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SUFFICIENT SECOND ORDER OPTIMALITY CONDITIONS FOR C^1 MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this work, we use the notion of Approximate Hessian introduced by Jeyakumar and Luc [19], and a special scalarization to establish sufficient optimality conditions for constrained multiobjective optimization problems. Throughout this paper, the data are assumed to be of class C^1, but not necessarily of class $C^{1.1}$.

1. Introduction. A lot of research has been carried out in the realm of multiobjective optimization problems [3, 4, 8, 10, 24, 26, ...]. Corley [8] has given optimality conditions for convex and nonconvex multiobjective problems in terms of Clarke derivative. Luc [24] also gives optimality conditions when the data are upper semidifferentiable. Luc and Malivert [26] extend the concept of invex functions to invex multifunctions and study optimality conditions for multiobjective optimization with invex data in terms of contingent derivative.

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For many optimization problems, notably in mathematical programming, the characterization of optimal solutions, with the help of second order conditions, is always of a great interest in order to refine first order optimality conditions. The need for second order informations also appears in numerical algorithms. Quite few publications exist on second order conditions, among which we cite the papers [2, 5, 6, 11, 16, 23, 28] for problems with C^2 and $\mathrm{C}^{1.1}$ data, and [17, 25] for problems with only C^1 data.

In this paper, we are concerned with the multiobjective optimization problem

$$(P) : \begin{cases} \min F(x) \\ \text{subject to } 0 \in G(x) \end{cases}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ are C^1-set valued mappings and $Y^+ \subset \mathbb{R}^p$, $Z^+ \subset \mathbb{R}^k$ are closed convex and pointed cones with $\operatorname{Int} Y^+ \neq \emptyset$.

Our aim is to establish sufficient second order optimality conditions for (P) when the support functions of F and G are continuously differentiable or C^1 for short. As in [17] and [25], the main tool we are going to exploit is approximate Hessian of continuously differentiable functions and its recession matrices. Necessary optimality conditions have been established in [17, 25]. The notions of approximate Jacobian and approximate Hessian have been introduced and studied by Jeyakumar and Luc [19]. Further developments and applications of these concepts are found in [13, 19, 20, 21, 29]. It is important to notice that several known second order generalized derivatives of continuously differentiable functions are examples of approximate Hessian, including the generalized Hessian introduced in [18] by using the Clarke generalized Jacobian, Cominetti and Correa's generalized Hessian [7] and Murdukhovich's second order subdifferential [27]. Consequently, the results obtained by using approximate Hessian remain true when applied to the generalized second order subdifferentials mentioned above. Moreover, a $\mathrm{C}^{1.1}$ function with locally Lipschitz gradient map may admit an approximate Hessian, whose closed convex hull is strictly contained in Clarke's generalized Hessian or in Mordukhovich's second order subdifferential. Therefore, the optimality conditions we are going to establish by means of approximate Hessian are not only valid for $\mathrm{C}^{1.1}$ problems when the Clarke generalized Hessian or the Modukhovich second order subdifferential is used, but sometimes also yield sharper results. For more details, see [22] and [29].

The rest of the paper is written as follows: Section 2 contains basic definitions and preliminary results. Section 3 is devoted to the optimality conditions. Section 4 discusses an application to a mathematical programming problem.

2. Preliminaries. Let F be a set valued mapping defined from $\mathbb{R}^n$ into $\mathbb{R}^p$. In the sequel, we denote the domain and the graph of F respectively by

$$dom\,(F) := \{x \in \mathbb{R}^n : F\,(x) \neq \emptyset\}\,,$$

$$gr\,(F) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : y \in F\,(x)\}\,.$$

If V is a nonempty subset of $\mathbb{R}^n$, then

$$F\,(V) = \underset{x \in V}{\cup} F\,(x)\,.$$

Let $y^* \in \mathbb{R}^p$. The function

$$C_F\,(y^*, x) := \inf_{y \in F(x)} \langle y^*, y \rangle$$

is called the support function of F, where $\langle ., . \rangle$ denotes the inner product.

Assume that the barrier cone of $F\,(x)$, i.e., the set

$$Y_F := \{y^* \in \mathbb{R}^p : \inf y \in F\,(x)\,\langle y^*, y \rangle > -\infty\}$$

is closed and does not depend on x. This is the case, for example, when F is locally Lipschitz [9]. Denoting this cone by Y_F, we say that F is C^1-mapping if, for any $y^* \in Y_F$, $C_F\,(y^*, .)$ is a C^1-function.

To progress, we need the definitions of *approximate Jacobian* and *approximate Hessian.*

Definition 2.1 [19]. *Let f be a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$. A closed set of $(m \times n)$ -matrices $\partial f\,(x) \subseteq L\,(\mathbb{R}^n, \mathbb{R}^m)$ is said to be an* approximate Jacobian *of f at x if for every $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, one has*

$$(vf)^+\,(x, u) \leq \sup_{M \in \partial f(x)} \langle v, M\,(u) \rangle\,,$$

where vf is the real function $\sum_{i=1}^{m} v_i f_i$, here $v_1, \ldots, v_m$ are components of v and $f_1, \ldots, f_m$ are components of f, and $(vf)^+\,(x, u)$ is the upper Dini directional derivative of the function vf at x in the direction u, that is

$$(vf)^+\,(x, u) := \limsup_{t \searrow 0} \frac{(vf)\,(x + tu) - (vf)\,(x)}{t}.$$

If for every $x \in \mathbb{R}^n$, $\partial f(x)$ *is* approximate Jacobian *of* f *at* x, *then the set valued mapping* $x \longmapsto \partial f(x)$ *from* $\mathbb{R}^n$ *to* $L(\mathbb{R}^n, \mathbb{R}^m)$ *is called an* approximate Jacobian *of* f.

For the next concept, we consider a continuously differentiable function f defined on $\mathbb{R}^n$. The gradient map ∇f is then a continuous vector function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Definition 2.2 [25]. *A closed set* $\partial^2 f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ *is said to be an* approximate Hessian *of* f *at* x *if it is an* approximate Jacobian *of* ∇f *at* x. *If for each* x *we have some subset* $\partial^2 f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ *which is an* approximate Hessian *of* f *at* x, *then the set valued mapping* $x \longmapsto \partial^2 f(x)$ *is called an* approximate Hessian *of* f.

Remark 2.1. *Approximate Hessian* shares all properties of *approximate Jacobian.*

For the reader's convenience, we list some of them (see [19] for proofs).

i) If $\partial^2 f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ is *approximate Hessian* of f at x, then every closed subset of $L(\mathbb{R}^n, \mathbb{R}^n)$ which contains $\partial^2 f(x)$ is an *approximate Hessian* of f at x;

ii) If $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiables and if $\partial^2 f(x)$ and $\partial^2 g(x)$ are *approximate Hessians* of f at x respectively, then the closure of the set $\partial^2 f(x) + \partial^2 g(x)$ is an *approximate Hessian* of $f + g$ at x.

iii) (Generalized Taylor expansion) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable; let $x, y \in \mathbb{R}^n$. Suppose that for each $z \in [x, y]$, $\partial^2 f(z)$ is an *approximate Hessian* of f at z. Then there exists $\zeta \in (x, y)$ such that

$$f(y) \in f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \overline{co} \left\langle \partial^2 f(\zeta)(y - x), y - x \right\rangle . \tag{2.1}$$

We shall need some more terminologies. Let $A \subset \mathbb{R}^n$ be a nonempty set. The recession cone of A, which is denoted by A_∞, consists of all limits $\lim\limits_{i \to \infty} t_i a_i$ where $a_i \in A$ and $\{t_i\}$ is a sequence of positive numbers converging to 0. It is important to notice that a set is bounded if and only if its recession cone is trivial. The elements of the recession cone of $\partial^2 f(x)$ are called *recession Hessian matrices.*

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set valued mapping. It is said to be upper semicontinuous at $\overline{x}$ if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$F(\overline{x} + \delta \mathbb{B}_n) \subset F(\overline{x}) + \varepsilon \mathbb{B}_m,$$

where $\mathbb{B}_n$ and $\mathbb{B}_m$ denote the closed unit balls in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The Euclidean norm in $\mathbb{R}^n$ as well as in $L(\mathbb{R}^n, \mathbb{R}^n)$ is denoted by $\|\ldots\|$.

3. Second order optimality conditions. As it was mentioned in the introduction, we are concerned with the multiobjective optimization problem

$$(P) : \begin{cases} \min F(x) \\ \text{subject to } 0 \in G(x) \end{cases}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ are C^1-set valued mappings and $Y^+ \subset \mathbb{R}^p$, $Z^+ \subset \mathbb{R}^k$ are closed convex and pointed cones with $\operatorname{Int} Y^+ \neq \emptyset$.

Let A be a nonempty subset of $\mathbb{R}^p$. A point $\overline{y} \in A$ is said to be a Pareto (respectively. a weak Pareto) minimal point of A with respect to Y^+ if

$$\begin{aligned} (A - \overline{y}) \cap \left(-Y^+\right) &= \{0\}, \\ \text{(respectively. } (A - \overline{y}) \cap \left(-\operatorname{Int} Y^+\right) &= \emptyset \text{),} \end{aligned}$$

here Int denotes the topological interior. A point $(\overline{x}, \overline{y}) \in gr(F)$ with $\overline{x} \in \Omega := \{x \in \mathbb{R}^n : 0 \in G(x)\}$ is said to be a local weak Pareto minimal point with respect to Y^+ of the problem (P) if there exists a neighborhood V of $\overline{x}$ such that

$$F(V \cap \Omega) \subset \overline{y} + \mathbb{R}^p \setminus \left(-\operatorname{Int} Y^+\right).$$

First order necessary optimality conditions for problem (P) were derived in [14] under the hypothesis that F and G are locally Lipschitz. They were generalized in [15] to the case where the support functions of F and G are continuous and admit bounded convexificators. Here we focus our attention on second order optimality conditions for problem (P).

We begin by introducing

Definition 3.1. *Let $(\overline{x}, \overline{y}) \in gr(F)$ such that $\overline{x} \in \Omega$. We say that $(y^*, z^*) \in Y_F \times Z_G$ satisfy the first order optimality condition for problem (P) at $(\overline{x}, \overline{y})$ if*

$$\text{(3.1)} \qquad \begin{cases} \nabla C_F(y^*, \overline{x}) + \nabla C_G(z^*, \overline{x}) = 0, \\ C_G(z^*, \overline{x}) = 0 \text{ and } C_F(y^*, \overline{x}) = \langle y^*, \overline{y} \rangle, \\ y^* \in (Y^+)' \setminus \{0\}, \; z^* \in (Z^+)' \setminus \{0\}, \end{cases}$$

where $(Y^+)'$ is the positive polar cone of Y^+ defined by

$$\left(Y^+\right)' = \left\{y^* \in \mathbb{R}^p : \langle y^*, y \rangle \geq 0 \text{ for all } y \in Y^+\right\}.$$

The existence of such a (y^*, z^*) follows from Theorem 3.4 of [15] under a regularity condition. The interested reader in this regularity condition may consult either [15] or [10]. We denote by Λ the set of all $(y^*, z^*) \in Y_F \times Z_G$ such that (y^*, z^*) satisfies the first order optimality condition for problem (P) at $(\overline{x}, \overline{y})$.

Taking into account the continuity of $C(.,.)$, $\nabla C(.,.)$, and using the compactness of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^k$, it is easily verified that the regularity concept of [10] and [15] is equivalent to that given in

Definition 3.2 [11]. *The constraint set Ω is said to be regular at $\overline{x} \in \Omega$ if the system*

$$\begin{cases} \nabla C_G(z^*, \overline{x}) = 0, \\ C_G(z^*, \overline{x}) = 0, \end{cases}$$

has the unique solution $z^* = 0$.

We start with our first preliminary result which will be a crucial step in the sequel.

Lemma 3.1. *Let $\overline{y} \in F(\overline{x})$. If there is a vector $y^* \in (Y^+)' \setminus \{0\}$ such that $\overline{x}$ is a local solution of the constrained mathematical programming problem*

$$(P_1) \qquad \text{Minimize } C_F(y^*, x) \text{ subject to } 0 \in G(x).$$

Then $(\overline{x}, \overline{y})$ is a local weak Pareto minimal point of (P).

Proof. Suppose the contrary. Then, there exist $x_n \in \Omega$, $x_n \to \overline{x}$, and $y_n \in F(x_n)$ such that

$$\overline{y} - y_n \in \operatorname{Int} Y^+ \text{ for all } n \in \mathbb{N}.$$

Hence $y_n = \overline{y} - s_n$ for some $s_n \in \operatorname{Int} Y^+$. Consequently, $\langle y^*, y_n \rangle = \langle y^*, \overline{y} \rangle - \langle y^*, s_n \rangle$.

Since $s_n \in \operatorname{Int} Y^+$ and $y^* \in (Y^+)' \setminus \{0\}$, we have $\langle y^*, s_n \rangle > 0$. Then,

$$\langle y^*, y_n - \overline{y} \rangle < 0 \text{ for all } n.$$

Finally, $C_F(y^*, x_n) < C_F(y^*, \overline{x})$. A contradiction. □

To provide sufficient optimality conditions, we shall need additional terminologies. Let $I(\overline{x})$, $D(\overline{x})$ and $P^0(\overline{x})$ be defined as follows

$$I(\overline{x}) = \{z^* \in Z_G : C_G(z^*, \overline{x}) = 0\},$$

$$D(\overline{x}) = \{d \in \mathbb{R}^n : \langle \nabla C_G(z^*, \overline{x}), d\rangle \leq 0,\ \forall z^* \in I(\overline{x})\}.$$

$$P^0(\overline{x}) = \{d \in T(\Omega, \overline{x}) : \langle \nabla C_G(z^*, \overline{x}), d\rangle = 0\}.$$

The following result has been proved by Dien and Sach [11]. It gives an estimate of the contingent cone to Ω at $\overline{x}$.

Proposition 3.2.

$$T(\Omega, \overline{x}) \subset D(\overline{x}), \tag{3.2}$$

where $T(\Omega, \overline{x})$ is the contingent cone to Ω at $\overline{x}$.

Proof. Inclusion (3.2) is clear if $\overline{x} \notin cl\Omega$, since $T(\Omega, \overline{x}) = \emptyset$. Now, let $\overline{x} \in cl\Omega$, $v \in T(\Omega, \overline{x})$ and $z^* \in I(\overline{x})$. There exist $h_n \to 0$ and $u_n \to v$ such that

$$x_n := \overline{x} + h_n u_n \in \Omega.$$

Since $C_G(z^*, .)$ is continuously differentiable,

$$\begin{aligned} \langle \nabla C_G(z^*, \overline{x}), v\rangle &= \lim_{n\to\infty} h_n^{-1}\left[C_G(z^*, \overline{x} + h_n v) - C_G(z^*, \overline{x})\right], \\ &= \lim_{n\to\infty} h_n^{-1}\left[C_G(z^*, \overline{x} + h_n u_n) - C_G(z^*, \overline{x})\right], \\ &= \lim_{n\to\infty} h_n^{-1} C_G(z^*, \overline{x} + h_n u_n). \end{aligned}$$

As $\overline{x} + h_n u_n \in \Omega$, $C_G(z^*, \overline{x} + h_n u_n) \leq 0$ and the desired inequality follows. □

The converse of this inclusion is not true in general. For it to hold, an additional condition is required. As such a condition, we can take the regularity condition of Definition 3.2.

Let $(y^*, z^*) \in \Lambda$. Denoting $L(.) := C_F(y^*, .) + C_G(z^*, .)$, one has the following theorem.

Theorem 3.3. *$(\overline{x}, \overline{y}) \in gr(F)$, with $\overline{x} \in \Omega$, is a local weak Pareto minimal point of (P) if there exists a vector $(y^*, z^*) \in Y_F \times Z_G$ satisfying the first order optimality condition at $(\overline{x}, \overline{y})$ and if there is an approximate Hessian map $\partial^2 L$ of L which is upper semicontinuous at $\overline{x}$ such that*

$$\forall u \in P^0(\overline{x}) \setminus \{0\},\quad \forall M \in \partial^2 L(\overline{x}) \cup \left(\left[\partial^2 L(\overline{x})\right]_\infty \setminus \{0\}\right),\quad \langle u, M(u)\rangle > 0. \tag{3.3}$$

P r o o f. By contrast, suppose that $(\overline{x}, \overline{y})$ is not a local weak Pareto minimal point of (P). By Lemma 3.1, $\overline{x}$ is not a local solution of the constrained mathematical programming problem

$$(P_2) \qquad \text{Minimize } C_F(y^*, x) \text{ subject to } 0 \in G(x).$$

Then, there exists $x_n \in \Omega \backslash \{\overline{x}\}$ such that $x_n \to \overline{x}$ and $C_F(y^*, x_n) < C_F(y^*, \overline{x})$ for all n. By taking a subsequence if necessary, we may assume that the sequence $u_n := \|x_n - \overline{x}\|^{-1}(x_n - \overline{x})$ is convergent. Since $\|u_n\| = 1$, there exists $d_1 \in \mathbb{R}^n$, $\|d_1\| = 1$ such that $\dfrac{x_n - \overline{x}}{\|x_n - \overline{x}\|} \to d_1$. Remark that $d_1 \in T(\Omega, \overline{x}) \setminus \{0\}$.

On the one hand, from (3.2) we have also $d_1 \in D(\overline{x}) \setminus \{0\}$ and consequently

$$\langle \nabla C_G(z^*, \overline{x}), d_1 \rangle \leq 0. \tag{3.4}$$

On the other hand, by the Generalized Taylor expansion (2.1), one has

$$\langle \nabla C_F(y^*, \overline{x}), d_1 \rangle \leq 0. \tag{3.5}$$

Hence, by (3.1),

$$\langle \nabla C_G(z^*, \overline{x}), d_1 \rangle = -\langle \nabla C_F(y^*, \overline{x}), d_1 \rangle \geq 0$$

So, (3.4) implies that both sides are zeros. Since $C_F(y^*, x_n) < C_F(y^*, \overline{x})$, as we supposed at the beginning, the Taylor expansion for $C_F(y^*, x_n)$ gives the existence of $M_n \in \partial^2 L(\widehat{x}_n)$ such that

$$\langle x_n - \overline{x}, M_n(x_n - \overline{x}) \rangle \leq 0.$$

If the sequence $\{M_n\}$ is bounded, then we may assume that it converges to some $M \in \partial^2 L(\overline{x})$, due to the upper semicontinuity of the approximate Hessian mapping $\partial^2 L$. Therefore,

$$\langle d_1, M(d_1) \rangle = \lim_{n \to \infty} \left\langle \frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}, M_n\left(\frac{x_n - \overline{x}}{\|x_n - \overline{x}\|}\right) \right\rangle \leq 0,$$

which contradicts the hypothesis.

If the sequence $\{M_n\}$ is unbounded, then due to the upper semicontinuity of $\partial^2 L$, we may assume that

$$\lim_{n \to \infty} \|M_n\| = \infty$$

and

$$\lim_{n\to\infty}\frac{M_n}{\|M_n\|}=M_0\in\left[\partial^2L(\overline{x})\right]_\infty\setminus\{0\}.$$

Then, similarly as for M,

$$\langle d_1, M_0(d_1)\rangle\le 0$$

which again contradicts the hypothesis. The proof is complete. □

From (3.2), we get the following consequence.

Corollary 3.4. *$(\overline{x},\overline{y})\in gr(F)$, with $\overline{x}\in\Omega$, is a local weak Pareto minimal point of (P) if there exists a vector $(y^*,z^*)\in Y_F\times Z_G$ satisfying the first order optimality condition at $(\overline{x},\overline{y})$ and if there is an upper semicontinuous* approximate Hessian *of L such that*

$$\forall u\in T(\Omega,\overline{x})\setminus\{0\},\ \ \forall M\in\partial^2L(\overline{x})\cup\left(\left[\partial^2L(\overline{x})\right]_\infty\setminus\{0\}\right),\quad \langle u, M(u)\rangle>0$$

As a special case, take the following optimization problem

$$(P_3):\begin{cases}\min f(x)\\ \text{subject to } 0\in G(x)\end{cases}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ and the support function of $G:\mathbb{R}^n\rightrightarrows\mathbb{R}^k$ are $C^{1.1}$ functions.

Dien and Sach's sufficient optimality conditions [11] can be derived from Theorem 3.3. It suffices to replace the *approximate Hessian* by the *generalized Hessian* and eliminate the recession parts of the formulas.

Remark 3.1. By using the Taylor expansion, one can show that a function is $C^{1.1}$ if and only if it admits a locally bounded approximate Hessian. The recession part in the inclusions above is a very characteristic feature of those problems that have C^1, but not $C^{1.1}$ data. Without this part, the inclusions may fail. For examples and details, see [25].

4. Application. In this section we are concerned with the mathematical programming problem

$$(P^*):\quad \begin{array}{l}\min\ f(x)\\ \begin{cases} g_i(x)\le 0 & i=1,\ 2,\ \ldots,\ m,\\ h_j(x)=0 & j=1,\ 2,\ \ldots,\ k,\end{cases}\end{array}$$

where f, g_i, and h_j are C^1 functions. Denotes by $\mathbb{R}^m_+$ the nonnegative orthant of $\mathbb{R}^m$.

Setting $\Omega := \{x : g_i(x) \leq 0, h_j(x) = 0 \text{ for all } i, j\}$, $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))$ and $h(x) = (h_1(x), h_2(x), \ldots, h_k(x))$, problem (P^*) is reduced to the problem (P), when the set valued mappings F and G from $\mathbb{R}^n$ into $\mathbb{R}$ and $Z = \mathbb{R}^m \times \mathbb{R}^k$ are respectively defined by

$$F(x) = f(x), \ G(x) := (g(x), h(x)) + \mathbb{R}^m_+ \times \{0_{\mathbb{R}^k}\}$$

where $\mathbb{R}^m_+$ is the nonnegative orthant of $\mathbb{R}^m$.

Obviously, in that case $Z_G = \mathbb{R}^m_+ \times \mathbb{R}^k$ and for any $z^* = (\lambda, \mu) \in Z_G$ we have

$$C_G(z^*, x) = \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle \text{ and } L(z^*, x) = f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle .$$

Take $\overline{x} \in \Omega$ and $z^* = (\lambda, \mu) = (\lambda_1, \lambda_2, \ldots, \lambda_m, \mu) \in \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^k$.

Setting $I = \{1, 2, \ldots, m\}$ and $q(\overline{x}) = \{i \in I : g_i(\overline{x}) = 0\}$ we get

$$P^0(\overline{x}) = \left\{ d : \text{such that} \begin{array}{c} \langle \nabla g_i(\overline{x}), d \rangle \leq 0, \ \forall i \in q(\overline{x}) \backslash p(\lambda), \\ \langle \nabla g_i(\overline{x}), d \rangle = 0, \ \forall i \in p(\lambda), \\ \langle \nabla h_j(\overline{x}), d \rangle = 0, \ j = 1, \ 2, \ldots, k \end{array} \right\}$$

where $p(\lambda) = \{i \in I : \lambda_i > 0\}$.

It can be verified that $C_G(z^*, \overline{x}) = 0$ if and only if $\lambda_i g_i(\overline{x}) = 0$ for all $i \in I$.

From Theorem 3.3, we deduce sufficient optimality conditions for problem (P^*).

Theorem 4.1 *Assume that the following conditions hold:*
i. *The functions f, g_i and h_j are continuously differentiables;*
ii. *There exists a vector $(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k$ satisfying the first order optimality condition at $\overline{x} \in \Omega$;*
iii. *There is an* approximate Hessian map *$\partial^2 L(\overline{x})$ of L which is upper semicontinuous at $\overline{x}$ such that for every $d \in P^0(\overline{x}) \backslash \{0\}$ and $M \in \partial^2 L(\overline{x}) \cup \left(\left[\partial^2 L(\overline{x}) \right]_\infty \setminus \{0\} \right)$, one has*

$$\langle d, M(d) \rangle > 0.$$

Then $\overline{x}$ is a strict local solution to problem (P^*).

Remark 4.1. Since $P^0(\overline{x}) \subset T(\Omega, \overline{x})$, the above result implies Theorem 3.2 of [25]. This inclusion may be strict (see the following example).

Consider the following problem

$$\begin{array}{ll} \min & x^2 + y^2 \\ \text{s.t.} & -x + y \le 0. \end{array}$$

It is clear that $(0,0)$ is a local solution of the problem. Since $\Omega := \left\{(d_1, d_2) \in \mathbb{R}^2 : -d_1 + d_2 \le 0\right\}$ is a closed cone, one has

$$T(\Omega, (0,0)) = \left\{(d_1, d_2) \in \mathbb{R}^2 : -d_1 + d_2 \le 0\right\}.$$

Moreover,

$$P^0(0,0) = \left\{(d_1, d_2) \in \mathbb{R}^2 : d_1 = d_2\right\}.$$

Observing that $(1,0) \in T(\Omega, (0,0))$ and $(1,0) \notin P^0(0,0)$, we deduce that

$$P^0(0,0) \subsetneqq T(\Omega, (0,0)).$$

Remark 4.2. When the functions f, g_i and h_j are C^2 functions, we get from Theorem 4.1 a well-known result of [12].

Remark 4.3. The constraint set Ω is regular at $\overline{x} \in \Omega$ if the vectors $\nabla g_i(\overline{x})$, $i \in q(\overline{x})$, $\nabla h_j(\overline{x})$, $j = 1, \ldots, k$ are lineary independent.

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