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UPPER AND LOWER BOUNDS IN RELATOR SPACES*

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ABSTRACT. An ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ consisting of a nonvoid set X and a nonvoid family \mathcal{R} of binary relations on X is called a relator space. Relator spaces are straightforward generalizations not only of uniform spaces, but also of ordered sets.

Therefore, in a relator space we can naturally define not only some topological notions, but also some order theoretic ones. It turns out that these two, apparently quite different, types of notions are closely related to each other through complementations.

1. Introduction. A nonvoid family \mathcal{R} of binary relations on a nonvoid set X is called a relator on X , and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space. Relator spaces are straightforward generalizations not only of uniform spaces, but also of ordered sets. Therefore, in a relator space we can naturally define not only some topological notions, but also some order theoretic ones.

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For instance, the families of all adherence and interior points of a subset A of $X(\mathcal{R})$ can be briefly defined by

$$\text{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}(A) \quad \text{and} \quad \text{int}_{\mathcal{R}}(A) = \text{cl}_{\mathcal{R}}(A^c)^c,$$

where $A^c = X \setminus A$.

While, the families of all lower and upper bounds of a subset A of $X(\mathcal{R})$ can be briefly defined by

$$\text{lb}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} R^{-1}(a) \quad \text{and} \quad \text{ub}_{\mathcal{R}}(A) = \text{lb}_{\mathcal{R}^{-1}}(A),$$

where $\mathcal{R}^{-1} = \{R^{-1} : R \in \mathcal{R}\}$.

The relations $\text{cl}_{\mathcal{R}}$ and $\text{lb}_{\mathcal{R}}$ are closely related to each other. Namely, by using the relator $\mathcal{R}^c = \{R^c : R \in \mathcal{R}\}$, where $R^c = X^2 \setminus R$, we can prove that

$$\text{lb}_{\mathcal{R}}(A) = \text{cl}_{\mathcal{R}^c}(A)^c \quad \text{and} \quad \text{cl}_{\mathcal{R}}(A) = \text{lb}_{\mathcal{R}^c}(A)^c.$$

These formulas resemble, in spirit, to those of Euler on elementary functions.

Now, by making use of the relation $\text{lb}_{\mathcal{R}}$, the members of the families

$$\min_{\mathcal{R}}(A) = A \cap \text{lb}_{\mathcal{R}}(A) \quad \text{and} \quad \max_{\mathcal{R}}(A) = \min_{\mathcal{R}^{-1}}(A)$$

may be naturally called the minima and the maxima of the set A in the relator space $X(\mathcal{R})$, respectively.

Moreover, analogously to the family $\mathcal{T}_{\mathcal{R}} = \{A \subset X : A \subset \text{int}_{\mathcal{R}}(A)\}$ of all open subsets of $X(\mathcal{R})$, we may also naturally define the families

$$\mathcal{L}_{\mathcal{R}} = \{A \subset X : A \subset \text{lb}_{\mathcal{R}}(A)\} \quad \text{and} \quad \mathcal{U}_{\mathcal{R}} = \mathcal{L}_{\mathcal{R}^{-1}}.$$

Thus, we also have

$$\mathcal{L}_{\mathcal{R}} = \{A \subset X : A = \min_{\mathcal{R}}(A)\} \quad \text{and} \quad \mathcal{L}_{\mathcal{R}} = \{\min_{\mathcal{R}}(A) : A \subset X\}.$$

Moreover, concerning the unicity of minima in a relator space $X(\mathcal{R})$, we can prove that the following assertions are equivalent:

- (1) $\bigcup \mathcal{R}$ is antisymmetric;
- (2) $\text{lb}_{\mathcal{R}}(x) \cap \text{ub}_{\mathcal{R}}(x) \subset \{x\}$ for all $x \in X$;
- (3) $\text{card}(A) \leq 1$ for all $A \in \mathcal{L}_{\mathcal{R}}$;
- (4) $\text{card}(\min_{\mathcal{R}}(A)) \leq 1$ for all $A \subset X$.

1. A few basic facts on relations. A subset F of a product set $X \times Y$ is called a relation on X to Y . In particular, the relations $\Delta_X = \{(x, x) : x \in X\}$ and $X^2 = X \times X$ are called the identity and the universal relations on X .

Namely, if in particular $X = Y$, then we may simply say that F is a relation on X . Note that if F is a relation on X to Y , then F is also a relation on $X \cup Y$. Therefore, it is sometimes not a severe restriction to assume that $X = Y$.

If F is a relation on X to Y and $x \in X$ and $A \subset X$, then the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F , respectively. Whenever $A \in X$ seems unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X to Y , then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$.

If F is a relation on X to Y , then the sets $D_F = F^{-1}(Y)$ and $R_F = F(X)$ are called the domain and range of F , respectively. If in particular, $D_F = X$ (and $R_F = Y$), then we say that F is a relation of X into (onto) Y .

A relation F is said to be a function if for each $x \in D_F$ there exists $y \in R_F$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If F is a relation on X to Y and G is a relation on Y to Z , then the composition $G \circ F$ of G and F can be defined such that $(G \circ F)(x) = G(F(x))$ for all $x \in X$. Note that thus we have $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

Moreover, if F and G are relations on X to Y , then we may also naturally consider the relations $F \cap G$, $F \cup G$ and $F \setminus G$. Moreover, when confusion seems unlikely, we may briefly write F^c in place of $X \times Y \setminus F$.

Concerning the complement relation F^c we can easily establish the following theorems.

Theorem 1.1. *If F is a relation on X to Y , and $x \in X$ and $A \subset X$, then*

$$F^c(x) = F(x)^c \quad \text{and} \quad F^c(A)^c = \bigcap_{a \in A} F(a).$$

Hint. To prove the second assertion, note that

$$F^c(A)^c = \left(\bigcup_{a \in A} F^c(a) \right)^c = \bigcap_{a \in A} F^c(a)^c = \bigcap_{a \in A} F(a). \quad \square$$

Theorem 1.2. *If F is a relation on X to Y and $A \subset X$, then*

$$F(A)^c \subset F^c(A) \quad \text{if} \quad A \neq \emptyset \quad \text{and} \quad F(A)^c \subset F(A^c) \quad \text{if} \quad Y = R_F.$$

Hint. To prove the first assertion, note that if $A \neq \emptyset$, then

$$F(A)^c = (F^c)^c(A)^c = \bigcap_{a \in A} F^c(a) \subset \bigcup_{a \in A} F^c(a) = F^c(A). \quad \square$$

Theorem 1.3. *If F is a relation on X to Y , then*

$$(F^c)^{-1} = (F^{-1})^c.$$

Theorem 1.4. *If F is a relation on X to Y and G is a relation on Y to Z , then*

$$(G \circ F)^c \subset G^c \circ F \quad \text{if } X = D_F \quad \text{and} \quad (G \circ F)^c \subset G \circ F^c \quad \text{if } Z = R_G.$$

Proof. Note that if $X = D_F$, then

$$(G \circ F)^c(x) = (G \circ F)(x)^c = G(F(x))^c \subset G^c(F(x)) = (G^c \circ F)(x)$$

for all $x \in X$. While, if $Z = R_G$, then

$$(G \circ F)^c(x) = G(F(x))^c \subset G(F(x)^c) = G(F^c(x)) = (G \circ F^c)(x)$$

for all $x \in X$. \square

Remark 1.5. By Theorem 1.1, we also have

$$(G^c \circ F)^c(x) = (G^c \circ F)(x)^c = G^c(F(x))^c = \bigcap_{y \in F(x)} G(y)$$

for all $x \in X$.

2. A few basic facts on relators. A nonvoid family \mathcal{R} of relations on one nonvoid set X to another Y is called a relator on X to Y . Moreover, the ordered pair $(X, Y)(\mathcal{R})$ is called a relator space. Particular cases of relators have been intensively studied by several authors.

If in particular $X = Y$, then we may simply say that \mathcal{R} is a relator on X . Moreover, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Note that if \mathcal{R} is a relator on X to Y , then \mathcal{R} is also a relator on $X \cup Y$. However, if F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then it seems quite unnatural to consider the families $F \circ \mathcal{R}$ and $\mathcal{S} \circ F$ as relators on $X \cup Y$.

Relator spaces of the simpler type $X(\mathcal{R})$ are already substantial generalizations of the various ordered sets and uniform spaces [17]. They deserve to be widely investigated because of the following facts.

If \mathcal{D} is a nonvoid family of certain distance functions on X , then the relator $\mathcal{R}_{\mathcal{D}}$ consisting of all surroundings $B_r^d = \{(x, y) : d(x, y) < r\}$, where $d \in \mathcal{D}$ and $r > 0$, is a more convenient mean of defining the basic notions of analysis in the space $X(\mathcal{D})$ than the family of all open subsets of $X(\mathcal{D})$, or even the family \mathcal{D} itself.

Moreover, all reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters, and convergences, for instance) can be easily derived from relators (according to the results of [21] and [16]), and thus they need not be studied separately.

For instance, if \mathcal{A} is a certain generalized topology or a nonvoid stack (ascending system) in X , then \mathcal{A} can be easily derived (according to the forthcoming definitions of the families $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$) from the Davis–Pervin relator $\mathcal{R}_{\mathcal{A}}$ consisting of all preorders $R_A = A^2 \cup A^c \times X$, where $A \in \mathcal{A}$.

Note that in contrast to these preorders R_A , the surroundings B_r^d are usually tolerances (reflexive and symmetric relations) on X . Therefore, beside preorder relators, tolerance relators are also important particular cases of reflexive relators.

Unfortunately, the class of all reflexive relators proved to be insufficient for several important purposes. For instance, if F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then we have to consider the relators $F \circ \mathcal{R}$ and $\mathcal{S} \circ F$ too.

In the sequel, we shall be frequently dealing with relations on families of sets. In particular, for any $A \subset X$ we write:

$$\mathcal{C}(A) = X \setminus A \quad \text{and} \quad \mathcal{P}(A) = \{B : B \subset A\}.$$

Moreover, if \mathcal{R} is a relator on X to Y , then for any $A \subset X$, $B \subset Y$, $x \in X$ and $y \in Y$ we write:

- (1) $A \in \text{Int}_{\mathcal{R}}(B)$ if $R(A) \subset B$ for some $R \in \mathcal{R}$;
- (2) $A \in \text{Cl}_{\mathcal{R}}(B)$ if $R(A) \cap B \neq \emptyset$ for all $R \in \mathcal{R}$;
- (3) $x \in \text{int}_{\mathcal{R}}(B)$ if $\{x\} \in \text{Int}_{\mathcal{R}}(B)$; (4) $x \in \text{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \text{Cl}_{\mathcal{R}}(B)$;
- (5) $B \in \mathcal{E}_{\mathcal{R}}$ if $\text{int}_{\mathcal{R}}(B) \neq \emptyset$; (6) $B \in \mathcal{D}_{\mathcal{R}}$ if $\text{cl}_{\mathcal{R}}(B) = X$;

Furthermore, if in particular \mathcal{R} is a relator on X , then for any $A \subset X$ we also write:

- (7) $A \in \tau_{\mathcal{R}}$ if $A \in \text{Int}_{\mathcal{R}}(A)$; (8) $A \in \mathcal{F}_{\mathcal{R}}$ if $A^c \notin \text{Cl}_{\mathcal{R}}(A)$;
 (9) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \text{int}_{\mathcal{R}}(A)$; (10) $A \in \mathcal{F}_{\mathcal{R}}$ if $\text{cl}_{\mathcal{R}}(A) \subset A$.

The relations $\text{Int}_{\mathcal{R}}$ and $\text{int}_{\mathcal{R}}$ are called the proximal and the topological interiors induced by \mathcal{R} . While, the members of the families, $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, the topologically open and the fat subsets of $X(\mathcal{R})$, respectively.

The fat sets are frequently more important tools than the open sets. For instance, if \prec is a certain order relation on X , then \mathcal{T}_{\prec} and \mathcal{E}_{\prec} are just the families of all ascending and residual subsets of the ordered set $X(\prec)$, respectively. And the residual sets are certainly more important than the ascending ones.

Moreover, it is also worth mentioning that if for instance R is a relation on \mathbb{R} such that $R(x) =]-\infty, x] \cup \{x+1\}$ for all $x \in \mathbb{R}$, then $\mathcal{T}_R = \{\emptyset, \mathbb{R}\}$, but $\mathcal{E}_R \neq \{\mathbb{R}\}$. Therefore, in contrast to the open sets, the fat sets may be useful tools even in a topologically indiscrete relator space.

Hence, it is not surprising that if \mathcal{R} is a relator on X to Y , then besides the relations

$$\delta_{\mathcal{R}} = \bigcap \mathcal{R} \quad \text{and} \quad \sigma_{\mathcal{R}} = \bigcup \mathcal{R},$$

sometimes we also need the sets

$$E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} \quad \text{and} \quad D_{\mathcal{R}} = \bigcup (\mathcal{P}(Y) \setminus \mathcal{D}_{\mathcal{R}}).$$

3. Proximal upper and lower bounds.

Definition 3.1. *If \mathcal{R} is a relator on X to Y , then we define two relations $\text{Ub}_{\mathcal{R}}$ on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ and $\text{Lb}_{\mathcal{R}}$ on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ such that for all $A \subset X$ and $B \subset Y$*

$$\text{Ub}_{\mathcal{R}}(A) = \{D \subset Y : \exists R \in \mathcal{R} : A \times D \subset R\}$$

and

$$\text{Lb}_{\mathcal{R}}(B) = \{C \subset X : \exists R \in \mathcal{R} : C \times B \subset R\}.$$

The members of the families $\text{Ub}_{\mathcal{R}}(A)$ and $\text{Lb}_{\mathcal{R}}(B)$ are called the proximal upper and lower bounds of the sets A and B in the relator space $(X, Y)(\mathcal{R})$, respectively.

Remark 3.2. To see the appropriateness of the above definition, we can note that $A \times B \subset R$ if and only if $(a, b) \in R$ or equivalently aRb for all $a \in A$ and $b \in B$.

Therefore, if \prec is a certain order relation on X , then for any $A, B \subset X$ we have $B \in \text{Ub}_{\prec}(A)$ if and only if $A \prec B$ in the sense that $a \prec b$ for all $a \in A$ and $b \in B$.

Note that, by writing $\prec_{\mathcal{R}}$ in place of $\text{Ub}_{\mathcal{R}}$, we could also write $A \prec_{\mathcal{R}} B$ in place of $B \in \text{Ub}_{\mathcal{R}}(A)$. However, the latter notation is usually more convenient than the former one.

By the corresponding definitions, we evidently have the following

Theorem 3.3. *If \mathcal{R} is a relator on X to Y , then*

$$\text{Ub}_{\mathcal{R}} = \text{Lb}_{\mathcal{R}^{-1}} = \text{Lb}_{\mathcal{R}}^{-1} \quad \text{and} \quad \text{Lb}_{\mathcal{R}} = \text{Ub}_{\mathcal{R}^{-1}} = \text{Ub}_{\mathcal{R}}^{-1}.$$

Hint. Note that the second statement of the theorem can be immediately derived from the first one by writing \mathcal{R}^{-1} in place of \mathcal{R} . \square

Remark 3.4. By the above theorem, it is clear that the relations $\text{Ub}_{\mathcal{R}}$ and $\text{Lb}_{\mathcal{R}}$ are equivalent tools in the relator space $(X, Y)(\mathcal{R})$.

Moreover, by using the corresponding definitions, we can easily prove

Theorem 3.5. *If \mathcal{R} is a relator on X to Y and $A \subset X$, then*

$$\text{Ub}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} \mathcal{P}(R^c(A)^c) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}(R(a)).$$

Proof. By Definition 3.1, for any $B \subset Y$, we have

$$B \in \text{Ub}_{\mathcal{R}}(A) \iff \exists R \in \mathcal{R} : A \times B \subset R.$$

Moreover, we can easily see that

$$A \times B \subset R \iff \forall a \in A : B \subset R(a) \iff B \in \bigcap_{a \in A} \mathcal{P}(R(a)).$$

Therefore, we actually have

$$B \in \text{Ub}_{\mathcal{R}}(A) \iff B \in \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}(R(a)).$$

On the other hand, by Theorem 1.1, it is clear that

$$\bigcap_{a \in A} \mathcal{P}(R(a)) = \mathcal{P}\left(\bigcap_{a \in A} R(a)\right) = \mathcal{P}(R^c(A)^c).$$

Therefore, the required inequalities are true. \square

Now, as an immediate consequence of Theorems 3.5 and 3.3, we can also state

Theorem 3.6. *If \mathcal{R} is a relator on X to Y , then*

$$\text{Ub}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{Ub}_R \quad \text{and} \quad \text{Lb}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{Lb}_R.$$

Remark 3.7. These simple facts will guarantee the existence of a largest relator \mathcal{R}^{\square} on X to Y such that $\text{Ub}_{\mathcal{R}} = \text{Ub}_{\mathcal{R}^{\square}}$ ($\text{Lb}_{\mathcal{R}} = \text{Lb}_{\mathcal{R}^{\square}}$).

Moreover, by Theorem 3.6, it is clear that we also have the following

Theorem 3.8. *If \mathcal{R}_i is a relator on X to Y for all $i \in I$, with $I \neq \emptyset$, and $\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i$, then*

$$\text{Ub}_{\mathcal{R}} = \bigcup_{i \in I} \text{Ub}_{\mathcal{R}_i} \quad \text{and} \quad \text{Lb}_{\mathcal{R}} = \bigcup_{i \in I} \text{Lb}_{\mathcal{R}_i}.$$

However, it is now more interesting to note that, by using Theorem 3.5, we can also easily prove the following counterparts of Euler's famous formulas on exponential and trigonometric functions [15, p. 227].

Theorem 3.9. *If \mathcal{R} is a relator on X to Y , then*

$$\text{Lb}_{\mathcal{R}} = (\text{Cl}_{\mathcal{R}^c})^c \quad \text{and} \quad \text{Cl}_{\mathcal{R}} = (\text{Lb}_{\mathcal{R}^c})^c.$$

Proof. By Theorems 3.3 and 3.5, for any $A \subset X$ and $B \subset Y$, we have

$$\begin{aligned} A \in \text{Lb}_{\mathcal{R}}(B) &\iff A \in \text{Ub}_{\mathcal{R}}^{-1}(B) \iff B \in \text{Ub}_{\mathcal{R}}(A) \iff \\ &\iff B \in \bigcup_{R \in \mathcal{R}} \mathcal{P}(R^c(A)^c) \iff \exists R \in \mathcal{R} : B \in \mathcal{P}(R^c(A)^c) \end{aligned}$$

Moreover, we can easily see that

$$B \in \mathcal{P}(R^c(A)^c) \iff B \subset R^c(A)^c \iff R^c(A) \cap B = \emptyset.$$

Therefore, we actually have

$$\begin{aligned} A \in \text{Lb}_{\mathcal{R}}(B) &\iff \exists R \in \mathcal{R} : R^c(A) \cap B = \emptyset \iff \\ &\iff A \notin \text{Cl}_{\mathcal{R}^c}(B) \iff A \in \text{Cl}_{\mathcal{R}^c}(B)^c \iff A \in (\text{Cl}_{\mathcal{R}^c})^c(B). \end{aligned}$$

Thus, $\text{Lb}_{\mathcal{R}}(B) = (\text{Cl}_{\mathcal{R}^c})^c(B)$, and therefore the first statement of the theorem is true. The second statement of the theorem is immediate from the first one. \square

From the above theorem, by the equality $\text{Cl}_{\mathcal{R}} = (\text{Int}_{\mathcal{R}} \circ \mathcal{C})^c$, it is clear that we also have the following

Theorem 3.10. *If \mathcal{R} is a relator on X to Y , then*

$$\text{Lb}_{\mathcal{R}} = \text{Int}_{\mathcal{R}^c} \circ \mathcal{C} \qquad \text{and} \qquad \text{Int}_{\mathcal{R}} = \text{Lb}_{\mathcal{R}^c} \circ \mathcal{C}.$$

Remark 3.11. The above properties can also be expressed in the forms that

$$\text{Cl}_{\mathcal{R}^c} = (\text{Lb}_{\mathcal{R}})^c \qquad \text{and} \qquad \text{Int}_{\mathcal{R}^c} = \text{Lb}_{\mathcal{R}} \circ \mathcal{C}.$$

By Definition 3.1 and Theorem 3.3, we evidently have the following

Theorem 3.12. *If \mathcal{R} is a relator on X to Y , then*

- (1) $\text{Ub}_{\mathcal{R}}(\emptyset) = \mathcal{P}(Y)$ and $\text{Ub}_{\mathcal{R}}^{-1}(\emptyset) = \mathcal{P}(X)$;
- (2) $\text{Ub}_{\mathcal{R}}(A) \subset \text{Ub}_{\mathcal{R}}(C)$ for all $C \subset A \subset X$ and $\text{Ub}_{\mathcal{R}}^{-1}(B) \subset \text{Ub}_{\mathcal{R}}^{-1}(D)$ for all $D \subset B \subset Y$.

Remark 3.13. The above characteristic properties can also be expressed in the forms that:

- (1) $\emptyset \in \text{Ub}_{\mathcal{R}}(A)$ and $B \in \text{Ub}_{\mathcal{R}}(\emptyset)$ for all $A \subset X$ and $B \subset Y$;
- (2) $B \in \text{Ub}_{\mathcal{R}}(A)$ implies $D \in \text{Ub}_{\mathcal{R}}(C)$ for all $C \subset A \subset X$ and $D \subset B \subset Y$.

By Theorem 3.12, we evidently have the following

Theorem 3.14. *If \mathcal{R} is a relator on X to Y and $A_i \subset X$ for all $i \in I$, then*

$$\text{Ub}_{\mathcal{R}}\left(\bigcup_{i \in I} A_i\right) \subset \bigcap_{i \in I} \text{Ub}_{\mathcal{R}}(A_i) \qquad \text{and} \qquad \bigcup_{i \in I} \text{Ub}_{\mathcal{R}}(A_i) \subset \text{Ub}_{\mathcal{R}}\left(\bigcap_{i \in I} A_i\right).$$

Remark 3.15. Note that if in particular \mathcal{R} is a singleton, then the equality is also true in the first statement of the above theorem.

Moreover, by using Theorems 3.12 and 1.1, we can also easily prove

Theorem 3.16. *If \mathcal{R} is a relator on X to Y , then*

- (1) $\text{Ub}_{\mathcal{R}} = \text{Ub}_{\mathcal{R}} \circ \mathcal{P}^{-1} = \mathcal{P} \circ \text{Ub}_{\mathcal{R}}$;
- (2) $\text{Ub}_{\mathcal{R}} = ((\text{Ub}_{\mathcal{R}})^c \circ \mathcal{P})^c = (\mathcal{P}^{-1} \circ (\text{Ub}_{\mathcal{R}})^c)^c$.

Proof. By Theorems 3.12 and 1.1, it is clear that

$$\text{Ub}_{\mathcal{R}}(A) = \bigcup_{A \subset C} \text{Ub}_{\mathcal{R}}(C) = \text{Ub}_{\mathcal{R}}(\mathcal{P}^{-1}(A)) = (\text{Ub}_{\mathcal{R}} \circ \mathcal{P}^{-1})(A)$$

and

$$\text{Ub}_{\mathcal{R}}(A) = \bigcap_{C \subset A} \text{Ub}_{\mathcal{R}}(C) = (\text{Ub}_{\mathcal{R}})^c(\mathcal{P}(A))^c = ((\text{Ub}_{\mathcal{R}})^c \circ \mathcal{P})^c(A)$$

for all $A \subset X$. Therefore, $\text{Ub}_{\mathcal{R}} = \text{Ub}_{\mathcal{R}} \circ \mathcal{P}^{-1}$ and $\text{Ub}_{\mathcal{R}} = ((\text{Ub}_{\mathcal{R}})^c \circ \mathcal{P})^c$. Hence, by using Theorems 3.3 and 1.1, we can infer that

$$\text{Ub}_{\mathcal{R}} = (\text{Ub}_{\mathcal{R}}^{-1})^{-1} = (\text{Ub}_{\mathcal{R}^{-1}})^{-1} = (\text{Ub}_{\mathcal{R}^{-1}} \circ \mathcal{P}^{-1})^{-1} = (\text{Ub}_{\mathcal{R}}^{-1} \circ \mathcal{P}^{-1})^{-1} = \mathcal{P} \circ \text{Ub}_{\mathcal{R}}$$

and

$$\begin{aligned} \text{Ub}_{\mathcal{R}} &= (\text{Ub}_{\mathcal{R}^{-1}})^{-1} = \left(((\text{Ub}_{\mathcal{R}^{-1}})^c \circ \mathcal{P})^c \right)^{-1} = \\ &= \left(((\text{Ub}_{\mathcal{R}}^{-1})^c \circ \mathcal{P})^c \right)^{-1} = \left((((\text{Ub}_{\mathcal{R}})^c)^{-1} \circ \mathcal{P})^{-1} \right)^c = (\mathcal{P}^{-1} \circ (\text{Ub}_{\mathcal{R}})^c)^c. \quad \square \end{aligned}$$

By Theorems 3.12 and 3.3, it is clear that in particular we also have

Theorem 3.17. *If \mathcal{R} is a relator on X to Y , and $A \subset X$ and $B \subset Y$, then*

$$\text{Ub}_{\mathcal{R}}(A) \subset \bigcap_{a \in A} \text{Ub}_{\mathcal{R}}(a) \quad \text{and} \quad \text{Lb}_{\mathcal{R}}(B) \subset \bigcap_{b \in B} \text{Lb}_{\mathcal{R}}(b).$$

Remark 3.18. Note that if in particular \mathcal{R} is a singleton, then the corresponding equalities are also true.

However, it is now more important to prove the following

Theorem 3.19. *If \mathcal{R} is a relator on X to Y , and $A \subset X$ and $B \subset Y$, then*

$$\text{Ub}_{\mathcal{R}}(A) = \{D \subset Y : \mathcal{P}(A) \subset \text{Lb}_{\mathcal{R}}(D)\}$$

and

$$\text{Lb}_{\mathcal{R}}(B) = \{C \subset X : \mathcal{P}(B) \subset \text{Ub}_{\mathcal{R}}(C)\}.$$

Proof. If $D \in \text{Ub}_{\mathcal{R}}(A)$, then by Definition 3.1 we have $D \subset Y$. Moreover, by Theorem 3.12, for any $C \subset A$, we have $D \in \text{Ub}_{\mathcal{R}}(C)$, and hence $C \in \text{Ub}_{\mathcal{R}}^{-1}(D)$. This, by Theorem 3.3, implies that $C \in \text{Lb}_{\mathcal{R}}(D)$. Therefore, $\mathcal{P}(A) \subset \text{Lb}_{\mathcal{R}}(D)$ is also true.

On the other hand, if D is a subset of Y such that $\mathcal{P}(A) \subset \text{Lb}_{\mathcal{R}}(D)$, then in particular we also have $A \in \text{Lb}_{\mathcal{R}}(D)$, and hence $D \in \text{Lb}_{\mathcal{R}}^{-1}(A)$. Therefore, by Theorem 3.3, $D \in \text{Ub}_{\mathcal{R}}(A)$ is also true. \square

Remark 3.20. The first statement of the latter theorem is actually a reformulation of the second statement of Theorem 3.16.

Namely, for any $A \subset X$ and $D \subset Y$, we have $\mathcal{P}(A) \subset \text{Lb}_{\mathcal{R}}(D)$ if and only if $C \in \text{Lb}_{\mathcal{R}}(D)$, and hence $D \in \text{Ub}_{\mathcal{R}}(C)$ for all $C \subset A$.

4. Topological upper and lower bounds.

Definition 4.1. If \mathcal{R} is a relator on X to Y , then we define two relations $\text{ub}_{\mathcal{R}}$ on $\mathcal{P}(X)$ to Y and $\text{lb}_{\mathcal{R}}$ on $\mathcal{P}(Y)$ to X such that for all $A \subset X$ and $B \subset Y$

$$\text{ub}_{\mathcal{R}}(A) = \{y \in Y : \{y\} \in \text{Ub}_{\mathcal{R}}(A)\}$$

and

$$\text{lb}_{\mathcal{R}}(B) = \{x \in X : \{x\} \in \text{Lb}_{\mathcal{R}}(B)\}.$$

The members of the families $\text{ub}_{\mathcal{R}}(A)$ and $\text{lb}_{\mathcal{R}}(B)$ are called the topological upper and lower bounds of the sets A and B in the relator space $(X, Y)(\mathcal{R})$, respectively.

Remark 4.2. Hence, by Remark 3.2, it is clear that if \prec is a certain order relation on X , then for any $A \subset X$ and $b \in X$ we have $b \in \text{ub}_{\prec}(A)$ if and only if $A \prec b$ in the sense that $a \prec b$ for all $a \in A$.

By Definitions 3.1 and 4.1, we evidently have the following

Theorem 4.3. If \mathcal{R} is a relator on X to Y , and $A \subset X$ and $B \subset Y$, then

$$\text{ub}_{\mathcal{R}}(A) = \{y \in Y : \exists R \in \mathcal{R} : A \times \{y\} \subset R\}$$

and

$$\text{lb}_{\mathcal{R}}(B) = \{x \in X : \exists R \in \mathcal{R} : \{x\} \times B \subset R\}.$$

Hence, it is clear that, analogously to Theorem 3.3, we also have

Theorem 4.4. If \mathcal{R} is a relator on X to Y , then

$$\text{ub}_{\mathcal{R}} = \text{lb}_{\mathcal{R}^{-1}} \qquad \text{and} \qquad \text{lb}_{\mathcal{R}} = \text{ub}_{\mathcal{R}^{-1}}.$$

Furthermore, as an immediate consequence of the corresponding definitions, we can also state the following

Theorem 4.5. *If \mathcal{R} is a relator on X to Y , and $x \in X$ and $y \in Y$, then*

$$\text{lb}_{\mathcal{R}}^{-1}(x) = \text{Lb}_{\mathcal{R}}^{-1}(x) \quad \text{and} \quad \text{ub}_{\mathcal{R}}^{-1}(y) = \text{Ub}_{\mathcal{R}}^{-1}(y).$$

Hint. To prove the first statement, note that for any $B \subset Y$ we have

$$B \in \text{lb}_{\mathcal{R}}^{-1}(x) \iff x \in \text{lb}_{\mathcal{R}}(B) \iff \{x\} \in \text{Lb}_{\mathcal{R}}(B) \iff B \in \text{Lb}_{\mathcal{R}}^{-1}(x). \quad \square$$

Concerning the inverses of the relations $\text{lb}_{\mathcal{R}}$ and $\text{ub}_{\mathcal{R}}$, it is also worth proving

Theorem 4.6. *If \mathcal{R} is a relator on X to Y , and $A \subset X$ and $B \subset Y$, then*

$$\text{Ub}_{\mathcal{R}}(A) \subset \bigcap_{a \in A} \text{lb}_{\mathcal{R}}^{-1}(a) \quad \text{and} \quad \text{Lb}_{\mathcal{R}}(B) \subset \bigcap_{b \in B} \text{ub}_{\mathcal{R}}^{-1}(b).$$

Proof. By Theorems 3.17, 3.3 and 4.5, we have

$$\text{Ub}_{\mathcal{R}}(A) \subset \bigcap_{a \in A} \text{Ub}_{\mathcal{R}}(a) = \bigcap_{a \in A} \text{Lb}_{\mathcal{R}}^{-1}(a) = \bigcap_{a \in A} \text{lb}_{\mathcal{R}}^{-1}(a).$$

Hence, by Theorems 3.3 and 4.4, it is clear that the second statement of the theorem is also true. \square

However, it is now more important to prove the following

Theorem 4.7. *If \mathcal{R} is a relator on X to Y , and $A \subset X$ and $B \subset Y$, then*

$$\text{Ub}_{\mathcal{R}}(A) \subset \mathcal{P}(\text{ub}_{\mathcal{R}}(A)) \quad \text{and} \quad \text{Lb}_{\mathcal{R}}(B) \subset \mathcal{P}(\text{lb}_{\mathcal{R}}(B)).$$

Proof. If $D \in \text{Ub}_{\mathcal{R}}(A)$, then by Remark 3.13 and Definition 4.1 we evidently have $\{d\} \in \text{Ub}_{\mathcal{R}}(A)$, and hence $d \in \text{ub}_{\mathcal{R}}(A)$ for all $d \in D$. Therefore, $D \subset \text{ub}_{\mathcal{R}}(A)$, and thus the first statement of the theorem is true.

The second statement of the theorem is again immediate from the first one by Theorems 3.3 and 4.4. \square

Remark 4.8. The second statement of Theorem 4.7 can also be derived from the first statement of Theorem 4.6.

Namely, by Theorem 3.3, $A \in \text{Lb}_{\mathcal{R}}(B)$ implies $B \in \text{Ub}_{\mathcal{R}}(A)$. Moreover, for any $A \subset X$ and $B \subset Y$, we have

$$\begin{aligned} B \in \bigcap_{a \in A} \text{lb}_{\mathcal{R}}^{-1}(a) &\iff \forall a \in A : B \in \text{lb}_{\mathcal{R}}^{-1}(a) \iff \\ &\iff \forall a \in A : a \in \text{lb}_{\mathcal{R}}(B) \iff A \subset \text{lb}_{\mathcal{R}}(B) \iff A \in \mathcal{P}(\text{lb}_{\mathcal{R}}(B)). \end{aligned}$$

By Theorem 3.5 and Definition 4.1, we evidently have the following

Theorem 4.9. *If \mathcal{R} is a relator on X to Y and $A \subset X$, then*

$$\text{ub}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} R^c(A)^c = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} R(a).$$

Now, as an immediate consequence of Theorems 4.9 and 4.4, we can also state

Theorem 4.10. *If \mathcal{R} is a relator on X to Y , then*

$$\text{ub}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{ub}_R \quad \text{and} \quad \text{lb}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{lb}_R.$$

From Theorems 3.9 and 3.10, by the corresponding definitions, it is clear that we also have the following two theorems.

Theorem 4.11. *If \mathcal{R} is a relator on X to Y , then*

$$\text{lb}_{\mathcal{R}} = (\text{cl}_{\mathcal{R}^c})^c \quad \text{and} \quad \text{cl}_{\mathcal{R}} = (\text{lb}_{\mathcal{R}^c})^c.$$

Theorem 4.12. *If \mathcal{R} is a relator on X to Y , then*

$$\text{lb}_{\mathcal{R}} = \text{int}_{\mathcal{R}^c} \circ \mathcal{C} \quad \text{and} \quad \text{int}_{\mathcal{R}} = \text{lb}_{\mathcal{R}^c} \circ \mathcal{C}.$$

Remark 4.13. The first statement of Theorem 4.11 can also be easily derived from Theorem 4.9 by noticing that

$$\begin{aligned} \text{lb}_{\mathcal{R}}(B) &= \text{ub}_{\mathcal{R}^{-1}}(B) = \bigcup_{R \in \mathcal{R}} (R^{-1})^c(B)^c = \bigcup_{R \in \mathcal{R}} (R^c)^{-1}(B)^c = \\ &= \left(\bigcap_{R \in \mathcal{R}} (R^c)^{-1}(B) \right)^c = \text{cl}_{\mathcal{R}^c}(B)^c = (\text{cl}_{\mathcal{R}^c})^c(B) \end{aligned}$$

for all $B \subset Y$. Moreover, Theorem 4.12 can also be easily derived from Theorem 4.11 by using that $\text{cl}_{\mathcal{R}} = (\text{int}_{\mathcal{R}} \circ \mathcal{C})^c$.

As a very particular cases of Theorem 4.9, we can also state the following

Theorem 4.14. *If \mathcal{R} is a relator on X to Y and $x \in X$, then*

$$\sigma_{\mathcal{R}}(x) = \text{ub}_{\mathcal{R}}(x).$$

Proof. By the corresponding definitions and Theorem 4.9, it is clear that

$$\sigma_{\mathcal{R}}(x) = \left(\bigcup \mathcal{R} \right) (x) = \left(\bigcup_{R \in \mathcal{R}} R \right) (x) = \bigcup_{R \in \mathcal{R}} R(x) = \text{ub}_{\mathcal{R}}(x). \quad \square$$

Hence, by using that $\sigma_{\mathcal{R}} = (\delta_{\mathcal{R}^c})^c$, we can immediately get the following

Theorem 4.15. *If \mathcal{R} is a relator on X to Y and $x \in X$, then*

$$\text{ub}_{\mathcal{R}}(x) = \delta_{\mathcal{R}^c}(x)^c \quad \text{and} \quad \delta_{\mathcal{R}}(x) = \text{ub}_{\mathcal{R}^c}(x)^c.$$

Remark 4.16. Note that this theorem is actually a particular case of Theorem 4.11 since we have $\delta_{\mathcal{R}}(x) = \text{cl}_{\mathcal{R}^{-1}}(x)$ for all $x \in X$.

Now, by using Theorems 4.9 and 4.15, we can also easily prove the following two theorems.

Theorem 4.17. *If \mathcal{R} is a relator on X to Y , then*

$$E_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} R^c(X)^c = \bigcap_{R \in \mathcal{R}} \text{ub}_R(X).$$

Proof. By the corresponding definitions and Theorem 4.9, it is clear that

$$E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \bigcap_{x \in X} R(x) = \bigcap_{R \in \mathcal{R}} R^c(X)^c = \bigcap_{R \in \mathcal{R}} \text{ub}_R(X). \quad \square$$

Theorem 4.18. *If \mathcal{R} is a relator on X to Y , then*

$$D_{\mathcal{R}} = (\delta_{\mathcal{R}})^c(X) = \bigcup_{x \in X} \text{ub}_{\mathcal{R}^c}(x).$$

Proof. By the corresponding definitions and Theorem 4.15, it is clear that

$$E_{\mathcal{R}} = \bigcap_{x \in X} \bigcap_{R \in \mathcal{R}} R(x) = \bigcap_{x \in X} \left(\bigcap \mathcal{R} \right) (x) = \bigcap_{x \in X} \delta_{\mathcal{R}}(x) = \bigcap_{x \in X} \text{ub}_{\mathcal{R}^c}(x)^c.$$

Hence, since $D_{\mathcal{R}} = X \setminus E_{\mathcal{R}}$, it is clear that the second statement of the theorem is also true. \square

From Theorems 3.12, 3.16 and 3.17, by Definition 4.1, it is clear that we also have the following three theorems.

Theorem 4.19. *If \mathcal{R} is a relator on X to Y , then*

- (1) $\text{ub}_{\mathcal{R}}(\emptyset) = Y$; (2) $\text{ub}_{\mathcal{R}}(A) \subset \text{ub}_{\mathcal{R}}(C)$ for all $C \subset A \subset X$.

Theorem 4.20. *If \mathcal{R} is a relator on X to Y , then*

$$\text{ub}_{\mathcal{R}} = \text{ub}_{\mathcal{R}} \circ \mathcal{P}^{-1} \qquad \text{and} \qquad \text{ub}_{\mathcal{R}} = ((\text{ub}_{\mathcal{R}})^c \circ \mathcal{P})^c.$$

Theorem 4.21. *If \mathcal{R} is a relator on X to Y , and $A \subset X$ and $B \subset Y$, then*

$$\text{ub}_{\mathcal{R}}(A) \subset \bigcap_{a \in A} \text{ub}_{\mathcal{R}}(a) \qquad \text{and} \qquad \text{lb}_{\mathcal{R}}(B) \subset \bigcap_{b \in B} \text{lb}_{\mathcal{R}}(b).$$

However, in contrast to Theorem 3.19, we can only prove the following

Theorem 4.22. *If \mathcal{R} is a relator on X to Y , and $A \subset X$ and $B \subset Y$, then*

$$\text{ub}_{\mathcal{R}}(A) \subset \{d \in Y : A \subset \text{lb}_{\mathcal{R}}(d)\} \quad \text{and} \quad \text{lb}_{\mathcal{R}}(B) \subset \{c \in X : B \subset \text{ub}_{\mathcal{R}}(c)\}.$$

Proof. By Definition 4.1 and Theorems 3.3 and 4.7, it is clear that

$$d \in \text{ub}_{\mathcal{R}}(A) \Rightarrow \{d\} \in \text{Ub}_{\mathcal{R}}(A) \Rightarrow A \in \text{Lb}_{\mathcal{R}}(d) \Rightarrow A \subset \text{lb}_{\mathcal{R}}(d). \quad \square$$

Remark 4.23. Note that if in particular \mathcal{R} is a singleton, then the equalities are also true in the assertions of Theorems 4.6, 4.7, 4.21 and 4.22.

5. Proximal maxima and minima.

Definition 5.1. *If \mathcal{R} is a relator on X , then we define two relations $\text{Max}_{\mathcal{R}}$ and $\text{Min}_{\mathcal{R}}$ on $\mathcal{P}(X)$ to itself such that for all $A \subset X$*

$$\text{Max}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \text{Ub}_{\mathcal{R}}(A) \qquad \text{and} \qquad \text{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \text{Lb}_{\mathcal{R}}(A).$$

The members of the families $\text{Max}_{\mathcal{R}}(A)$ and $\text{Min}_{\mathcal{R}}(A)$ are called the proximal maxima and minima of the set A in the relator space $X(\mathcal{R})$, respectively.

Remark 5.2. Hence, by Remark 3.2, it is clear that if \prec is a certain order relation on X , then for any $A, B \subset X$ we have $B \in \text{Max}_{\mathcal{R}}(A)$ if and only if $B \subset A$ and $A \prec B$.

Moreover, by Definitions 3.1 and 5.1, we evidently have the following

Theorem 5.3. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\text{Max}_{\mathcal{R}}(A) = \{B \subset A : \exists R \in \mathcal{R} : A \times B \subset R\}$$

and

$$\text{Min}_{\mathcal{R}}(A) = \{B \subset A : \exists R \in \mathcal{R} : B \times A \subset R\}.$$

By Theorem 3.3 and Definition 5.1, it is clear that we also have the following

Theorem 5.4. *If \mathcal{R} is a relator on X , then*

$$\text{Max}_{\mathcal{R}} = \text{Min}_{\mathcal{R}^{-1}} \quad \text{and} \quad \text{Min}_{\mathcal{R}} = \text{Max}_{\mathcal{R}^{-1}}.$$

Moreover, as an immediate consequence of the corresponding definitions and Theorem 3.3, we can also state the following

Theorem 5.5. *If \mathcal{R} is a relator on X , then*

$$\text{Max}_{\mathcal{R}}^{-1} = \mathcal{P}^{-1} \cap \text{Lb}_{\mathcal{R}} \quad \text{and} \quad \text{Min}_{\mathcal{R}}^{-1} = \mathcal{P}^{-1} \cap \text{Ub}_{\mathcal{R}}.$$

Hint. To prove the first statement, note that $\text{Max}_{\mathcal{R}} = \mathcal{P} \cap \text{Ub}_{\mathcal{R}}$, and thus

$$\text{Max}_{\mathcal{R}}^{-1} = (\mathcal{P} \cap \text{Ub}_{\mathcal{R}})^{-1} = \mathcal{P}^{-1} \cap \text{Ub}_{\mathcal{R}}^{-1} = \mathcal{P}^{-1} \cap \text{Lb}_{\mathcal{R}}. \quad \square$$

From Theorem 3.5, by using Definition 5.1, we can easily get the following

Theorem 5.6. *If \mathcal{R} is a relator on X and $\emptyset \neq A \subset X$, then*

$$\text{Max}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} \mathcal{P}(A \setminus R^c(A)) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}(A \cap R(a)).$$

Proof. By the corresponding definitions and Theorem 3.5, we have

$$\begin{aligned} \text{Max}_{\mathcal{R}}(A) &= \mathcal{P}(A) \cap \text{Ub}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \bigcup_{R \in \mathcal{R}} \mathcal{P}(R^c(A)^c) = \\ &= \bigcup_{R \in \mathcal{R}} \mathcal{P}(A) \cap \mathcal{P}(R^c(A)^c) = \bigcup_{R \in \mathcal{R}} \mathcal{P}(A \cap R^c(A)^c) = \bigcup_{R \in \mathcal{R}} \mathcal{P}(A \setminus R^c(A)). \end{aligned}$$

Moreover, it is clear that we also have

$$\begin{aligned} \text{Max}_{\mathcal{R}}(A) &= \mathcal{P}(A) \cap \text{Ub}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}(R(a)) = \\ &= \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}(A) \cap \mathcal{P}(R(a)) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} \mathcal{P}(A \cap R(a)). \quad \square \end{aligned}$$

Remark 5.7. Note that the first equality in the above theorem is also true for the case $A = \emptyset$. However, if $A = \emptyset$, then the left hand side of the second equality is $\{\emptyset\}$, while its right hand side is $\mathcal{P}(X)$.

Now, as an immediate consequence of Theorems 5.6 and 5.4, we can also state

Theorem 5.8. *If \mathcal{R} is a relator on X , then*

$$\text{Max}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{Max}_R \quad \text{and} \quad \text{Min}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{Min}_R.$$

From Theorems 3.9 and 3.10, by using Definition 5.1, we can easily derive the following two theorems.

Theorem 5.9. *If \mathcal{R} is a relator on X , then*

$$\text{Min}_{\mathcal{R}} = \mathcal{P} \setminus \text{Cl}_{\mathcal{R}^c} \quad \text{and} \quad (\text{Min}_{\mathcal{R}^c})^c = \mathcal{P}^c \cup \text{Cl}_{\mathcal{R}}.$$

Proof. For any $A \subset X$, we have

$$\begin{aligned} \text{Min}_{\mathcal{R}}(A) &= \mathcal{P}(A) \cap \text{Lb}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap (\text{Cl}_{\mathcal{R}^c})^c(A) = \\ &= \mathcal{P}(A) \cap \text{Cl}_{\mathcal{R}^c}(A)^c = \mathcal{P}(A) \setminus \text{Cl}_{\mathcal{R}^c}(A) = (\mathcal{P} \setminus \text{Cl}_{\mathcal{R}^c})(A). \end{aligned}$$

Moreover, it is clear that we also have

$$\begin{aligned} (\text{Min}_{\mathcal{R}^c})^c(A) &= \text{Min}_{\mathcal{R}^c}(A)^c = (\mathcal{P}(A) \cap \text{Lb}_{\mathcal{R}^c}(A))^c = \mathcal{P}(A)^c \cup \text{Lb}_{\mathcal{R}^c}(A)^c = \\ &= \mathcal{P}^c(A) \cup (\text{Lb}_{\mathcal{R}^c})^c(A) = \mathcal{P}^c(A) \cup \text{Cl}_{\mathcal{R}}(A) = (\mathcal{P}^c \cup \text{Cl}_{\mathcal{R}})(A). \quad \square \end{aligned}$$

Theorem 5.10. *If \mathcal{R} is a relator on X , then*

$$\text{Min}_{\mathcal{R}} = \mathcal{P} \cap (\text{Int}_{\mathcal{R}^c} \circ \mathcal{C}) \quad \text{and} \quad \text{Min}_{\mathcal{R}^c} \circ \mathcal{C} = (\mathcal{P} \circ \mathcal{C}) \cap \text{Int}_{\mathcal{R}}.$$

Hint. To prove the second statement, note that for any $A \subset X$ we have

$$\begin{aligned} (\text{Min}_{\mathcal{R}^c} \circ \mathcal{C})(A) &= \text{Min}_{\mathcal{R}^c}(\mathcal{C}(A)) = \mathcal{P}(\mathcal{C}(A)) \cap \text{Lb}_{\mathcal{R}^c}(\mathcal{C}(A)) = \\ &(\mathcal{P} \circ \mathcal{C})(A) \cap (\text{Lb}_{\mathcal{R}^c} \circ \mathcal{C})(A) = (\mathcal{P} \circ \mathcal{C})(A) \cap \text{Int}_{\mathcal{R}}(A) = ((\mathcal{P} \circ \mathcal{C}) \cap \text{Int}_{\mathcal{R}})(A). \quad \square \end{aligned}$$

From Theorem 5.3, it is clear that the mapping $A \mapsto \text{Max}_{\mathcal{R}}(A)$ need not be monotonic. However, by Theorem 5.5 and the dual of Theorem 3.12, we evidently have the following

Theorem 5.11. *If \mathcal{R} is a relator on X , then*

- (1) $\text{Max}_{\mathcal{R}}^{-1}(\emptyset) = \mathcal{P}(X)$;
- (2) $\text{Max}_{\mathcal{R}}^{-1}(A) \subset \text{Max}_{\mathcal{R}}^{-1}(B)$ for all $B \subset A \subset X$.

Therefore, analogously to Theorem 3.16, we can also prove the following

Theorem 5.12. *If \mathcal{R} is a relator on X , then*

$$\text{Max}_{\mathcal{R}} = \mathcal{P} \circ \text{Max}_{\mathcal{R}} \quad \text{and} \quad \text{Max}_{\mathcal{R}} = (\mathcal{P}^{-1} \circ (\text{Max}_{\mathcal{R}})^c)^c.$$

Hint. To prove the second statement, note that by Theorem 5.11 we have $\text{Max}_{\mathcal{R}}^{-1} = ((\text{Max}_{\mathcal{R}}^{-1})^c \circ \mathcal{P})^c$. Hence, by using Theorem 1.3, we can infer that

$$\begin{aligned} \text{Max}_{\mathcal{R}} &= (\text{Max}_{\mathcal{R}}^{-1})^{-1} = \left(\left((\text{Max}_{\mathcal{R}}^{-1})^c \circ \mathcal{P} \right)^c \right)^{-1} = \\ &= \left(\left((\text{Max}_{\mathcal{R}})^c \right)^{-1} \circ \mathcal{P} \right)^{-1}{}^c = (\mathcal{P}^{-1} \circ (\text{Max}_{\mathcal{R}})^c)^c. \end{aligned} \quad \square$$

Moreover, as an immediate consequence of Theorem 3.19 and Definition 5.1, we can also state the following

Theorem 5.13. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\text{Max}_{\mathcal{R}}(A) = \{B \subset A : \mathcal{P}(A) \subset \text{Lb}_{\mathcal{R}}(B)\}$$

and

$$\text{Min}_{\mathcal{R}}(A) = \{B \subset A : \mathcal{P}(A) \subset \text{Ub}_{\mathcal{R}}(B)\}.$$

6. Topological maxima and minima.

Definition 6.1. *If \mathcal{R} is a relator on X , then we define two relations $\text{max}_{\mathcal{R}}$ and $\text{min}_{\mathcal{R}}$ on $\mathcal{P}(X)$ to X such that for all $A \subset X$*

$$\text{max}_{\mathcal{R}}(A) = \{b \in X : \{b\} \in \text{Max}_{\mathcal{R}}(A)\}$$

and

$$\text{min}_{\mathcal{R}}(A) = \{b \in X : \{b\} \in \text{Min}_{\mathcal{R}}(A)\}.$$

The members of families $\max_{\mathcal{R}}(A)$ and $\min_{\mathcal{R}}(A)$ are called the topological maxima and minima of the set A in the relator space $X(\mathcal{R})$, respectively.

Remark 6.2. Hence, by Remark 4.2, it is clear that if \prec is a certain order relation on X , then for any $A \subset X$ and $b \in X$ we have $b \in \max_{\prec}(A)$ if and only if $b \in A$ and $A \prec b$.

Moreover, by Definition 6.1 and Theorem 4.3, we evidently have the following

Theorem 6.3. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\max_{\mathcal{R}}(A) = \{b \in A : \exists R \in \mathcal{R} : A \times \{b\} \subset R\}$$

and

$$\min_{\mathcal{R}}(A) = \{b \in A : \exists R \in \mathcal{R} : \{b\} \times A \subset R\}.$$

From Theorem 5.4, by Definition 6.1, it is clear that we also have the following

Theorem 6.4. *If \mathcal{R} is a relator on X , then*

$$\max_{\mathcal{R}} = \min_{\mathcal{R}^{-1}} \quad \text{and} \quad \min_{\mathcal{R}} = \max_{\mathcal{R}^{-1}}.$$

Moreover, as an immediate consequence of the corresponding definitions, we can also state the following

Theorem 6.5. *If \mathcal{R} is a relator on X and $x \in X$, then*

$$\max_{\mathcal{R}}^{-1}(x) = \text{Max}_{\mathcal{R}}^{-1}(x) \quad \text{and} \quad \min_{\mathcal{R}}^{-1}(x) = \text{Min}_{\mathcal{R}}^{-1}(x).$$

However, it is now more important to note that in particular we also have

Theorem 6.6. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\max_{\mathcal{R}}(A) = A \cap \text{ub}_{\mathcal{R}}(A) \quad \text{and} \quad \min_{\mathcal{R}}(A) = A \cap \text{lb}_{\mathcal{R}}(A).$$

H i n t. To prove the first statement, note that for any $b \in X$ we have

$$\begin{aligned} b \in \max_{\mathcal{R}}(A) &\iff \{b\} \in \text{Max}_{\mathcal{R}}(A) \iff \{b\} \in \mathcal{P}(A), \quad \{b\} \in \text{Ub}_{\mathcal{R}}(A) \\ &\iff b \in A, \quad b \in \text{ub}_{\mathcal{R}}(A) \iff b \in \max_{\mathcal{R}}(A). \end{aligned} \quad \square$$

Now, as an immediate consequence of Theorems 4.7 and 6.6, we can also state

Theorem 6.7. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\text{Max}_{\mathcal{R}}(A) \subset \mathcal{P}(\text{max}_{\mathcal{R}}(A)) \quad \text{and} \quad \text{Min}_{\mathcal{R}}(A) \subset \mathcal{P}(\text{min}_{\mathcal{R}}(A)).$$

Hint. To prove the first inclusion, note that

$$\begin{aligned} \text{Max}_{\mathcal{R}}(A) &= \mathcal{P}(A) \cap \text{Ub}_{\mathcal{R}}(A) \subset \mathcal{P}(A) \cap \mathcal{P}(\text{ub}_{\mathcal{R}}(A)) = \\ &= \mathcal{P}(A \cap \text{ub}_{\mathcal{R}}(A)) = \mathcal{P}(\text{max}_{\mathcal{R}}(A)). \end{aligned} \quad \square$$

Remark 6.8. Note that if in particular \mathcal{R} is a singleton, then the corresponding equalities are also true.

From Theorem 4.9, by Definition 6.1, it is clear that we also have the following

Theorem 6.9. *If \mathcal{R} is a relator on X and $\emptyset \neq A \subset X$, then*

$$\text{max}_{\mathcal{R}}(A) = \bigcup_{R \in \mathcal{R}} (A \setminus R^c(A)) = \bigcup_{R \in \mathcal{R}} \bigcap_{a \in A} A \cap R(a).$$

Remark 6.10. Note that the first equality in the above theorem is also true for the case $A = \emptyset$. However, if $A = \emptyset$, then the left hand side of the second equality is \emptyset , while the right hand side is X .

Now, as an immediate consequence of Theorems 6.9 and 6.4, we can also state

Theorem 6.11. *If \mathcal{R} is a relator on X , then*

$$\text{max}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{max}_R \quad \text{and} \quad \text{min}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \text{min}_R.$$

From Theorems 5.9 and 5.10, by using Definition 6.1, we can easily get the following two theorems.

Theorem 6.12. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\text{min}_{\mathcal{R}}(A) = A \setminus \text{cl}_{\mathcal{R}^c}(A) \quad \text{and} \quad \text{min}_{\mathcal{R}^c}(A)^c = A^c \cup \text{cl}_{\mathcal{R}}(A).$$

Theorem 6.13. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\min_{\mathcal{R}}(A) = A \cap \text{int}_{\mathcal{R}^c}(A^c) \quad \text{and} \quad \min_{\mathcal{R}^c}(A^c) = A^c \cap \text{int}_{\mathcal{R}}(A).$$

Moreover, from Theorems 4.14, 4.15, 4.17 and 4.22, by Theorem 6.6, it is clear that we also have the following three theorems.

Theorem 6.14. *If \mathcal{R} is a relator on X and $x \in X$, then*

$$\max_{\mathcal{R}}(x) = \{x\} \cap \sigma_{\mathcal{R}}(x) = \{x\} \setminus \delta_{\mathcal{R}^c}(x).$$

Theorem 6.15. *If \mathcal{R} is a relator on X , then*

$$E_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \max_R(X).$$

Theorem 6.16. *If \mathcal{R} is a relator on X and $A \subset X$, then*

$$\max_{\mathcal{R}}(A) \subset \{b \in A : A \subset \text{lb}_{\mathcal{R}}(b)\} \quad \text{and} \quad \min_{\mathcal{R}}(A) \subset \{b \in A : A \subset \text{ub}_{\mathcal{R}}(b)\}.$$

Remark 6.17. Note that if in particular \mathcal{R} is a singleton, then the corresponding equalities are also true.

7. Proximal self upper and lower bound sets.

Definition 7.1. *If \mathcal{R} is a relator on X , then we define*

$$u_{\mathcal{R}} = \{A \subset X : A \in \text{Ub}_{\mathcal{R}}(A)\}.$$

The members of the family $u_{\mathcal{R}}$ are called the proximal self upper bound subsets of the relator space $X(\mathcal{R})$.

Remark 7.2. Hence, by Remark 3.2, it is clear that if \prec is a certain order relation on X , then for any $A \subset X$ we have $A \in u_{\mathcal{R}}$ if and only if $A \prec A$.

Moreover, by Definitions 3.1 and 7.1, we evidently have the following

Theorem 7.3. *If \mathcal{R} is a relator on X , then*

$$u_{\mathcal{R}} = \{A \subset X : \exists R \in \mathcal{R} : A^2 \subset R\}.$$

Remark 7.4. Hence, it is clear that if d is a certain distance function on X , then $u_{\mathcal{R}_d}$ is just the family of all bounded subsets of the space $X(d)$.

From Theorem 7.3, it is clear that the following two theorems are also true.

Theorem 7.5. *If \mathcal{R} is a relator on X , then*

$$u_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} u_R.$$

Theorem 7.6. *If \mathcal{R} is a relator on X , then $u_{\mathcal{R}}$ is a nonvoid descending family such that*

$$u_{\mathcal{R}} = u_{\mathcal{R}^{-1}}.$$

By the corresponding definitions and Theorem 3.3, it is clear that we also have

Theorem 7.7. *If \mathcal{R} is a relator on X and $A \subset X$, then the following assertions are equivalent:*

- | | |
|--|--|
| (1) $A \in u_{\mathcal{R}};$ | (2) $A \in \text{Lb}_{\mathcal{R}}(A);$ |
| (3) $A \in \text{Max}_{\mathcal{R}}(A);$ | (4) $A \in \text{Min}_{\mathcal{R}}(A).$ |

Now, by using Theorems 5.3 and 7.3, we can also easily prove the following

Theorem 7.8. *If \mathcal{R} is a relator on X , then*

$$u_{\mathcal{R}} = \text{Max}_{\mathcal{R}}(\mathcal{P}(X)) \quad \text{and} \quad u_{\mathcal{R}} = \text{Min}_{\mathcal{R}}(\mathcal{P}(X)).$$

Proof. If $A \in u_{\mathcal{R}}$, then by Theorem 7.7 we also have $A \in \text{Max}_{\mathcal{R}}(A)$. Hence, since $A \in \mathcal{P}(A)$, it is clear that $A \in \text{Max}_{\mathcal{R}}(\mathcal{P}(X))$ is also true.

On the other hand, if $A \in \text{Max}_{\mathcal{R}}(\mathcal{P}(X))$, then there exists $B \subset X$ such that $A \in \text{Max}_{\mathcal{R}}(B)$. Hence, by Theorem 5.3, it follows that $A \subset B$, and there exist $R \in \mathcal{R}$ such that $B \times A \subset R$. These, in particular, imply that $A^2 \subset R$. Hence, by Theorem 7.3, it is clear that $A \in u_{\mathcal{R}}$ is also true.

Therefore, the first statement of the theorem is true. The second statement of the theorem is immediate from the first one by Theorems 7.6 and 5.4. \square

Moreover, by using Theorems 3.9 and 3.10, we can easily establish the following

Theorem 7.9. *If \mathcal{R} is a relator on X and $A \subset X$, then the following assertions are equivalent:*

- (1) $A \in u_{\mathcal{R}}$; (2) $A \notin Cl_{\mathcal{R}^c}(A)$; (3) $A \in Int_{\mathcal{R}^c}(A^c)$.

8. Topological self upper and lower bound sets.

Definition 8.1. *If \mathcal{R} is a relator on X , then we define*

$$\mathcal{U}_{\mathcal{R}} = \{A \subset X : A \subset \text{ub}_{\mathcal{R}}(A)\} \quad \text{and} \quad \mathcal{L}_{\mathcal{R}} = \{A \subset X : A \subset \text{lb}_{\mathcal{R}}(A)\}.$$

The members of the families $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{L}_{\mathcal{R}}$ are called the topological self upper and lower bound subsets of the relator space $X(\mathcal{R})$, respectively.

Remark 8.2. Hence, by Remark 4.2, it is clear that if \prec is a certain order relation on X , then for any $A \subset X$ we have $A \in \mathcal{U}_{\mathcal{R}}$ if and only if $A \prec a$ for all $a \in A$.

Moreover, by Theorem 4.3 and Definition 8.1, we evidently have the following

Theorem 8.3. *If \mathcal{R} is a relator on X , then*

$$\mathcal{U}_{\mathcal{R}} = \{A \subset X : \forall a \in A : \exists R \in \mathcal{R} : A \times \{a\} \subset R\}$$

and

$$\mathcal{L}_{\mathcal{R}} = \{A \subset X : \forall a \in A : \exists R \in \mathcal{R} : \{a\} \times A \subset R\}.$$

Hence, is clear that we also have the following

Theorem 8.4. *If \mathcal{R} is a relator on X , then $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{L}_{\mathcal{R}}$ are nonvoid descending families such that*

$$\mathcal{U}_{\mathcal{R}} = \mathcal{L}_{\mathcal{R}^{-1}} \quad \text{and} \quad \mathcal{L}_{\mathcal{R}} = \mathcal{U}_{\mathcal{R}^{-1}}.$$

Moreover, by using Theorems 4.7, 7.6 and 8.4, we can easily prove the following

Theorem 8.5. *If \mathcal{R} is a relator on X , then*

$$u_{\mathcal{R}} \subset \mathcal{U}_{\mathcal{R}} \cap \mathcal{L}_{\mathcal{R}}.$$

Proof. By the corresponding definitions and Theorem 4.7, it is clear that

$$A \in u_{\mathcal{R}} \Rightarrow A \in \text{Ub}_{\mathcal{R}}(A) \Rightarrow A \in \mathcal{P}(\text{ub}_{\mathcal{R}}(A)) \Rightarrow A \in \mathcal{U}_{\mathcal{R}}.$$

Therefore, $u_{\mathcal{R}} \subset \mathcal{U}_{\mathcal{R}}$. Hence, by using Theorems 7.6 and 8.4, we can see that

$$u_{\mathcal{R}} = u_{\mathcal{R}^{-1}} \subset \mathcal{U}_{\mathcal{R}^{-1}} = \mathcal{L}_{\mathcal{R}}. \quad \square$$

By the corresponding definitions and the dual Remark 4.8, it is clear that we also have the following

Theorem 8.6. *If \mathcal{R} is a relator on X and $A \subset X$, then the following assertions are equivalent:*

$$(1) \ A \in \mathcal{U}_{\mathcal{R}}; \quad (2) \ A = \max_{\mathcal{R}}(A); \quad (3) \ A \in \bigcap_{a \in A} \text{ub}_{\mathcal{R}}^{-1}(a).$$

In addition to this theorem, it is also worth proving the following

Theorem 8.7. *If \mathcal{R} is a relator on X , then*

$$\mathcal{U}_{\mathcal{R}} = \{\max_{\mathcal{R}}(A) : A \subset X\} \quad \text{and} \quad \mathcal{L}_{\mathcal{R}} = \{\min_{\mathcal{R}}(A) : A \subset X\}.$$

Proof. By Theorem 8.6, we evidently have $\mathcal{U}_{\mathcal{R}} \subset \{\max_{\mathcal{R}}(A) : A \subset X\}$. Moreover, if $A \subset X$ and $B = \max_{\mathcal{R}}(A)$, then by Theorem 6.3, it follows that $B \subset A$, and for each $b \in B$ there exists $R \in \mathcal{R}$ such that $A \times \{b\} \subset R$. This, in particular, implies that for each $b \in B$ there exists $R \in \mathcal{R}$ such that $B \times \{b\} \subset R$. Hence, by Theorem 8.3, it follows that $B \in u_{\mathcal{R}}$.

Therefore, the first statement of the theorem is true. The second statement of the theorem is immediate from the first one by Theorems 6.4 and 8.4. \square

Moreover, by Theorems 4.11 and 4.12, it is clear that we also have the following

Theorem 8.8. *If \mathcal{R} is a relator on X and $A \subset X$, then the following assertions are equivalent:*

$$(1) \ A \in \mathcal{L}_{\mathcal{R}}; \quad (2) \ \text{cl}_{\mathcal{R}^c}(A) \subset A^c; \quad (3) \ A \subset \text{int}_{\mathcal{R}^c}(A^c).$$

9. The unicity of topological maxima and minima.

Definition 9.1. *A relator \mathcal{R} on X is called antisymmetric if for all $R, S \in \mathcal{R}$*

$$R \cap S^{-1} \subset \Delta_X.$$

Remark 9.2. More precisely, in this case, we should rather say that the relator \mathcal{R} is uniformly antisymmetric.

Namely, by using some basic operations on relators [12], the above condition can be expressed in the form that $(\mathcal{R} \wedge \mathcal{R}^{-1})^c \subset (\{\Delta_X\}^c)^*$.

By Definition 9.1, we evidently have the following

Theorem 9.3. *If \mathcal{R} is a relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) \mathcal{R}^{-1} is antisymmetric.

Moreover, as a useful characterization of antisymmetric relators, we can prove

Theorem 9.4. *If \mathcal{R} is a relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) $\bigcup \mathcal{R}$ is antisymmetric;
- (3) $\text{ub}_{\mathcal{R}}(x) \cap \text{lb}_{\mathcal{R}}(x) \subset \{x\}$ for all $x \in X$.

Proof. If the assertion (1) holds, then by the corresponding definitions we have

$$\begin{aligned} (\bigcup \mathcal{R}) \cap (\bigcup \mathcal{R})^{-1} &= (\bigcup \mathcal{R}) \cap (\bigcup \mathcal{R}^{-1}) = \\ &= \left(\bigcup_{R \in \mathcal{R}} R \right) \cap \left(\bigcup_{S \in \mathcal{R}} S^{-1} \right) = \bigcup_{R \in \mathcal{R}} \bigcup_{S \in \mathcal{R}} (R \cap S^{-1}) \subset \Delta_X. \end{aligned}$$

Therefore, the assertion (2) also holds.

While, if the assertion (2) holds, then we have

$$\left(\bigcup \mathcal{R} \right) \cap \left(\bigcup \mathcal{R}^{-1} \right) = \left(\bigcup \mathcal{R} \right) \cap \left(\bigcup \mathcal{R} \right)^{-1} \subset \Delta_X.$$

Hence, by using Theorems 4.14 and 4.4, we can infer that

$$\begin{aligned} \text{ub}_{\mathcal{R}}(x) \cap \text{lb}_{\mathcal{R}}(x) &= \left(\bigcup \mathcal{R} \right)(x) \cap \left(\bigcup \mathcal{R}^{-1} \right)(x) = \\ &= \left(\left(\bigcup \mathcal{R} \right) \cap \left(\bigcup \mathcal{R}^{-1} \right) \right)(x) \subset \Delta_X(x) = \{x\} \end{aligned}$$

for all $x \in X$. Therefore, the assertion (3) also holds.

Finally, if the assertion (3) holds and $R, S \in \mathcal{R}$, then again by Theorems 4.14 and 4.4 it is clear that

$$\begin{aligned} (R \cap S^{-1})(x) &= R(x) \cap S^{-1}(x) \subset \\ &\subset \left(\bigcup \mathcal{R} \right)(x) \cap \left(\bigcup \mathcal{R}^{-1} \right)(x) = \text{ub}_{\mathcal{R}}(x) \cap \text{lb}_{\mathcal{R}}(x) \subset \{x\} = \Delta_X(x) \end{aligned}$$

for all $x \in X$. Therefore, $R \cap S^{-1} \subset \Delta_X$, and thus the assertion (1) also holds. \square

From Theorem 9.4, it is clear that in particular we also have

Corollary 9.5. *If \mathcal{R} is a relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) there exists an antisymmetric relation V on X such that $\mathcal{R} \subset \mathcal{P}(V)$.

Proof. If the assertion (1) holds, then by defining $V = \bigcup \mathcal{R}$ and using Theorem 9.4 we can at once see that the assertion (2) also holds.

While, if the assertion (2) holds and $R, S \in \mathcal{R}$, then we can also at once see that $R \cap S^{-1} \subset V \cap V^{-1} \subset \Delta_X$. Therefore, the assertion (1) also holds. \square

The importance of antisymmetric relators lies mainly in the following

Theorem 9.6. *If \mathcal{R} is a reflexive relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) $\text{card}(A) \leq 1$ for all $A \in \mathcal{U}_{\mathcal{R}}$;
- (3) $\text{card}(\max_{\mathcal{R}}(A)) \leq 1$ for all $A \subset X$.

Proof. If the assertion (2) does not hold, then there exists $A \in \mathcal{U}_{\mathcal{R}}$ such that $\text{card}(A) \geq 2$. Therefore, there exist $x, y \in A$ such that $x \neq y$. Hence, by Definition 8.1, it follows that $x, y \in \text{ub}_{\mathcal{R}}(A)$. Therefore, by Theorem 4.3, there exist $R, S \in \mathcal{R}$ such that $A \times \{x\} \subset R$ and $A \times \{y\} \subset S$. Thus, in particular, we also have $(y, x) \in R$ and $(x, y) \in S$. However, this implies that $(y, x) \in R \cap S^{-1}$. Therefore, the assertion (1) does not also hold. Thus, the implication (1) \Rightarrow (2) is true.

While, if the assertion (1) does not hold, then there exist $R, S \in \mathcal{R}$ such that $R \cap S^{-1} \not\subset \Delta_X$. Therefore, there exist $x, y \in X$, with $x \neq y$, such that $(x, y) \in R \cap S^{-1}$. This implies that $(x, y) \in R$ and $(y, x) \in S$. Now, by defining $A = \{x, y\}$ and using the reflexivity of R and S , we can see that $A \times \{y\} \subset R$ and $A \times \{x\} \subset S$. Therefore, by Theorem 4.3, we have $A \subset \text{ub}_{\mathcal{R}}(A)$. Hence, by Definition 8.1, it follows that $A \in \mathcal{U}_{\mathcal{R}}$. Therefore, the assertion (2) does not also hold. Thus, the implication (2) \Rightarrow (1) is also true.

Finally, to complete the proof, we note that the equivalence (2) \iff (3) is immediate from Theorem 8.7. \square

Remark 9.7. From the above proof, we can see that the implications (1) \Rightarrow (2) \iff (3) do not require the relator \mathcal{R} to be reflexive.

From Theorem 9.6, by using Theorems 9.3, 8.4 and 6.4, we can easily get

Theorem 9.8. *If \mathcal{R} is a reflexive relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) $\text{card}(A) \leq 1$ for all $A \in \mathcal{L}_{\mathcal{R}}$;
- (3) $\text{card}(\min_{\mathcal{R}}(A)) \leq 1$ for all $A \subset X$.

Remark 9.9. By Remark 9.7, it is clear that the implications $(1) \Rightarrow (2) \iff (3)$ do not require the relator \mathcal{R} to be reflexive.

10. The unicity of proximal maxima and minima. From Theorem 9.6, we can also easily get the following

Theorem 10.1. *If \mathcal{R} is a reflexive relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) $\text{card}(\text{Max}_{\mathcal{R}}(A)) \leq 2$ for all $A \subset X$.

Proof. If the assertion (2) does not hold, then there exists $A \subset X$ such that $\text{card}(\text{Max}_{\mathcal{R}}(A)) \geq 3$. Hence, since $\emptyset \in \text{Max}_{\mathcal{R}}(A)$, we can infer that there exist $B, C \in \text{Max}_{\mathcal{R}}(A)$, with $B \neq \emptyset$ and $C \neq \emptyset$, such that $B \neq C$. Here, we may assume, without loss of generality, that $B \not\subset C$. That is, there exists $b \in B$ such that $b \notin C$. Therefore, if $c \in C$, then $b \neq c$. Moreover, by Theorem 6.7, it is clear that $b, c \in \max_{\mathcal{R}}(A)$. Therefore, by Theorem 9.6, the assertion (1) does not also hold. Thus, the implication $(1) \Rightarrow (2)$ is true.

On the other hand, if the assertion (1) does not hold, then by Theorem 9.6 there exist $A \subset X$ such that $\text{card}(\max_{\mathcal{R}}(A)) \geq 2$. Therefore, there exist $x, y \in \max_{\mathcal{R}}(A)$ such that $x \neq y$. Hence, by Definition 6.1, it follows that $\{x\}, \{y\} \in \text{Max}_{\mathcal{R}}(A)$. Therefore, since $\emptyset \in \text{Max}_{\mathcal{R}}(A)$, we necessarily have $\text{card}(\text{Max}_{\mathcal{R}}(A)) \geq 3$. Thus, the assertion (2) does not also hold. Therefore, the implication $(2) \Rightarrow (1)$ is also true. \square

Remark 10.2. Note that the assertions 9.6(3) and 10.1(2) are actually equivalent for any relator \mathcal{R} on X .

Therefore, by Remark 9.7, the implication $(1) \Rightarrow (2)$ in Theorem 10.1 does not also require the relator \mathcal{R} to be reflexive.

Now, by Theorems 10.1, 9.3 and 5.4, it is clear that we also have

Theorem 10.3. *If \mathcal{R} is a reflexive relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) $\text{card}(\text{Min}_{\mathcal{R}}(A)) \leq 2$ for all $A \subset X$.

Remark 10.4. Moreover, by Remark 10.2, it is clear that the implication $(1) \Rightarrow (2)$ does not require the relator \mathcal{R} to be reflexive.

Analogously to Definition 9.1, we may also have the following

Definition 10.5. A relator \mathcal{R} on X is called *antifiltered* if for any $R, S \in \mathcal{R}$ and $x \in X$ there exists $T \in \mathcal{R}$ such that

$$(R \cap S^{-1})(x) \subset (T \cap T^{-1})(x).$$

Remark 10.6. More precisely, in this case we should rather say that the relator \mathcal{R} is topologically antifiltered.

Namely, by using some basic operations on relators [12], the above condition can be expressed in the form that $(\mathcal{R} \wedge \mathcal{R}^{-1})^c \subset ((\mathcal{R} \Delta \mathcal{R}^{-1})^c)^\wedge$.

By Definition 10.5, we evidently have the following

Theorem 10.7. If \mathcal{R} is a relator on X , then the following assertions are equivalent:

- (1) \mathcal{R} is antifiltered;
- (2) \mathcal{R}^{-1} is antifiltered.

Moreover, analogously to Theorem 9.6, we can also prove the following

Theorem 10.8. If \mathcal{R} is a reflexive and antifiltered relator on X , then the following assertions are equivalent:

- (1) \mathcal{R} is antisymmetric;
- (2) $\text{card}(A) \leq 1$ for all $A \in u_{\mathcal{R}}$;
- (3) $\text{card}(A) \leq 1$ for all $A \in \text{Max}_{\mathcal{R}}(\mathcal{P}(X))$.

Proof. If the assertion (2) does not hold, then there exists $A \in u_{\mathcal{R}}$ such that $\text{card}(A) \geq 2$. Hence, by Theorem 8.5, it follows that $A \in \mathcal{U}_{\mathcal{R}}$. Therefore, by Theorem 9.6, the assertion (1) does not also holds. Thus, the implication (1) \Rightarrow (2) is true.

While, if the assertion (1) does not hold, then by the second part of the proof of Theorem 9.6 there exist $R, S \in \mathcal{R}$ and $x, y \in X$, with $x \neq y$, such that $y \in (R \cap S^{-1})(x)$. Moreover, since \mathcal{R} is antifiltered, there exists $T \in \mathcal{R}$ such that $(R \cap S^{-1})(x) \subset (T \cap T^{-1})(x)$. Therefore, we also have $y \in (T \cap T^{-1})(x)$, and hence $(x, y) \in T$ and $(y, x) \in T$. Now, by defining $A = \{x, y\}$ and using the reflexivity of T , we can see that $A^2 \subset T$. Hence, by Theorem 7.3, it follows that $A \in u_{\mathcal{R}}$. Therefore, the assertion (2) does not also hold. Thus the implication (2) \Rightarrow (1) is also true.

Finally, to complete the proof, we note that the equivalence (2) \iff (3) is immediate from Theorem 7.8. \square

Remark 10.9. From the above proof and Remark 9.7, we can see that the implications (1) \Rightarrow (2) \iff (3) do not require the relator \mathcal{R} to be reflexive or antifiltered.

Now, by Theorems 10.8, 9.3 and 5.4, it is clear that we also have

Theorem 10.10. *If \mathcal{R} is a reflexive and antifiltered relator on X , then the following assertions are equivalent:*

- (1) \mathcal{R} is antisymmetric;
- (2) $\text{card}(A) \leq 1$ for all $A \in \text{Min}_{\mathcal{R}}(\mathcal{P}(X))$.

Remark 10.11. Moreover, by Remark 10.9, it is clear that the implication (1) \Rightarrow (2) does not require the relator \mathcal{R} to be reflexive or antifiltered.

11. Some supplementary notes and comments. If \mathcal{R} is a relator on X and $A \subset X$, then the members of the families

$$\text{Ub}_{\mathcal{R}}^*(A) = \{B \subset X : \mathcal{P}(A) \cap \text{Ub}_{\mathcal{R}}(B) \subset \text{Lb}_{\mathcal{R}}(B)\}$$

and

$$\text{Lb}_{\mathcal{R}}^*(A) = \{B \subset X : \mathcal{P}(A) \cap \text{Lb}_{\mathcal{R}}(B) \subset \text{Ub}_{\mathcal{R}}(B)\}$$

may be called the proximal quasi upper and lower bounds of the set A in the relator space $X(\mathcal{R})$, respectively. Namely, by Theorem 3.19, we have

$$\text{Ub}_{\mathcal{R}}(A) \subset \text{Ub}_{\mathcal{R}}^*(A) \quad \text{and} \quad \text{Lb}_{\mathcal{R}}(A) \subset \text{Lb}_{\mathcal{R}}^*(A).$$

Quite similarly, by Theorem 4.22, the members of the families

$$\text{ub}_{\mathcal{R}}^*(A) = \{b \in X : A \cap \text{ub}_{\mathcal{R}}(b) \subset \text{lb}_{\mathcal{R}}(b)\}$$

and

$$\text{lb}_{\mathcal{R}}^*(A) = \{b \in X : A \cap \text{lb}_{\mathcal{R}}(b) \subset \text{ub}_{\mathcal{R}}(b)\}$$

may be called the topological quasi upper and lower bounds of the set A in the relator space $X(\mathcal{R})$, respectively. However, if \mathcal{R} is not a singleton, then in contrast to Definition 6.1 we can only prove that

$$\{b : \{b\} \in \text{Ub}_{\mathcal{R}}^*(A)\} \subset \text{ub}_{\mathcal{R}}^*(A) \quad \text{and} \quad \{b : \{b\} \in \text{Lb}_{\mathcal{R}}^*(A)\} \subset \text{lb}_{\mathcal{R}}^*(A).$$

Now, analogously to Definitions 5.1 and 6.1, the members of the families

$$\text{Max}_{\mathcal{R}}^*(A) = \mathcal{P}(A) \cap \text{Ub}_{\mathcal{R}}^*(A) \quad \text{and} \quad \text{Min}_{\mathcal{R}}^*(A) = \mathcal{P}(A) \cap \text{Lb}_{\mathcal{R}}^*(A)$$

may be called the proximal quasi maxima and minima of the set A in the relator space $X(\mathcal{R})$, respectively. Moreover, the members of the families

$$\text{max}_{\mathcal{R}}^*(A) = A \cap \text{ub}_{\mathcal{R}}^*(A) \quad \text{and} \quad \text{min}_{\mathcal{R}}^*(A) = A \cap \text{lb}_{\mathcal{R}}^*(A).$$

may called the topological quasi maxima and minima of the set A in the relator space $X(\mathcal{R})$, respectively.

The members of the latter families may also be called the maximal and the minimal elements of the set A in the relator space $X(\mathcal{R})$, respectively. Namely, if for instance \prec is a certain order relation on X , then for any $A \subset X$ and $b \in X$ we have $b \in \max_{\mathcal{R}}^*(A)$ if and only if $b \in A$, and $b \prec a$ implies $a \prec b$ for all $a \in A$. That is, b is a maximal element of A in the usual sense [4, p. 30]. Therefore, in view of the various maximality principles (such as those of Zorn, Bourbaki [1], Bishop–Phelps, Brøndsted [2], Brézis–Browder, Altman and the present author [29], for instance), it seems to be of particular importance to find some algebraic or analytical conditions in order that the set $\max_{\mathcal{R}}^*(X)$ be nonempty.

Analogously to the families $u_{\mathcal{R}}$, $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{L}_{\mathcal{R}}$, we may also naturally consider the families

$$u_{\mathcal{R}}^* = \{A \subset X : A \in \text{Ub}_{\mathcal{R}}^*(A)\} \quad \text{and} \quad l_{\mathcal{R}}^* = \{A \subset X : A \in \text{Lb}_{\mathcal{R}}^*(A)\}.$$

and

$$\mathcal{U}_{\mathcal{R}}^* = \{A \subset X : A \subset \text{ub}_{\mathcal{R}}^*(A)\} \quad \text{and} \quad \mathcal{L}_{\mathcal{R}}^* = \{A \subset X : A \subset \text{lb}_{\mathcal{R}}^*(A)\}.$$

Concerning the above families, for instance, we can also prove that $u_{\mathcal{R}} \subset u_{\mathcal{R}}^*$ and $\mathcal{U}_{\mathcal{R}} \subset \mathcal{U}_{\mathcal{R}}^*$, and moreover

$$u_{\mathcal{R}} = \{A \subset X : \text{Max}_{\mathcal{R}}(A) \subset \text{Lb}_{\mathcal{R}}(A)\} = \{A \subset X : \text{Max}_{\mathcal{R}}(A) \subset \text{Min}_{\mathcal{R}}(A)\}.$$

Finally, we note that the members of the families

$$\text{Sup}_{\mathcal{R}}(A) = \text{Min}_{\mathcal{R}}(\text{Ub}_{\mathcal{R}}(A)) \quad \text{and} \quad \text{sup}_{\mathcal{R}}^*(A) = \text{min}_{\mathcal{R}}^*(\text{ub}_{\mathcal{R}}^*(A))$$

may, for instance, be called the proximal and the quasi topological suprema of the set A in the relator space $X(\mathcal{R})$, respectively. Concerning proximal suprema, for instance we can prove that

$$\text{Sup}_{\mathcal{R}}(A) = u_{\mathcal{R}} \cap \text{Ub}_{\mathcal{R}}(A) \quad \text{and} \quad u_{\mathcal{R}} = \{A \subset X : A \in \text{Sup}_{\mathcal{R}}(A)\}.$$

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