transformations (see e.g. [11, 16]). In Sec. 2 we list some known results associated with the symplectic expansion formulas for the operator $L(v)$ in a suitable form the proofs of which can be found, for example, in [9, 10]. We should mention here that, along with the transformations (1.10)-(1.12) usually called single commutation relations [17], there is also the so-called double commutation relation for adding (or removing) bound states associated with the Gelfand-Levitan-Marchenko equation [2, 3]. In our case it is given by the expression

$$r(x) = v(x) - 2\frac{d^2}{dx^2}\ln(1 + \alpha\int_{-\infty}^{x} f^{(-)^2}(s,k_0)\,ds), \quad \alpha > 0.$$

This transform is suitable for generalizing the scheme considered here to the general case of expansions in products of solutions of two Schrödinger operators with different spectra, similar to [15]. The theory of the double commutation method in the general case ($v \in L_{\mathrm{loc}}$) is given in [18].

2. Symplectic expansion formulas and biorthogonality relations. Recall that the solutions $f^{(\pm)}(x,k)$ are analytic in the upper half-plane $\operatorname{Im} k > 0$ and continuous in $\operatorname{Im} k \geq 0$. Denote the squared Jost solutions by

$$F^{(+)}(v;x,k) = f^{(+)^2}(x,k), \quad F^{(-)}(v;x,k) = f^{(-)^2}(x,k), \tag{2.1}$$

where (here and further on) the dependence on v is suppressed if it does not lead to ambiguity.

Now we determine the system $\{P(v), Q(v)\}$ as follows [5, 9, 10]:

$$(2.2)\qquad \begin{aligned} &P(v;x,k) = r^{+}(k)F^{(+)}(x,k) - r^{+}(-k)F^{(+)}(x,-k),\\ &Q(v;x,k) = r^{-}(k)F^{(-)}(x,k) + r^{+}(k)F^{(+)}(x,k), \quad k \in \mathbf{R}^{+},\\ &P_j(v;x) = M_j^{+}F_j^{(+)}(x) = M_j^{-}F_j^{(-)}(x),\\ &Q_j(v;x) = (\tau_j \dot{a}(k_j))^{-1} f^{(-)}(x,k_j) m(x,k_j), \quad k_j \in \sigma(v). \end{aligned}$$

Here the reflection coefficients are $r^{\pm}(k) = b(\mp k)/a(k)$, with the respective norming constants being

$$(2.3)\qquad M_j^{\pm} \stackrel{\text{def}}{=} \|f^{(\pm)}(\,.\,,k_j)\|^{-2} = -iC_j^{\mp 1}\dot{a}^{-1}(k_j), \quad (\dot{\ } = \partial/\partial k).$$

The function $m(x,k_j) = \dot{f}^{(+)}(x,k_j) - C_j \dot{f}^{(-)}(x,k_j)$ is a solution to the equation $l(v)m(x,k_j) = k_j^2 m(x,k_j)$. Let

$$(2.4)\qquad \beta(k) = 2ikb(k)b(-k).$$

We recall the following result.

Theorem 2.1 [5]**.** *The variational derivatives of the scattering data* $\mathbf{s}(v)$ *are expressed through the system* $\{P, Q\}$ *as follows:*

$$(2.5)\qquad \begin{aligned} &\frac{\partial p}{\partial v}(x,k) = (2i\pi)^{-1}P(x,k), \qquad &&\frac{\partial p_j}{\partial v}(x) = P_j(x),\\ &\frac{\partial q}{\partial v}(x,k) = (2i\beta(k))^{-1}Q(x,k), &&\frac{\partial q_j}{\partial v}(x) = -2^{-1}Q_j(x). \end{aligned}$$

Next we have

Lemma 2.1 [9, 10, 11]**.** *The functions* (2.2) *are the eigenfunctions of the operator* $L(v)$*:*

$$(2.6)\qquad \begin{aligned} &LP(x,k) = k^2P(x,k), \quad &&LQ(x,k) = k^2Q(x,k), \quad k \in \mathbf{R}^{+},\\ &LP_j(x) = k_j^2P_j(x), &&LQ_j(x) = k_j^2Q_j(x), \quad k_j \in \sigma. \end{aligned}$$

Let us introduce the skew scalar product

$$[f,g] = (f, D\overline{g}) = -[g,f].$$

The next theorem shows that the system $\{P, Q\}$ is complete in the space L_2^1.

Theorem 2.2 [9, 10]. **(i).** *For any absolutely continuous function* $f \in L_2^1$, *the following expansion formulas are valid:*

$$f(x) = -\frac{1}{2\pi}\int_0^\infty \{P(x,k)[f,Q(k)] - Q(x,k)[f,P(k)]\}\frac{dk}{\beta(k)}$$

$$-\sum_{j=1}^{N}\{P_j(x)[f,Q_j] - Q_j(x)[f,P_j]\} \tag{2.7}$$

and

$$f(x) = \frac{1}{2\pi}\int_0^\infty \{P'(x,k)(f,Q(k)) - Q'(x,k)(f,P(k))\}\frac{dk}{\beta(k)} + \dots \tag{2.8}$$

where the dots stand for the discrete spectrum portion of the expansion.

(ii). *The expansion formula* (2.7) *is a decomposition of unity for the operator* L *(with* $v' \in L_1$*):*

$$Lf(x) = -\frac{1}{2\pi}\int_0^\infty \{k^2P(x,k)[f,Q(k)] - k^2Q(x,k)[f,P(k)]\}\frac{dk}{\beta(k)} + \dots. \tag{2.9}$$

We finish the present section with the next lemma.

Lemma 2.2. *For the system* $\{P,\ Q\}$ *we have the following biorthogonal relations*

$$[P(k), P(\mu)] = [Q(k), Q(\mu)] = 0, \quad k,\ \mu \in \mathbf{R}^+ \cup \sigma,$$

$$[P(k), Q(\mu)] = 2\pi\,\beta(k)\delta(k-\mu),\ k,\ \mu \in \mathbf{R}^+,\ [P_j, Q_l] = \delta_{j,l},\ k_j, k_l \in \sigma,$$

which, in combination with (2.7), (2.8), *show that the system* $\{P, Q\}$ *is a symplectic basis in the space* L_2^1.

Remark 2.1. Following [5], we state Lemma 2.2 with simplified biorthogonality relations sufficient for our purposes. A detailed study of these relations with the pole $k = 0$ of $a(k)$ and $b(k)$ included can be found in [9].

3. The operators $\mathbf{A}$, $\tilde{\mathbf{A}}$ and $\mathbf{B}_{\boldsymbol{a}}$, $\tilde{\mathbf{B}}_{\boldsymbol{a}}$. The basic idea behind our construction here (as well as in [14]) is the following simple observation.

Lemma 3.1. *Consider the equations*

$$y'' + (\lambda - v(x))\, y = 0, \quad a < x < b, \tag{3.1}$$

and

$$y_1'' + (\lambda - r(x))\, y_1 = 0, \quad r(x) = v(x) - 2\frac{d^2}{dx^2} \ln z(x), \tag{3.2}$$

where $z(x)$ is a solution of (3.1) for $\lambda = \lambda_0, z(x) \neq 0, a < x < b$. Then

$$W(Z_1(x), Y_1(x,\lambda)) = (\lambda_0 - \lambda)^{-1} \frac{d}{dx}\left(Z_1(x) Y(x,\lambda)\right), \quad (W(f,g) = fg' - f'g)$$

where $Y(x,\lambda) = y^2(x,\lambda)$, $Y_1(x,\lambda) = y_1^2(x,\lambda)$ and $Z_1(x) = z^{-2}(x)$.

This Lemma is a direct consequence of the next Darboux-Crum lemma in view of the identity [5]

$$W(Y(x,\lambda), Z(x,\lambda_0)) = \frac{1}{\lambda - \lambda_0} \frac{d}{dx} W^2(y(x,\lambda), z(x,\lambda_0)).$$

Lemma 3.2 [13]**.** *Consider equation* (3.1) *and let $z(x)$ be the same as in Lemma* 3.1*. Then the function*

$$y_1(x,\lambda) = \frac{W(z(x), y(x,\lambda))}{(\lambda_0 - \lambda) z(x)}$$

satisfies equation (3.2) *for $\lambda \neq \lambda_0$. In the case $\lambda = \lambda_0$, the function $z_1(x) = z^{-1}(x)$ is a solution to* (3.2)*.*

By using the last lemma in [4], the following statement is proved.

Lemma 3.3. *Let the potential r be constructed as in* (1.10)*,* (1.11)*, $r = \mathcal{G}(v)$. Then the discrete spectrum is $\sigma(r) = \sigma(v) \cup k_0$ where $h(x) = g^{-1}(x) \neq 0$ is the eigenfunction corresponding to k_0,*

$$f^{(+)}(r; x, k_0) = \alpha a(v; k_0) h(x), \quad f^{(-)}(r; x, k_0) = a(v; k_0) h(x), \tag{3.3}$$

the norming constants are

$$M_j^+(r) = \frac{\tau_0 + \tau_j}{\tau_0 - \tau_j} M_j^+(v),\ (k_j \in \sigma(v)),\ M_0^+(r) = \frac{2\tau_0}{\alpha a(v; k_0)},$$

the scattering coefficients are

$$a(r; k) = \frac{k - k_0}{k + k_0} a(v; k), \quad b(r; k) = -b(v; k), \tag{3.4}$$

and the Jost solutions $f^{(\pm)}(r;x,k)$ *and* $f^{(\pm)}(v;x,k)$ *are, respectively*

$$f^{(\pm)}(r;x,k) = \pm\frac{W(g(x),f^{(\pm)}(v;x,k))}{i(k+k_0)g(x)}, \tag{3.5}$$

$$f^{(\pm)}(v;x,k) = \pm\frac{W(h(x),f^{(\pm)}(r;x,k))}{i(k-k_0)h(x)}.$$

Let us denote

$$H(x) = P_0(x) = 2\tau_0\alpha a(k_0)h^2(x) \quad G(x) = P_0^{-1}(x). \tag{3.6}$$

From the asymptotics [4]

$$g(x) =\sim \alpha a(k_0)e^{\tau_0 x},\ x\to\infty, \quad g(x)\sim a(k_0)e^{-\tau_0 x},\ x\to-\infty \tag{3.7}$$

we obtain the following estimates:

$$H(x)\sim\frac{2\tau_0}{\alpha a(k_0)}e^{-2\tau_0 x},\ x\to\infty, \quad H(x)\sim\frac{2\tau_0\alpha}{a(k_0)}e^{2\tau_0 x},\ x\to-\infty. \tag{3.8}$$

The next lemma is a simple consequence of lemmas 1.1 and 3.1.

Lemma 3.4. *For* $v\in\Omega(N)$ *and* $r=\mathcal{G}(v)$ *we have*

$$W(H(x),F^{(\pm)}(r;x,k)) = \frac{k-k_0}{k+k_0}\frac{d}{dx}(H(x)F^{(\pm)}(v;x,k)). \tag{3.9}$$

Now we introduce the operators

$$\begin{aligned}\mathbf{A}^{(\pm)}f &= f(x) - 2G(x)\int_{\pm\infty}^{x} H'(s)f(s)\,ds, \quad f\in L_1, L_\infty\\ \tilde{\mathbf{A}}^{(\pm)}f &= f(x) + 2G'(x)\int_{\pm\infty}^{x} H(s)f(s)\,ds, \quad f\in L_1, L_\infty.\end{aligned} \tag{3.10}$$

where $L_1 = L_1(\mathbf{R})$,$L_\infty = L_\infty(\mathbf{R})$. (We should mention here that if $L_1(\mathbf{R})$ is replaced by L_2^1 then all the results below still remain valid.) Note that

$$D\mathbf{A}^{(\pm)}f(x) = \tilde{\mathbf{A}}^{(\pm)}Df(x) \tag{3.11}$$

and

$$\mathbf{A}^{(\pm)}H(x) = \tilde{\mathbf{A}}^{(\pm)}H'(x) = 0. \tag{3.12}$$

Let $L_1(H)$, $L_1(H')$ be the subspaces

$$L_1(H) = \{f \in L_1 : (f, H) = 0\}, \quad L_1(H') = \{f \in L_1 : (f, H) = 0\},$$

with similar definitions for $L_\infty(H)$, $L_\infty(H')$. Obviously, the relations

$$\begin{aligned} &\mathbf{A}^{(+)} f(x) = \mathbf{A}^{(-)} f(x) \stackrel{def}{=} \mathbf{A} f(x), \quad f \in L_1(H'), L_\infty(H'), \\ &\tilde{\mathbf{A}}^{(+)} f(x) = \tilde{\mathbf{A}}^{(-)} f(x) \stackrel{def}{=} \tilde{\mathbf{A}} f(x), \quad f \in L_1(H), L_\infty(H) \end{aligned} \tag{3.13}$$

take place. Also, from (3.8) it is easy to obtain that

$$\mathbf{A} \in \mathcal{L}(L_1(H'), L_1), \mathcal{L}(L_\infty(H'), L_\infty), \quad \tilde{\mathbf{A}} \in \mathcal{L}(L_1(H), L_1), \mathcal{L}(L_\infty(H), L_\infty).$$

Here as usual $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators defined in X with values in Y. Note that if $f \in L_\infty(H')$, $g \in L_1(H)$, $g' \in L_1(H')$ then

$$(\mathbf{A} f, \tilde{\mathbf{A}} g) = (f, g), \quad [\mathbf{A} f, \mathbf{A} g] = [f, g]. \tag{3.14}$$

Theorem 3.1. *Let the potential $r \in \Omega(N+1)$, with $\{P(r), Q(r)\}$ being the respective symplectic system, and let $v = \mathcal{F}(r)$. Then for the system $\{P(v), Q(v)\}$, the representations*

$$\begin{aligned} &P(v; x, k) = -\mathbf{A} P(r; x, k), \quad Q(v; x, k) = -\mathbf{A} Q(r; x, k), \ k \in \mathbf{R}^+, \\ &P_j(v; x) = -\mathbf{A} P_j(r; x), \qquad Q_j(v; x) = -\mathbf{A} Q_j(r; x), \ k_j \in \sigma(v) \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &P'(v; x, k) = -\tilde{\mathbf{A}} P'(r; x, k), \quad Q'(v; x, k) = -\tilde{\mathbf{A}} Q'(r; x, k), \\ &P'_j(v; x) = -\tilde{\mathbf{A}} P'_j(r; x), \qquad Q'_j(v; x) = -\tilde{\mathbf{A}} Q'_j(r; x) \end{aligned} \tag{3.16}$$

take place. Moreover,

$$\beta(v; k) = \beta(r; k), \quad k \in \mathbf{R}^+, \tag{3.17}$$

where $\mathbf{A}$, $\tilde{\mathbf{A}}$ *are as in* (3.13), *and* $\beta(k)$ *as in* (2.4).

Proof. As a result of (3.12), the representations (3.16) follow from (3.15). Also, the equality (3.17) follows from (3.4). The proof of (3.15) can easily be derived by using the next Lemma 3.5, in view of the definition (2.2)-(2.5) of $\{P, Q\}$ and Lemma 3.3. $\square$

Lemma 3.5. *If the conditions of Theorem* 3.1 *hold then we have the representations*

$$F^{(\pm)}(v;x,k)=\frac{k+k_0}{k-k_0}\mathbf{A}F^{(\pm)}(r;x,k),\qquad k\in\mathbf{R}\cup\sigma(v), \tag{3.18}$$

and

$$\dot F^{(\pm)}(v;x,k)=\frac{k_j+k_0}{k_j-k_0}\mathbf{A}(\dot F^{(\pm)}(r;x,k_j)-\frac{2k_0}{k_j^2-k_0^2}F^{(\pm)}(r;x,k_j)) \tag{3.19}$$

for $k_j\in\sigma(v)$.

P r o o f. By integrating (3.9) from x to $\pm\infty$ and in view of

$$(H,F^{(\pm)'}(k))=-(H',F^{(\pm)}(k))=0,\quad k\in\mathbf{R}\cup\sigma(v)$$

we get (3.18). In order to obtain (3.19) we differentiate (3.9) with respect to k for $k=k_j$ and then integrate from x to $\pm\infty$ while using the equality $(H,\dot F^{(\pm)'}(k_j))=-(H',\dot F^{(\pm)}(k_j))=0$. The latter follows from the asymptotics (3.8) and

$$\dot f^{(+)}(x,k_j)\sim ixe^{ik_jx},\ x\to\infty,\quad \dot f^{+}(x,k_j)\sim\dot a(k_j)e^{ik_jx},\ x\to-\infty.$$

Now, in order to obtain the formulas inverse to those in Theorem 3.1, we introduce the operators

$$\begin{aligned}\mathbf{B}_af&=f(x)-2H(x)\int_a^x G'(s)f(s)\,ds,\\ \tilde{\mathbf{B}}_af&=f(x)+2H'(x)\int_a^x G(s)f(s)\,ds.\end{aligned} \tag{3.20}$$

Define the functions $H_a^{(\pm)}(x)$ and $H_a^{(\pm)'}(x)$ as follows:

$$H_a^{(\pm)}(x)=\{\mp2H^{-2}(a)H(x),\ x\in[a,\pm\infty);0,\ x\in[a,\mp\infty)\},$$

$$H_a^{(\pm)'}(x)=\{\mp2H^{-2}(a)H'(x),\ x\in[a,\pm\infty);0,\ x\in[a,\mp\infty)\}.$$

It is easy to check that

$$(H_a^{(\pm)},H')=(H_a^{(\pm)'},H)=1, \tag{3.21}$$

$$(\tilde{\mathbf{B}}_af,H_a^{(\pm)})=0,\quad(\mathbf{B}_af,H_a^{(\pm)'})=0,\quad f\in L_1,L_\infty. \tag{3.22}$$

Therefore

$$(\tilde{\mathbf{B}}_a f, H) = 0, \quad (\mathbf{B}_a f, H') = 0, \quad f \in L_1, L_\infty. \tag{3.23}$$

Next, we introduce the subspaces

$$L_1(H_a^{(\pm)}) = \{f \in L_1 : (f, H_a^{(\pm)}) = 0\},\ L_1(H_a^{(\pm)'}) = \{f \in L_1 : (f, H_a^{(\pm)'}) = 0\},$$

with similar expressions for $L_\infty(H_a^{(\pm)})$, $L_\infty(H_a^{(\pm)'})$. In view of (3.8), (3.22) implies that

$$\mathbf{B}_a \in \mathcal{L}(L_1, L_1(H_a^{(\pm)})), \quad \tilde{\mathbf{B}}_a \in \mathcal{L}(L_1, L_1(H_a^{(\pm)'})),$$

and $\mathbf{B}_a \in \mathcal{L}(L_\infty, L_\infty(H_a^{(\pm)}))$, $\tilde{\mathbf{B}}_a \in \mathcal{L}(L_\infty, L_\infty(H_a^{(\pm)'}))$ as well. Let us define the projection operators $\mathbf{P}_a^{(\pm)} : L_1 \to L_1(H_a^{(\pm)'})$, $(L_\infty \to L_\infty(H_a^{(\pm)'}))$ via

$$\mathbf{P}_a^{(\pm)} f = f(x) - C_a^{(\pm)}(f) H(x), \quad C_a^{(\pm)}(f) = (f, H_a^{(\pm)'}),$$

and $\tilde{\mathbf{P}}_a^{(\pm)} : L_1 \to L_1(H_a^{(\pm)})$, $(L_\infty \to L_\infty(H_a^{(\pm)}))$ in a similar manner:

$$\tilde{\mathbf{P}}_a^{(\pm)} f = f(x) - \tilde{C}_a^{(\pm)}(f) H'(x), \quad \tilde{C}_a^{(\pm)}(f) = (f, H_a^{(\pm)}).$$

Note that

$$\begin{aligned} C_a^{(\pm)}(f) &\stackrel{def}{=} C_a(f), \quad f \in L_1(H'),\ L_\infty(H') \\ \tilde{C}_a^{(\pm)}(f) &\stackrel{def}{=} \tilde{C}_a(f), \quad f \in L_1(H),\ L_\infty(H). \end{aligned} \tag{3.24}$$

As a result, one can define

$$\mathbf{P}_a^{(\pm)} f \stackrel{def}{=} \mathbf{P}_a f, \quad f \in L_1(H'),\ L_\infty(H'), \tag{3.25}$$

$$\tilde{\mathbf{P}}_a^{(\pm)} f \stackrel{def}{=} \tilde{\mathbf{P}}_a f, \quad f \in L_1(H),\ L_\infty(H). \tag{3.26}$$

The main connections between the operators $\mathbf{A}$, $\tilde{\mathbf{A}}$ and $\mathbf{B_a}$, $\tilde{\mathbf{B}}_a$ are given in the next theorem.

Theorem 3.2. (i) *The operators introduced above satisfy the relations*

$$\mathbf{A}\mathbf{B}_a f = f, \quad \tilde{\mathbf{A}}\tilde{\mathbf{B}}_a f = f, \quad f \in L_1,\ L_\infty \tag{3.27}$$

and

$$\begin{aligned} \mathbf{B}_a \mathbf{A} f &= \mathbf{P}_a f, \quad f \in L_1(H'),\ L_\infty(H') \\ \tilde{\mathbf{B}}_a \tilde{\mathbf{A}} f &= \tilde{\mathbf{P}}_a f, \quad f \in L_1(H),\ L_\infty(H). \end{aligned} \tag{3.28}$$

(ii) *In the subspaces* $L_1(H)$, $L_\infty(H)$, *the adjoint operator* $\mathbf{B}_a^*$ *equals* $\tilde{\mathbf{A}}$, *i.e.*

$$(3.29)\qquad (\mathbf{B}_a f, g) \overset{def}{=} (f, \mathbf{B}_a^* g) = (f, \tilde{\mathbf{A}} g), \quad f \in L_1, L_\infty,\ g \in L_\infty(H), L_1(H).$$

Similarly, in the subspaces $L_1(H')$, $L_\infty(H')$, *the adjoint operator* $\tilde{\mathbf{B}}_a^*$ *equals* $\mathbf{A}$, *i.e.*

$$(3.30)\qquad (\tilde{\mathbf{B}}_a f, g) \overset{def}{=} (f, \tilde{\mathbf{B}}_a^* g) = (f, \mathbf{A} g), \quad f \in L_1, L_\infty,\ g \in L_\infty(H'), L_1(H').$$

Proof. Consider the equation

$$h(x) + 2G'(x)\int_a^x G^{-1}(s)h(s)\,ds = f(x).$$

By using the substitution $y(x) = \int_a^x G^{-1}(s)h(s)\,ds$ we obtain

$$y'(x) + 2G'(x)G^{-1}(x)y(x) = G^{-1}(x)f(x).$$

The general solution of this equation is

$$y(x) = CG^{-2}(x) + G^{-2}(x)\int_b^x G(s)f(s)\,ds.$$

Since $h(x) = G(x)y'(x)$, the last equality shows that any solution $h \in L_1(H')$ of the equation $\mathbf{A}h(x) = f(x)$ $(f \in L_1)$ is given by the expression $h(x) = CH(x) + \mathbf{B}_a f(x)$, $(C = \tilde{C}_a^{(\pm)}(h, H_a^{(\pm)'}))$. This is equivalent to the first formula in (3.28). The second formula follows from the first one in view of (3.26) and (3.12). The relations (3.27) can be verified directly. Finally, the proof of (3.29) results from

$$\int_{-\infty}^{\infty} g(x)(\int_a^x h(s)\,ds)\,dx = \int_{-\infty}^{\infty} h(x)(\int_x^{\infty} g(s)\,ds)\,dx.$$

where $f, g \in L_1$ and $\int_{-\infty}^{\infty} g(x)\,dx = 0$. □

The next Theorem 3.3 follows directly from Theorems 3.1-2.

Theorem 3.3. *For the formulas inverse to those in Theorem* 3.1 *we have the representations*

$$(3.31)\qquad \begin{aligned} &\mathbf{P}_a P(r;x,k) = -\mathbf{B}_a P(v;x,k), \quad &&\mathbf{P}_a Q(r;x,k) = -\mathbf{B}_a Q(v;x,k), k \in \mathbf{R}^+ \\ &\mathbf{P}_a P_j(r;x) = -\mathbf{B}_a P_j(v;x), &&\mathbf{P}_a Q_j(r;x) = -\mathbf{B}_a Q_j(v;x),\ k_j \in \sigma(v), \end{aligned}$$

with similar expressions for the system $\{P', Q'\}$*:* $\tilde{\mathbf{P}}_a P'(r; x, k) = -\tilde{\mathbf{B}}_a P'(v; x, k), \ldots$

4. Differential properties of the operators $\mathcal{G}(v)$ and $\mathcal{F}(r)$. Here we shall show that the operators $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}_a$ introduced in §3 naturally arise as functional derivatives of the operators $\mathcal{G}(v)$ and $\mathcal{F}(r)$. The main result is the following statement.

Theorem 4.1. (i) *At any point* $r \in \Omega(N+1)$*, the functional derivative of the operator* $\mathcal{F}(r)$ *is expressed via the operator* $\tilde{\mathbf{A}}^+$ (3.10) *as follows:*

$$\frac{d}{d\varepsilon}\mathcal{F}(r+\varepsilon f)|_{\varepsilon=0} \stackrel{def}{=} \mathcal{F}'(r)f = -\tilde{\mathbf{A}}^+ f(x) - 2M'(x)(P_0(r), f), \tag{4.1}$$

where

$$M(x) = H^{-1}(x)\int_x^\infty H(s)\,ds,\; P_0(x) = H(x).$$

(ii) *At any point* $v \in \Omega(N)$*, the derivative of the operator* $\mathcal{G}(v)$ *is:*

$$\frac{d}{d\varepsilon}\mathcal{G}(v+\varepsilon f)|_{\varepsilon=0} \stackrel{def}{=} \mathcal{G}'(v)f = -\tilde{\mathbf{B}}_a f(x) \tag{4.2}$$

and

$$\partial\mathcal{G}/\partial k_0 = k_0 Q_0'(r; x), \quad \partial\mathcal{G}/\partial\alpha = -2\alpha^{-1}P_0'(r; x), \tag{4.3}$$

where the operator $\tilde{\mathbf{B}}_a$ *is defined as in (3.20), and the function*

$$Q_0(r; x) = -2H(x)\int_a^x H^{-1}(s)\,ds + K_a H(x), \quad K_a = \frac{W(g(a,k_0), \dot{g}(a,k_0))}{2\tau_0 a(v; k_0)}. \tag{4.4}$$

(iii) *The null space of the operator* $\mathcal{F}'(r)$ *consists of the functions* $P_0'(r; x)$ *and* $Q_0'(r; x)$*, i.e.*

$$\mathcal{F}'(r)P_0'(r; x) = 0, \quad \mathcal{F}'(r)Q_0'(r; x) = 0. \tag{4.5}$$

Remark 4.1. As a result of (3.5) and (4.6), the function $Q_0(r; x)$ defined in (2.2) coincides with (4.4) because

$$m(r; x, k_0) = -(2\tau_0)^{-1}W(g(x,k_0), \dot{g}(x,k_0))h(x).$$

P r o o f o f T h e o r e m 4.1. Eq. (3.4) yields $\dfrac{d}{dx}\log h(x) = \dfrac{d}{dx}\log f^{(+)}(x)$, therefore

$$\mathcal{F}'(r)f = \frac{d}{d\varepsilon}\left\{r(x) + \varepsilon f(x) - 2\frac{d^2}{dx^2}\log f^{(+)}(r+\varepsilon f; x, k_0(r+\varepsilon f))\right\}\Bigg|_{\varepsilon=0}$$

$$= f(x) - 2\frac{d^2}{dx^2}\left(f^{(+)^{-1}}(r;x,k_0)\frac{\partial f^{(+)}(r;x,k_0)}{\partial r}f(x)\right)$$

$$-2\frac{d^2}{dx^2}\left(\frac{\dot f^{(+)}(r;x,k_0(r))}{f^{(+)}(r;x,k_0(r))}\right)\left(\frac{\partial k_0}{\partial r}, f\right).$$

Now, by using the identity

$$\frac{d}{dx}W(\dot y(x,k), z(x,k)) = 2ky(x,k)z(x,k) \tag{4.6}$$

we find $d(\dot f^{(+)}(r;x,k_0(r))/f^{(+)}(r;x,k_0(r))/dx = 2k_0 M(x)$ resulting in

$$-2(\partial k_0/\partial r, f)\frac{d^2}{dx^2}\frac{\dot f^{(+)}(r;x,k_0(r))}{f^{(+)}(r;x,k_0(r))} = -2M'(x)(P_0(r), f).$$

The formula

$$\frac{\partial f^{(+)}(r;x,k_0)}{\partial r}f(x) = f^{(+)}(r;x)\int_x^\infty H^{-1}(s)\int_s^\infty H(\xi)f(\xi)\,d\xi\,ds.$$

for the derivative $\partial f^{(+)}(r;x,k_0)/\partial r$ (see e.g. [10]) ends the proof of (4.1). The expression (4.2) can be derived in a similar way starting from

$$\frac{\partial g(v;x,k_0)}{\partial v}f(x) = g(x)\int_a^x H(s)\int_a^s H^{-1}(\xi)f(\xi)\,d\xi\,ds.$$

Directly from (1.11) ($g = g_\alpha(v;x) = g(v;x,k_0,\alpha)$) we obtain

$$\frac{\partial \mathcal{G}}{\partial \alpha} = -2\frac{d^2}{dx^2}\frac{\partial}{\partial\alpha}\log g_\alpha(v;x,k_0) = -2\frac{d^2}{dx^2}\frac{f^{(-)}(v;x)}{g_\alpha(v;x)}$$

$$= -2\frac{d}{dx}\frac{W(g_\alpha(v;x), f^{(-)}(v;x))}{g_\alpha^2(v;x)} = -2\alpha^{-1}P_0'(x),$$

since $W(g(v;x), f^{(-)}(v;x)) = -2ik_0 a(v;k_0)$. Now, the first formula in (4.3) results from

$$\frac{\partial \mathcal{G}}{\partial k_0} = -2\frac{d^2}{dx^2}\frac{\dot{g}(v;x,k_0)}{g^2(v;x,k_0)} = \frac{d}{dx}\frac{W(\dot{g}(v;x,k_0), g(v;x,k_0))}{g_\alpha^2(v;x,k_0)} = Q_0'(x).$$

Relations (4.5) can be verified directly by using (3.8). □

Remark 4.2. From (4.1) and Lemma 2.2 it follows that the formulas (3.16) in Theorem 3.1 can be written in the form

$$P'(v;x,k) = \mathcal{F}'(r)P'(r;x,k),\ Q'(v;x,k) = \mathcal{F}'(r)Q'(r;x,k),\ k \in \mathbf{R}^+,$$

$$P_j'(v;x) = \mathcal{F}'(r)P_j'(r;x),\ Q_j'(v;x) = \mathcal{F}'(r)Q_j'(r;x),\ k_j \in \sigma(v).$$

5. Transformation of the expansion formulas. We begin with the following statement.

Theorem 5.1. *Let the system* $\{P(r), Q(r)\}$ *corresponds to a potential* $r \in \Omega(N+1)$, *and let the expansion*

$$f(x) = \frac{1}{2\pi}\int_0^\infty \{P'(r;x,k)(f,Q(r;k)) - Q'(r;x,k)(f,P(r;k))\}\frac{dk}{\beta(r;k)}$$

$$-\sum_{j=0}^{N}\{P_j'(r;x)(f,Q_j(r)) - Q_j'(r;x)(f,P_j(r))\}$$

holds. Then if $v = \mathcal{F}(r)$, *we have also the expansion* (2.9) *where the system* $\{P(v), Q(v)\}$ *is constructed as in Theorem* 3.1.

Proof. If for brevity we rewrite (2.8) as

$$f(x) = \mathbf{EF}\{P'(v;x,k)(f,Q(v,k)),\ k \in \mathbf{R}^+ \cup \sigma(v)\}$$

then the expansion (5.1) takes the form

$$f(x) + P_0'(r;x)(f,Q_0(r)) - Q_0'(r;x)(f,P_0(r)) =$$

$$= \mathbf{EF}\{P'(r;x,k)(f,Q(r,k)),\ k \in \mathbf{R}^+ \cup \sigma(v)\}. \tag{5.2}$$

Now we apply the operator $\mathcal{F}'(v)$ to both sides of (5.2). Due to (4.5) and Theorem 3.1 (see also Remark 4.2) we have

$$\mathcal{F}'(v)f(x) = \mathbf{EF}\{P'(v;x,k)(f,Q(r,k)),\ k \in \mathbf{R}^+ \cup \sigma(v)\}. \tag{5.3}$$

Next, we replace $f(x)$ in (5.3) by $f(x) = -\tilde{\mathbf{B}}_a h(x)$ where $h(x)$ is a function in L_2^1. Eqs. (3.23) and (3.27) lead to

$$-\mathcal{F}'(v)\tilde{\mathbf{B}}_a h(x) = \tilde{\mathbf{A}}\tilde{\mathbf{B}}_a h(x) = h(x).$$

On the other hand, (3.30) and (3.15) result in

$$(\tilde{\mathbf{B}}_a h, P(r;k)) = (h, \mathbf{A}P(r;k)) = (h, P(v;k)), \dots$$

since $P(r,x,k) \in L_\infty(H'), \dots$ due to Lemma 2.2. □

The transformation inverse to the one in Theorem 5.1 is given in the next theorem.

Theorem 5.2. *Let us associate the expansion* (2.7) *with Eq.* (1.1) *where* $v \in \Omega(N)$, *and let* $r = \mathcal{G}(v)$. *Then we also have the expansion* (5.1) *where the system* $\{P(r),\, Q(r)\}$ *is constructed by using Theorem* 3.3 *and the functions* P_0, Q_0 *are defined as in Theorem* 4.1.

Proof. From Theorem 3.1 we directly obtain

$$P'(v;x,k)(f,Q(v;k))\beta^{-1}(v;k) = \tilde{\mathbf{A}}P'(r;x,k)(f,\mathbf{A}Q(r;k))\beta^{-1}(r;k)\dots.$$

Thus, the expansion (2.7) can be written in the form

$$f(x) = \mathbf{EF}\{\tilde{\mathbf{A}}P'(r;x,k)(f,\mathbf{A}Q(r;k))\, k \in \mathbf{R}^+ \cup \sigma(v)\}.$$

Now we substitute $f(x) = \tilde{\mathbf{A}}h(x)$ where $h \in L_1(H)$ and, in view of (3.14), we obtain

$$\tilde{\mathbf{A}}h(x) = \mathbf{EF}\{\tilde{\mathbf{A}}P'(r;x,k)(h,Q(r;k)),\; k \in \mathbf{R}^+ \cup \sigma(v)\}, h \in L_1(P_0). \tag{5.4}$$

Next, we apply the operator $\tilde{\mathbf{B}}_a$ to both sides of (5.4). As a result of (3.28) and Theorem 3.3 we obtain the equation

$$\tilde{\mathbf{P}}_a g = 0 \tag{5.5}$$

where

$$g(x) = h(x) + \mathbf{EF}\{P'(r;x,k)(h,Q(r;k)),\; k \in \mathbf{R}^+ \cup \sigma(v)\}, \quad h \in L_2^1(P_0).$$

Since $(P_0', Q_0) = 1$, the solution of equation (5.5) is $g(x) = (g, Q_0)P_0'(x)$ which, in view of Lemma 2.2, can be written in the form

$$h(x) = -(h, Q_0)P_0'(r;x) + \mathbf{EF}\{P'(r;x,k)(h,Q(r;k)),\; k \in \mathbf{R}^+ \cup \sigma(v)\}. \tag{5.6}$$

Now, if we make the substitution $h(x) = f(x) - Q_0'(x)(f, P_0)$ in (5.6) where $f \in L_2^1$ then, by using the biorthogonality relations from Lemma 2.2, we obtain (5.1). $\square$

Lemma 5.3. *If $v = \mathcal{F}(r)$ then the operators $L(v)$ and $L(r)$ are related through*

$$L(v)f = \mathbf{A}L(r)\mathbf{B}_a f, \quad f \in L_2^1. \tag{5.7}$$

Proof. If we substitute the representation (3.31) in $L(r)P(r; x, k) = k^2 P(r; x, k)$ (cf. (2.7)) we obtain

$$-L(r)\mathbf{B}_a P(v; x, k) + L(r)H(x)C_a(P(r; k))$$

$$= -k^2 \mathbf{B}_a P(v; x, k) + k_0^2 H(x) C_a(P(r; k))$$

where $C_a(P(r; k))$ is as in (3.24). Now we apply the operator $\mathbf{A}$ to both sides getting as a result

$$\mathbf{A}L(r)\mathbf{B}_a P(v; x, k) = k^2 P(v; x, k) = L(v)P(v; x, k),$$

due to (3.12) and (3.27). In the same way, we derive analogous relations for the other functions of $\{P(v), Q(v)\}$.

Now it remains to use the expansion (2.9) in order to show that (5.7) takes place for any function $f(x) \in L_2^1$. $\square$

6. On the transformation of the KdV equation.

In the present section, we use the results obtained so far with $v \in \mathcal{S}$ in connection with applying the scheme to the KdV equation.

Lemma 6.1. *Let $v = \mathcal{F}(r)$, and $r \in \Omega(N + 1)$. Then we have*

$$\mathbf{A}r(x) = -v(x). \tag{6.1}$$

Proof. Let us recall that for any potential $r \in \Omega(N + 1)$ we have the representation, usually called trace formula (see e.g. [6, 9, 11]):

$$r(x) = \frac{2}{\pi i} \int_0^\infty P(r; x, k)\, dk + \sum_{j=0}^{N} 4ik_j P_j(r; x). \tag{6.2}$$

By applying the operator $\mathbf{A}$ to both sides of (6.2) and using the representations (3.15) and Eq.(3.12) we obtain (6.1). $\square$

Now, let us consider the KdV equation

$$r_t = 6rr_x - r_{xxx} \equiv 4DL(r)r, \quad r(x,t) \in \Omega(N+1). \tag{6.3}$$

Here, as in (1.7), we suppose that $r(x,.) \in \mathcal{S}$. The next theorem is known as Bäcklund transformation for the KdV equation (see, e.g. [11, 16], for a detailed bibliography).

Theorem 6.1. *If the solution $r = r(x,t)$ of* (6.3) *is transformed according to $v(x,t) = \mathcal{F}(r(x,t))$ then the function $v = v(x,t)$ is also a solution of the KdV equation* (1.7) *(with $v(x,t) \in \Omega(N)$). In the opposite direction,* (6.3) *with $r = G(v)$ follows from* (1.7) *if and only if*

$$\alpha_t + 8ik_0^3\alpha = 0 \tag{6.4}$$

where $\lambda_0(r(t)) = \lambda_0(r(0))$.

P r o o f. Recall that if r is a solution of (6.3) then $\lambda_0(r(t)) = \lambda_0(r(0))$ and, therefore, we have $(r_t, P_0(r;t)) = 0$. Also, note that (6.2) leads to $r \in L_1(H')$. As a result, by using (3.28) we get

$$r(x) = -\mathbf{B}_a v(x) + H(x)C_a(r).$$

Now from (4.1) and (6.3) we find

$$v_t|_{\lambda(r(t))=Const.} = -\tilde{\mathbf{A}}r_t = -4\tilde{\mathbf{A}}DL(r)r = 4DL(v)v$$

where, in the last equality, we also used (5.7).

It remains to show that, starting from Eq.(1.7) with $v \in \Omega(N)$, one can obtain (6.3). From the transformations (1.10) and (1.11),

$$\mathcal{G}(\alpha(t), k_0, v(t)) = r(t), \quad v(t) = \mathcal{F}(r(t); k_0), \tag{6.5}$$

where the time t is considered as a parameter we obtain

$$\mathcal{G}(\alpha(t), k_0, \mathcal{F}(r(t); k_0)) = r(t). \tag{6.6}$$

Then, by differentiating (6.6) with respect to t we find

$$\frac{\partial \mathcal{G}}{\partial \alpha}\alpha_t + \frac{\partial \mathcal{G}}{\partial v}\frac{\partial \mathcal{F}}{\partial r}r_t = r_t. \tag{6.7}$$

Now we use Theorem 4.1 to rewrite (6.7) in the form

$$-\frac{2}{\alpha}\alpha_t P_0'(r;x,t) + \tilde{\mathbf{B}}_a\tilde{\mathbf{A}}r_t(x,t) = r_t(x,t). \tag{6.8}$$

The condition $k_0(t) = k_0(0)$ leads to $(r_t(t), P_0(r(t)) = 0$ which means that $r_t(x,t) \in L_1(H)$. As a result, we can apply (3.28) with $f = r_t(x,t)$, thus obtaining $-2\alpha^{-1}\alpha_t = \tilde{C}_a(r_t)$ from (6.8). The dependence on a in the right hand side indicates only that α_t is arbitrary. In order to obtain the equation for $\alpha(t)$ we proceed as follows. We differentiate the first equality in (6.5) with respect to t and find, in view of Theorem 4.1 and Eq.(1.7), that

$$r_t(x,t) = \frac{\partial \mathcal{G}}{\partial \alpha}\alpha_t(t) + \frac{\partial \mathcal{G}}{\partial v}v_t(x,t) = P_0(r(t);x)\tilde{C}_a(r_t) - 4\tilde{\mathbf{B}}_a DL(v)v(x,t).$$

This equation leads to $\tilde{\mathbf{P}}_a(g) = 0$, $g = r_t - 4DL(r)r$ due to the transformations (5.7), (6.1) and (3.28). Its solution $g(x,t)$ is $g(x,t) = (g(t), Q_0(r(t)))\ P_0'(r(t);x)$ as pointed out in the proof of Theorem 5.2. As a result, $r(x,t)$ is a solution of (6.3) if $(r_t, Q_0((r(t)) - 4(DL(r)r, Q_0((r(t)) = 0$. This is equivalent to the condition (6.4) in view of (2.5) and (6.2). Note that Lemma 3.3 yields $C_0(r) = \alpha$. Now it is clear from (1.9) that (6.4) has to hold. □

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