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A BASIS FOR $\mathbb{Z}$-GRADED IDENTITIES OF MATRICES OVER INFINITE FIELDS

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Abstract. The algebra $M_n(K)$ of the matrices $n \times n$ over a field K can be regarded as a $\mathbb{Z}$-graded algebra. In this paper, it is proved that if K is an infinite field, all the $\mathbb{Z}$-graded polynomial identities of $M_n(K)$ follow from the identities:

$$\begin{array}{rcl} x & = & 0, \quad |\alpha(x)| \geq n, \\ xy & = & yx, \quad \alpha(x) = \alpha(y) = 0, \\ xyz & = & zyx, \quad \alpha(x) = -\alpha(y) = \alpha(z), \end{array}$$

where α is the degree of the corresponding variable. This is a generalization of a result of Vasilovsky about the $\mathbb{Z}$-graded identities of the algebra $M_n(K)$ over fields of characteristic 0.

Introduction. Let us denote by $M_n(K)$ the algebra of all square matrices of order n over a field K. The polynomial identities of the algebra $M_n(K)$

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play an important role in the theory of PI-algebras. For fields of characteristic zero, Razmyslov [14] described a finite basis for the identities of $M_2(K)$ (this result was improved by Drensky [5], who found a minimal basis of these identities). When K is a finite field, Maltsev and Kuzmin [12] found a basis of two identities for $M_2(K)$. Koshlukov [10] described a finite basis of the identities of $M_2(K)$, when K is an infinite field of characteristic $p > 2$. Recall that in [7, 8] finite bases for the identities of $M_3(K)$ and $M_4(K)$ were described when K is a finite field.

However, the problem of finding an explicit finite basis for the identities of the algebra $M_n(K)$, for $n \geq 3$ and K an infinite field, still has no solution even in the case of characteristic 0. Hence one is led to study other types of polynomial identities such as weak identities, identities with trace, graded identities etc. Thus for example the trace identities of the algebra $M_n(K)$ over a field of characteristic 0 were described by Procesi [13] and by Razmyslov [15]. The interest in the study of graded identities is justified by the relationship between the graded and ordinary polynomial identities which is one of the key components in the structure theory of T-ideals developed by Kemer, see for an account [9].

Although in positive characteristic there does not exist such relationship, the graded identities are still of interest, see for example [2, 3].

Let $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{N}$ denote the sets of the integers, the integers modulo n and the positive integers respectively. The algebra $M_n(K)$ can be equipped with a natural $\mathbb{Z}_n$-grading. When the characteristic of the field K equals 0, Di Vincenzo [4] described a finite basis for the $\mathbb{Z}_2$-graded polynomial identities of $M_2(K)$. This basis consists of two graded identities, namely $y_1y_2 = y_2y_1$ and $z_1z_2z_3 = z_3z_2z_1$ for y_i being even and z_i odd variables. Still in characteristic 0, Vasilovsky [17] found an explicit finite basis for the $\mathbb{Z}_n$-graded polynomial identities of $M_n(K)$ for every n. This last result holds for K an infinite field, see [1].

The algebra $M_n(K)$ has also a $\mathbb{Z}$-grading. When K is of characteristic 0, Vasilovsky [16] found a basis for the $\mathbb{Z}$-graded identities. In this paper, we prove that the result of Vasilovsky also holds for infinite fields. Our methods are similar to those of [1] and [17].

From now on, let K be an infinite field. The main theorem we prove is the following.

Theorem 1. *All graded polynomial identities of the $\mathbb{Z}$-graded algebra $M_n(K)$ follow from*

$$\begin{aligned} x &= 0, \quad |\alpha(x)| \geq n, \\ xy &= yx, \quad \alpha(x) = \alpha(y) = 0, \\ xyz &= zyx, \quad \alpha(x) = -\alpha(y) = \alpha(z), \end{aligned}$$

where α is the degree of the corresponding variable.

1. Definitions and preliminary results. A $\mathbb{Z}$-graded algebra $A = \sum_{\alpha\in\mathbb{Z}} A_\alpha$ is an associative algebra that can be expressed as the direct sum of the subspaces $\{A_\alpha \mid \alpha \in \mathbb{Z}\}$ of A such that $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. Further, $\mathbb{Z}$-graded homomorphisms, subalgebras, ideals and so on, are defined in the usual way. Sometimes the adjective homogeneous is used instead of $\mathbb{Z}$-graded.

Denote by e_{ij} the matrix units, i.e. the matrices whose only non-zero entry is 1 in the ith row and jth column. For $\alpha \in \mathbb{Z}$, let $M_n(K)_\alpha$ be the subspace of $M_n(K)$ spanned by all matrix units e_{ij} such that $j - i = \alpha$. Thus $M_n(K)_0$ consists of the diagonal matrices

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

with $a_1, a_2, \ldots, a_n \in K$; for $1 \le \alpha \le n-1$, $M_n(K)_\alpha$ consists of the matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & a_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-\alpha} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $a_1, a_2, \ldots, a_{n-\alpha} \in K$, while $M_n(K)_{-\alpha}$ consists of the matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-\alpha} & 0 & \cdots & 0 \end{pmatrix}$$

where $a_1, a_2, \ldots, a_{n-\alpha} \in K$. Finally $M_n(K)_\alpha = 0$ for $|\alpha| \ge n$. Since $e_{ij}e_{jl} = e_{il}$ and $e_{ij}e_{kl} = 0$ if $j \ne k$, it follows that $M_n(K)_\alpha M_n(K)_\beta \subseteq M_n(K)_{\alpha+\beta}$ for α and β in $\mathbb{Z}$, so the decomposition above defines a $\mathbb{Z}$-grading for the algebra $M_n(K)$.

Let $\Omega = K[y_i^{(k)} \mid i \in \mathbb{N}, 1 \le k \le n]$ be the commutative polynomial algebra generated by the variables $y_i^{(k)}$. Since the algebra $M_n(K) \otimes \Omega$ is isomorphic to $M_n(\Omega)$ and has a $\mathbb{Z}$-grading given by $(M_n(K) \otimes \Omega)_\alpha = M_n(K)_\alpha \otimes \Omega$, we can define in a natural way a $\mathbb{Z}$-grading for the algebra $M_n(\Omega)$. More exactly, the following decomposition is a $\mathbb{Z}$-grading for $M_n(\Omega)$. If $0 \leq \alpha \leq n-1$ then $M_n(\Omega)_\alpha$ consists of all matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 & f_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & f_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & f_{n-\alpha} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $f_1, \ldots, f_{n-\alpha} \in \Omega$, analogously $M_n(\Omega)_{-\alpha}$ consists of the matrices of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ f_1 & 0 & \cdots & 0 & 0 & \cdots & \\ 0 & f_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f_{n-\alpha} & 0 & \cdots & 0 \end{pmatrix}$$

where $f_1, \ldots, f_{n-\alpha} \in \Omega$, and if $|\alpha| \geq n$ then $M_n(\Omega)_\alpha = 0$.

Let $X = \{x_i \mid i \in \mathbb{N}\}$ be a set of variables and let $\{X_\alpha \mid \alpha \in \mathbb{Z}\}$ be a family of disjoint countable subsets of X such that $X = \cup_{\alpha \in \mathbb{Z}} X_\alpha$. A variable $x \in X$ is of homogeneous degree α, written $\alpha(x) = \alpha$, if $x \in X_\alpha$. Denote by $K\langle X \rangle$ the free associative algebra freely generated over K by the set X. We define a $\mathbb{Z}$-grading in $K\langle X \rangle$. The monomials

$$\{x_{i_1} x_{i_2} \ldots x_{i_k} \mid k \geq 1; x_{i_1}, x_{i_2}, \ldots, x_{i_k} \in X\}$$

form a basis of $K\langle X \rangle$ as a vector space. The homogeneous degree of a monomial $\mathsf{m} = x_{i_1} x_{i_2} \ldots x_{i_k}$ is $\alpha(\mathsf{m}) = \alpha(x_{i_1}) + \alpha(x_{i_2}) + \ldots + \alpha(x_{i_k})$. For $\alpha \in \mathbb{Z}$, denote by $K\langle X \rangle_\alpha$ the subspace of $K\langle X \rangle$ spanned by all monomials of homogeneous degree α. Notice that $K\langle X \rangle_\alpha K\langle X \rangle_\beta \subseteq K\langle X \rangle_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}$. Therefore this decomposition defines a $\mathbb{Z}$-grading of the algebra $K\langle X \rangle$. An ideal I of $K\langle X \rangle$ is said to be a $\mathbb{Z}$-ideal if it is invariant under all $\mathbb{Z}$-graded endomorphisms of $K\langle X \rangle$, i.e. $\phi(I) \subseteq I$ for every $\mathbb{Z}$-graded homomorphism $\phi : K\langle X \rangle \to K\langle X \rangle$.

Let $A = \sum_{\alpha \in \mathbb{Z}} A_\alpha$ be a $\mathbb{Z}$-graded algebra. A polynomial $f(x_1, \ldots, x_m)$, or the expression $f(x_1, \ldots, x_m) = 0$, is called a graded polynomial identity of the

$\mathbb{Z}$-graded algebra A if $f(a_1, \dots, a_m) = 0$ for all $a_1, \dots, a_m \in \cup_{\alpha \in \mathbb{Z}} A_\alpha$ such that $a_i \in A_{\alpha(x_i)}$, $i = 1, \dots, m$. The set $T_{\mathbb{Z}}(A)$ of all graded identities of a $\mathbb{Z}$-graded algebra A is a $T_{\mathbb{Z}}$-ideal of $K\langle X \rangle$.

It is well known that the generic matrix algebra of order n is isomorphic to the relatively free algebra $K\langle X \rangle / T(M_n(K))$ of the $n \times n$ matrix variety (see for example Section 7.2 in [6], pp. 86-87). We shall use a similar idea for graded algebras. Denote by F the $\mathbb{Z}$-graded subalgebra of $M_n(\Omega)$ generated by the matrices

$$A_i = \begin{pmatrix} 0 & \cdots & 0 & y_i^{(1)} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & y_i^{(2)} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & y_i^{(n-\alpha(x_i))} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

when $0 \leq \alpha(x_i) \leq n-1$,

$$A_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ y_i^{(1)} & 0 & \cdots & 0 & 0 & \cdots & \\ 0 & y_i^{(2)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & y_i^{(n+\alpha(x_i))} & 0 & \cdots & 0 \end{pmatrix}$$

when $-n+1 \leq \alpha(x_i) \leq -1$, and $A_i = 0$ when $|\alpha(x_i)| \geq n$.

Lemma 1. *The relatively free $\mathbb{Z}$-graded algebra $K\langle X \rangle / T_{\mathbb{Z}}(M_n(K))$ is isomorphic to the algebra F.*

Proof. The proof is analogous to that for the generic matrices. The map $\phi : K\langle X \rangle \to F$ defined by $\phi(f(x_1, \dots, x_m)) = f(A_1, \dots, A_m)$ is a $\mathbb{Z}$-graded homomorphism. Clearly, ϕ is onto. Moreover, an easy calculation shows that $\ker \phi = T_{\mathbb{Z}}(M_n(K))$ and ϕ induces an isomorphism, as required. $\square$

Thus we can work in the graded algebra F instead of the graded algebra $K\langle X \rangle / T_{\mathbb{Z}}(M_n(K))$.

Let I be the ideal of the $\mathbb{Z}$-graded identities of $K\langle X \rangle$ generated by the graded identities

$$x = 0, \quad |\alpha(x)| \geq n,$$

$$\begin{aligned} xy &= yx, \quad \alpha(x) = \alpha(y) = 0, \\ xyz &= zyx, \quad \alpha(x) = -\alpha(y) = \alpha(z), \end{aligned}$$

where x, y, z are variable of X.

Lemma 2. *The $\mathbb{Z}$-graded algebra $M_n(K)$ satisfies every graded identity from the $T_{\mathbb{Z}}$-ideal I.*

P r o o f. Since $M_n(K)_\alpha = 0$ whenever $|\alpha| \geq n$, $M_n(K)$ satisfies the graded identity $x = 0$ for $|\alpha(x)| \geq n$. Two diagonal matrices commute, therefore the graded identity $xy = yx$ with $\alpha(x) = \alpha(y) = 0$ holds in $M_n(K)$. As the identity $xyz = zyx$ with $\alpha(x) = -\alpha(y) = \alpha(z)$ is multilinear, it is sufficient to prove that it holds for $x = e_{ij} \in M_n(K)_\alpha$, $y = e_{rs} \in M_n(K)_{-\alpha}$ and $z = e_{kl} \in M_n(K)_\alpha$, with $|\alpha| \leq n - 1$. Obverse that $e_{ij}e_{rs}e_{kl} \neq 0$ if and only if $j = r$ and $s = k$; in this case $i = j - \alpha = r - \alpha = s = k$ and $j = i + \alpha = k + \alpha = l$. Hence $e_{ij}e_{rs}e_{kl} \neq 0$ if and only if $i = s = k$ and $j = r = l$. Similarly, we have that $e_{kl}e_{rs}e_{ij} \neq 0$ if and only if $k = s = i$ and $l = r = j$. Therefore, if $e_{ij}e_{rs}e_{kl} \neq 0$ then $e_{ij}e_{rs}e_{kl} = e_{il} = e_{kj} = e_{kl}e_{rs}e_{ij}$, else $e_{ij}e_{rs}e_{kl} = 0 = e_{kl}e_{rs}e_{ij}$. □

Lemma 3. *Let $\mathsf{m} = x_{i_1} \dots x_{i_q}$ be a monomial of $\mathbb{Z}$-degree α. If $A_{i_1} \dots A_{i_q} \neq 0$ then there exist $1 \leq s \leq t \leq n \in \mathbb{N}$ such that $A_{i_1} \dots A_{i_q} = \sum_{i=s}^{t} \omega_i e_{i,i+\alpha}$ where $\omega = y_{i_1}^{(h_{1,i})} \dots y_{i_q}^{(h_{q,i})}$ and $h_{j,i+1} = h_{j,i} + 1$ for all $s \leq i \leq t-1$, $1 \leq j \leq q$.*

P r o o f. We shall use induction on q. If $q = 1$, obviously we have the result. If $q > 1$, applying the hypothesis of induction to the monomial $x_{i_l} \dots x_{i_{q-1}}$ and multiplying the matrices $A_{i_1} \dots A_{i_{q-1}}$ and A_{i_q} we can conclude the proof. □

Lemma 4. *Let $\mathsf{m}(x_1, \dots, x_m)$ be a monomial of $K\langle X\rangle$. If $\mathsf{m} = 0$ is a $\mathbb{Z}$-graded polynomial identity of $M_n(K)$ then m lies in the ideal I.*

P r o o f. This result is true for multilinear monomials and its proof is the same as of Corollary 4 in [16] (that reasoning holds for any characteristic).

If $\mathsf{m} = x_{i_1} \dots x_{i_q}$, let $\mathsf{n} = x_{j_1} \dots x_{j_q}$ be a multilinear monomial such that $\alpha(x_{j_k}) = \alpha(x_{i_k})$. Each entry of the matrix $A_{i_1} \dots A_{i_q}$ is either 0 or a monomial of the form $y_{i_1}^{(\alpha_1)} \dots y_{i_q}^{(\alpha_q)}$ for some $\alpha_1, \dots, \alpha_q \in \{1, \dots, n\}$. The matrices A_{i_k} and A_{j_k} have zero at the same positions, and at a determined position the matrix A_{i_k} has $y_{i_k}^{(\alpha)}$ if and only if the matrix A_{j_k} has $y_{j_k}^{(\alpha)}$. Therefore, where the matrix $A_{i_1} \dots A_{i_q}$ has 0 the matrix $A_{j_1} \dots A_{j_q}$ has 0 too, and where the matrix $A_{i_1} \dots A_{i_q}$ has a monomial $y_{i_1}^{(\alpha_1)} \dots y_{i_q}^{(\alpha_q)}$ the matrix $A_{j_1} \dots A_{j_q}$ has the monomial $y_{j_1}^{(\alpha_1)} \dots y_{j_q}^{(\alpha_q)}$. Since $\mathsf{m} \in T_{\mathbb{Z}}(M_n(K)) = T_{\mathbb{Z}}(F)$ (Lemma 1), we have that $A_{i_1} \dots A_{i_q} = 0$ which implies $A_{j_1} \dots A_{j_q} = 0$. Hence $\mathsf{n} = 0$ is a $\mathbb{Z}$-graded polynomial identity of $M_n(K)$ and $\mathsf{n} \in I$. Substituting the variables $x_{j_k} \mapsto x_{i_k}$ it follows that $\mathsf{m} \in I$, because I is a $T_{\mathbb{Z}}$-ideal. □

Lemma 5. *Let $\mathsf{m}(x_1,\dots,x_m)$ and $\mathsf{n}(x_1,\dots,x_m)$ be two monomials of $K\langle X\rangle$. If the matrices $\mathsf{m}(A_1,\dots,A_m)$ and $\mathsf{n}(A_1,\dots,A_m)$ have at the same position the same non-zero entry then $\mathsf{m}(x_1,\dots,x_m)\equiv\mathsf{n}(x_1,\dots,x_m)(\mathrm{mod} I)$.*

Proof. Let (h,k) be the position where the matrices $\mathsf{m}(A_1,\dots,A_m)$ and $\mathsf{n}(A_1,\dots,A_m)$ have the same non-zero entry. Let q be the length of m. We shall use induction on q. If $q=1$, the result is obviously true. Now suppose $q>1$.

Suppose that x_p is a variable of $\mathsf{m}(x_1,\dots,x_m)$ and m_1 and m_2 are two monomials of $K\langle X\rangle$ such that $\mathsf{m}=\mathsf{m}_1x_p\mathsf{m}_2$. Denote $r=\alpha(\mathsf{m}_1)$, $s=\alpha(x_p)$, $t=\alpha(\mathsf{m}_2)$. Then (by Lemma 3) the (h,k)-entry in $\mathsf{m}(A_1,\dots,A_m)$ is obtained from the product

$$(\omega'_h e_{h,h+r})y_p^{(i)}e_{h+r,k-t}(\omega''_{k-t}e_{k-t,k})$$

where $i=h+r$ if $\alpha(x_p)\geq 0$, $i=k-t$ otherwise. Hence x_p occurs in n, and A_p in $\mathsf{n}(A_1,\dots,A_m)$. Notice that for every non-zero product $B:=A_{j_1}\dots A_{j_q}$ of the generic matrices, each matrix A_l contributes to a non-zero entry of B exactly once with one suitable variable, namely y_l^e. Therefore by the assumption there exist subwords n_1, n_2 of n such that $\mathsf{n}=\mathsf{n}_1x_p\mathsf{n}_2$ and A_p contributes with $y_p^{(i)}$ to the computation of the (h,k)-entry of $\mathsf{n}(A_1,\dots,A_m)$. (Observe that if $\mathsf{n}_1=1$ then $\mathsf{n}_1(A_1,\dots,A_m)$ is the identity matrix.) Using Lemma 3 once again, we know that the (h,k)-entry in $\mathsf{n}(A_1,\dots,A_m)$ is obtained from the product

$$(\eta'_h e_{h,h+r})y_p^{(i)}e_{h+r,k-t}(\eta''_{k-t}e_{k-t,k})$$

where η'_h is the $(h,h+r)$-entry of $\mathsf{n}_1(A_1,\dots,A_m)$ and η''_{k-t} is the $(k-t,k)$-entry of $\mathsf{n}_2(A_1,\dots,A_m)$. Observe that η'_h is not zero because the (h,k)-entry of $\mathsf{n}(A_1,\dots,A_m)$ is not zero; thus $n_1(A_1,\dots,A_m)$ has a non-zero entry at the position $(h,h+r)$. Hence the $\mathbb{Z}$-degree of $\mathsf{n}_1(A_1,\dots,A_m)$ in F is r and $\alpha(\mathsf{n}_1)=r=\alpha(\mathsf{m}_1)$. Therefore we can conclude that if x_p is a variable of $\mathsf{m}(x_1,\dots,x_m)$ and $\mathsf{m}_1,\dots,\mathsf{m}_l$ are monomials of $K\langle X\rangle$ such that $\mathsf{m}=\mathsf{m}_1x_p\mathsf{m}_2x_p\mathsf{m}_3\dots\mathsf{m}_{l-1}x_p\mathsf{m}_l$, then there exist monomials $\mathsf{n}_1,\dots,\mathsf{n}_l$ in $K\langle X\rangle$ and a bijection (1-1 correspondence) $\varphi:\{1,\dots,l\}\to\{1,\dots,l\}$ such that $\mathsf{n}=\mathsf{n}_1x_p\mathsf{n}_2x_p\mathsf{n}_3\dots\mathsf{n}_{l-1}x_p\mathsf{n}_l$ and $\alpha(\mathsf{m}_1x_p\mathsf{m}_2\dots\mathsf{m}_t)=\alpha(\mathsf{n}_1x_p\mathsf{n}_2\dots\mathsf{n}_{\varphi(t)})$.

We will show that there exist monomials w_1, w_2 such that $\mathsf{m}\equiv\mathsf{w}_1(\mathrm{mod}\ I)$, $\mathsf{n}\equiv\mathsf{w}_2(\mathrm{mod}\, I)$ and w_1, w_2 have the same starting variable. Let x_i be the first variable of m. Hence there exist two monomials n_1 and n_2 of $K\langle X\rangle$ such that $\mathsf{n}=\mathsf{n}_1x_i\mathsf{n}_2$ and $\alpha(\mathsf{n}_1)=0$. We have three possible cases:

Case 1. There exist two monomials m_1, m_2 such that $\mathsf{m}=x_i\mathsf{m}_1x_i\mathsf{m}_2$ and $\alpha(x_i\mathsf{m}_1)=0$. Then there exist three monomials n_3, n_4, n_5 in $K\langle X\rangle$ such that $\mathsf{n}=\mathsf{n}_3x_i\mathsf{n}_4x_i\mathsf{n}_5$ and $\alpha(\mathsf{n}_3)=\alpha(\mathsf{n}_3x_i\mathsf{n}_4)=0$. Hence $\alpha(x_i\mathsf{n}_4)=0$ and therefore $\mathsf{n}\equiv x_i\mathsf{n}_4\mathsf{n}_3x_i\mathsf{n}_5(\mathrm{mod} I)$.

Case 2. There exist two variables x_a and x_b, and six monomials m_1, m_2, n_3, n_4, n_5, n_6 such that $\mathsf{m}=\mathsf{m}_1x_ax_b\mathsf{m}_2$, $\mathsf{n}=\mathsf{n}_3x_a\mathsf{n}_4x_i\mathsf{n}_5x_b\mathsf{n}_6$, $\mathsf{n}_1=\mathsf{n}_3x_a\mathsf{n}_4$,

$\alpha(\mathsf{m}_1) = \alpha(\mathsf{n}_3)$ and $\alpha(\mathsf{m}_1 x_a) = \alpha(\mathsf{n}_3 x_a \mathsf{n}_4 x_i \mathsf{n}_5)$. Then an easy calculation gives us that $\alpha(\mathsf{n}_4 x_i \mathsf{n}_5) = 0$. Since $\alpha(\mathsf{n}_3 x_a \mathsf{n}_4) = \alpha(\mathsf{n}_1) = 0$, we may conclude that $\alpha(\mathsf{n}_3 x_a) = -\alpha(\mathsf{n}_4) = \alpha(x_i \mathsf{n}_5)$ and $\mathsf{n} \equiv x_i \mathsf{n}_5 \mathsf{n}_4 \mathsf{n}_3 x_a x_b \mathsf{n}_6 (\mathrm{mod} I)$.

Case 3. Neither of the previous cases holds. Consider $\mathsf{m} = x_{i_1} \dots x_{i_q}$. Let x_l be a variable occuring in n_1, that is $\mathsf{n}_1 = \mathsf{n}_3 x_l \mathsf{n}_4$. Then there exist $r \in \{1, \dots, q\}$ such that $x_l = x_{i_r}$ and $\alpha(\mathsf{n}_3) = \alpha(x_{i_1} \dots x_{i_{r-1}})$. Assume that $r \neq q$ and let n_5 and n_6 be two monomials such that $\mathsf{n} = \mathsf{n}_5 x_{i_{r+1}} \mathsf{n}_6$ and $\alpha(\mathsf{n}_5) = \alpha(x_{i_1} \dots x_{i_r})$. Then the length of n_5 is smaller than the length of n_1, lest the previous cases happen (if the length of n_5 equals the length of n_1 then it falls into case 1, if it is larger it falls into case 2). Thus $x_{i_{r+1}}$ appears in n_1 too. We conclude that there exists $r_0 \in \{1, \dots, q\}$ such that the monomials n_1 and $x_{i_{r_0}} x_{i_{r_0+1}} \dots x_{i_q}$ are multihomogeneous with respect to the $\mathbb{Z}$-degree. Let x_j be the first variable of n, then there exist m_3, m_4, m_5 monomials such that $\mathsf{m} = \mathsf{m}_3 \mathsf{m}_4 x_j \mathsf{m}_5$, $\alpha(\mathsf{m}_3 \mathsf{m}_4) = 0$ and $\mathsf{m}_4 x_j \mathsf{m}_5 = x_{i_{r_0}} x_{i_{r_0+1}} \dots x_{i_q}$. Therefore $\alpha(\mathsf{m}_4 x_j \mathsf{m}_5) = \alpha(x_{i_{r_0}} x_{i_{r_0+1}} \dots x_{i_q}) = \alpha(\mathsf{n}_1) = 0$. Then it follows that $\alpha(\mathsf{m}_3) = -\alpha(\mathsf{m}_4) = \alpha(x_j \mathsf{m}_5)$ and $\mathsf{m} \equiv x_j \mathsf{m}_5 \mathsf{m}_4 \mathsf{m}_3 (\mathrm{mod} I)$ which starts as n.

Now let x be the first variable of w_1 and w_2, and let w_1' and w_2' be two monomials such that $\mathsf{w}_1 = x\mathsf{w}_1'$ and $\mathsf{w}_2 = x\mathsf{w}_2'$. Since $\mathsf{m} - \mathsf{w}_1$ and $\mathsf{n} - \mathsf{w}_2$ belong to $I \subseteq T_{\mathbb{Z}}(M_n(K)) = T_{\mathbb{Z}}(F)$, we have $\mathsf{m}(A_1, \dots, A_m) = \mathsf{w}_1(A_1, \dots, A_m)$ and $\mathsf{n}(A_1, \dots, A_m) = \mathsf{w}_2(A_1, \dots, A_m)$. Then the matrices $\mathsf{w}_1'(A_1, \dots, A_m)$ and $\mathsf{w}_2'(A_1, \dots, A_m)$ have at the same position the same non-zero entry, because the same thing happens with $\mathsf{w}_1(A_1, \dots, A_m)$ and $\mathsf{w}_2(A_1, \dots, A_m)$. By the hypothesis of induction, we have $\mathsf{w}_1' \equiv \mathsf{w}_2' (\mathrm{mod} I)$, therefore $\mathsf{w}_1 \equiv \mathsf{w}_2 (\mathrm{mod} I)$, which concludes the proof. □

2. Proof of the theorem. By Lemma 2, we know that $I \subseteq T_{\mathbb{Z}}(M_n(K))$. Thus it is enough to show the other inclusion. Since the field K is infinite, a standard Vandermonde argument shows that every ideal of graded polynomial identities is generated by its multihomogeneous elements (see for example Proposition 4.2.3 in [6], pp. 39–40). Hence we need only prove that an arbitrary multihomogeneous graded polynomial identity $f(x_1, \dots, x_m) = 0$ of $M_n(K)$ lies in I. Let r be the least non-negative integer such that the polynomial f can be expressed modulo I as a linear combination of r multihomogeneous monomials:

$$f \equiv \sum_{q=1}^{r} a_q \mathsf{m}_q (\mathrm{mod} I)$$

where $0 \neq a_q \in K$, $\mathsf{m}_1, \mathsf{m}_2, \dots, \mathsf{m}_r \in K\langle X \rangle$. We shall show that $r = 0$. Suppose, on the contrary, $r > 0$. By Lemma 1, we have $f \in T_{\mathbb{Z}}(F)$. Since

$$a_1\mathsf{m}_1(A_1,\ldots,A_m) = -\sum_{q=2}^{r} a_q\mathsf{m}_q(A_1,\ldots,A_m)$$

it follows that there exists $p \in \{2,3,\ldots,r\}$ such that $\mathsf{m}_1(A_1,\ldots,A_m)$ and $\mathsf{m}_p(A_1,\ldots,A_m)$ have at the same position the same non-zero entry. (Observe that $\mathsf{m}_1(A_1,\ldots,A_m) \neq 0$, because otherwise by Lemma 4 $\mathsf{m}_1 \in I$.) Then, by Lemma 5, $\mathsf{m}_1 \equiv \mathsf{m}_p(\mathrm{mod} I)$ and

$$f \equiv (a_1 + a_p)\mathsf{m}_1 + \sum_{q=2}^{p-1} a_q\mathsf{m}_q + \sum_{q=p+1}^{r} a_q\mathsf{m}_q(\mathrm{mod} I).$$

Therefore f can be expressed modulo I as a linear combination of no more than $r-1$ multihomogeneous monomials, which contradicts our choice of the number r. Thus $f \equiv 0(\mathrm{mod} I)$. This completes the proof of the theorem.

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