Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **29** (2003), 271-290

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

A NEW ALGORITHM FOR MONTE CARLO FOR AMERICAN OPTIONS

Roland Mallier, Ghada Alobaidi

Communicated by S. T. Rachev

Abstract. We consider the valuation of American options using Monte Carlo simulation, and propose a new technique which involves approximating the optimal exercise boundary. Our method involves splitting the boundary into a linear term and a Fourier series and using stochastic optimization in the form of a relaxation method to calculate the coefficients in the series. The cost function used is the expected value of the option using the the current estimate of the location of the boundary. We present some sample results and compare our results to other methods.

1. Introduction and numerical method. Options are derivative financial instruments which give the holder the right but not the obligation to buy (or sell) the underlying asset. American options are options which can be exercised either on or before a pre-determined expiry date. For such options there is, therefore, the possibility of early exercise, and the issue of whether and when to exercise an American option is one of the best-known problems in

2000 *Mathematics Subject Classification:* 91B28, 65C05.
Key words: American options; Monte Carlo.

mathematical finance, leading to an optimal exercise boundary and an optimal exercise policy, the following of which will maximize the expected return from the option. Regrettably, despite considerable efforts on the part of many researchers, no closed form solution has yet been found for this optimal exercise boundary, except in one or two special cases. One such special case is the American call with no dividends, when exercise is never optimal, so that the value of the option is the same as that of a European call; indeed, the value of an American call will differ from that of the European only if there is a dividend of sufficient size to make early exercise worthwhile. Another special case is the Roll-Geske-Whaley formula [45, 27, 28, 48] for the American call with discrete dividends. For cases where exact solutions are not known, an investor wishing to know the location of this free boundary must rely either on approximations, for example the Geske-Johnson formula discussed below [34, 29] for the American put, or else solve the problem numerically. One popular approximation is the use of series expansions close to expiry [5, 35, 23, 2, 3, 38]. Obviously, the location of this optimal exercise boundary is critical in correctly pricing an American option. By contrast, for European options, which can only be exercised at expiry, the value of the option can be calculated using the Black-Scholes-Merton option pricing formula [8, 41], either in terms of error functions or equivalently the cumulative probability density function for the normal distribution.

For American options, as mentioned above, to date, no closed form solutions have been found, and practitioners usually price such options either by approximations or by numerically solving the underlying equations. Some of the more popular approximations include quadratic approximation method used by MacMillan [36] for the valuation of an American put on a non-dividend paying stock, which has extended to stocks with dividends by Barone-Adesi and co-workers [6, 7, 1]; this method, which approximates the early exercise premium, *i.e.* the amount by which the value of an American exceeds a European, is very popular amongst institutional investors. Another well-known approximation is the Geske-Johnson formula [34, 29, 17, 9] for the American put. Selby & Hodges [46] give an overview of the Roll-Geske-Whaley and Geske-Johnson formulae together with an complete analysis of American call options with an arbitrary number of (discrete) dividends and a suggestion as to how to improve the numerical implementation of the Geske-Johnson formula; a review of the current state of the art of the computational aspects of this problem is given in [26]. If the numerical approach is taken, there are two principal ways of doing this. One approach involves directly integrating the stochastic d.e. for the price of the underlying security, which is assumed to follow a log-normal random walk,

$$dS = (r - D_0)\, Sdt + \sigma S dX, \tag{1.1}$$

where dS is the change in the stock price in the time interval dt, r is the risk-free rate, D_0 is the dividend yield, σ is the volatility and dX is a random walk. Black & Scholes [8] derived this equation in the absence of dividends, and Merton [41] added the effect of a constant dividend yield. While the assumption of a constant dividend yield is questionable for an option on a single security, it is justifiable for other options, such as foreign exchange, index options and options on commodities. Typically, this stochastic d.e. (1.1) is integrated numerically, and then the option valued by calculating the pay-off, which is max $(S - E, 0)$ in the case of a vanilla call and max $(E - S, 0)$ in the case of a vanilla put. Merton [42] observed that it is the boundary conditions that distinguish options, and in the stochastic framework, the only difference between the put and the call is the pay-off. Binomial and trinomial trees [21, 12] are two popular methods for integrating this equation, both of which involve integrating backwards in time from expiration rather than forwards in time from the time of purchase of the option.

An alternative approach involves using a no-arbitrage argument to transform the problem into the Black-Scholes-Merton partial differential equation for the value $V(S,t)$ of the option

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S\frac{\partial V}{\partial S} - r = 0, \tag{1.2}$$

and solving this together with the constraint that the value of the option cannot be less than the pay-off from immediate exercise. A popular method used together with this approach is finite-difference [14, 20, 50, 49]. Broadie & Detemple [16] give a review of all numerical methods.

Although finite-difference methods and binomial/trinomial trees are both well-suited to tackle the valuation of American options, another numerical method, Monte Carlo simulation, despite being one of the most flexible and popular methods available to financial practitioners, appears to less well-suited to that problem. As with the tree methods, Monte Carlo simulation, pioneered by Boyle [11], involves integrating the underlying stochastic d.e. (1.1), but involves marching forwards rather than backwards in time and typically involves generating a large number of realizations of the possible stock price and then averaging over those realizations to obtain an average or expected price. Monte Carlo methods encounter problems with the free boundary: in the real world, an investor holding an American option constantly has to decide whether it is optimal to hold or optimal to exercise, and, just as in the real world, a Monte Carlo simulation needs to make the same decisions. In theory, this can be done using Monte Carlo, but it would involve each path being split into a multitude of other paths at each time-step, and the number of realizations required quickly becomes impractical, as pointed out in [31], while Wilmott [49] has opined that Monte Carlo for American options is "very, very hard". Because of this, a number of techniques have been proposed over the years to enable Monte Carlo to be adapted to the valuation of American options, ranging from Malliavin calculus to direct calculation of the location of the free boundary. Examples of early attempts to apply Monte Carlo methods to American options include [47, 10, 18, 30, 44, 15, 13]. For a more detail bibliography, the reader is referred to [32], whose work motivated the present study. Fu *et al.* [24] recently gave a partial survey of some of the existing methods, considering three classes of methods: methods which attempt to mimic backwards induction methods [47, 30], methods which write the early exercise boundary in terms of parameters and optimize over those parameters, such as [25] for discrete dividends, and methods which are based on finding upper and lower bounds for the optimal exercise boundary [15].

In this study, we shall take the path of direct calculation of the location of the free boundary coupled with Monte Carlo simulation. However, while others [32] have proposed considering the position of the free boundary at a number of points and optimizing the location of the boundary at those points to maximize the expected pay-off from the option, we have taken a slightly different approach and supposed that the boundary is composed of a number of basis functions and then optimized the coefficients accompanying those functions to find the location of the boundary. The principal advantage of doing this is that we have only a small number of coefficients to optimize (in our simulations, typically about 100) rather than a large number of grid points (in our simulations, we typically had 2000 grid points), and therefore the dimension of the problem is significantly smaller. Thus, if we denote the location of the free boundary as $S = S_f(t)$, then while others have found the location of the boundary by varying the position of $S_f(t_1), \cdots, S_f(t_n)$, our approach is instead to assume that we can write

$$S_f(t) = \sum_{n=1}^{\infty} c_n \phi_n(t), \tag{1.3}$$

for some set of basis functions $\phi_n(t)$, and then truncate the series (1.3), so that we assume we can write

$$S_f(t) \approx \sum_{n=1}^{N} c_n \phi_n(t), \tag{1.4}$$

and then find the location of the free boundary by varying the c_n. Typically, one would define a cost function $V(S_0, t_0, S_f)$ to be the value of the option if it was purchased at time t_0 when the initial stock price was S_0 and if we assume that the exercise boundary is given by $S_f(t)$. Given the location of the boundary and the initial stock price, the value of the option can be calculated by Monte Carlo simulation.

Turning to specifics, we consider the valuation of both a plain vanilla American call and put with constant volatility. For these problems, we know the location of the free boundary at two points [49]: at expiry T, we know that for the call, if $r > D_0 > 0$ then

$$S_f(T) = S_T = Er/D_0 > E, \tag{1.5}$$

while similarly for the put

$$S_f(T) = S_T = E. \tag{1.6}$$

If $D_0 > r$, this behavior is reversed and for the call

$$S_f(T) = S_T = E, \tag{1.7}$$

while for the put

$$S_f(T) = S_T = Er/D_0 < E. \tag{1.8}$$

The reasons for this stem from the put-call 'symmetry' condition [19, 40], namely that the prices of the call and put are related by

$$C[S, E, D_0, r] = P[E, S, r, D_0], \tag{1.9}$$

while the positions of the optimal exercise boundary for the call and put are related by

$$S_f^c[t, E, r, D_0] = E^2 / S_f^p[t, E, D_0, r]. \tag{1.10}$$

Also, as $t \to -\infty$, we can use the perpetual American call and put to give us the location of the boundary in that limit, finding that

$$S_f(t) \to S^* = \frac{E}{1 - 1/\alpha^{\pm}}, \tag{1.11}$$

where

$$\alpha^{\pm} = \frac{1}{2\sigma^2}\left[\sigma^2 - 2(r - D_0) \pm \sqrt{4\,(r - D_0)^2 + 4D_0\sigma^2\,(r + D_0) + \sigma^4}\right],$$

where we take "+" for the call and "-" for the put. The behavior of perpetual American options without dividends was discussed in [41], and the extension to options with a continuous dividend yield is straightforward, and is discussed further in [39]. In terms of $\tau = T - t$, the tenor or remaining life of the option, we know the location of the free boundary at the points where $\tau = 0$ and $\tau \to \infty$. In this paper, we wish to find the value at time t_0 of an American option which expires at time $T \geq t_0$, or equivalently the value at $\tau_0 = T - t_0$ of an option that expires at $\tau = 0$. Rather than work directly with the semi-infinite interval $0 \leq \tau \leq \infty$, we use a standard transformation,

$$\xi = \frac{\tau}{\tau + \tau_0}, \qquad \tau = \frac{\tau_0 \xi}{1 - \xi}, \tag{1.12}$$

to transform this interval onto the finite interval $0 \leq \xi \leq 1$, so that we wish to find the value at $\xi = 1/2$ of an option which expires at $\xi = 0$. We then make the assumption that the free boundary can be written as a linear term together with a Fourier sine series,

$$S_f = S_T + (S^* - S_T)\,\xi + \sum_{n=1}^{\infty} c_n \sin n\pi\xi\ . \tag{1.13}$$

We chose this form for two principal reasons: firstly, it gives the required behavior at the two ends and, secondly, it is fairly straightforward to evaluate. We then sought to chose the c_n that maximized the cost function mentioned above, namely, the value of the option. Typically in a multi-dimensional optimization problem such as this, the numerical algorithm is as follows:
(i) choose a direction in which to optimize
(ii) optimize in that direction
(iii) either stop or return to (i).
For the first part of this, namely choosing the directions in which to optimize, we used a standard and very popular scheme, conjugate gradients; in fact we used a popular package [43]. At the end of each step, the code returned the direction in which to optimize during the next step, although it was necessary to input a direction for the initial step. For the second part, we used a slightly unorthodox scheme for our line minimization. Normally, for this part of the algorithm, one might pick a scheme such as a golden search or a quasi-Newton scheme, such as

that used in [33]. However, our cost function (the value of the option) is somewhat unusual in that part of the routine (generation of the random walk, based on [22] is fairly expensive, but the remainder of the routine is not. The cost function, meaning the Monte Carlo portion of the code as opposed to the optimization portion, was essentially the same code used by us to study the effect of using sub-optimal exercise policies on the value of an option [4], and also to evaluate the series expansion mentioned earlier [38]. For our particular problem, evaluating the cost function for several possible boundaries simultaneously (using the same random walk for every boundary) costs only marginally more than evaluating it at for a single boundary, although obviously if too many possible boundaries were used, storage would become an issue. Because of this, we chose to use a comparatively primitive line minimization routine which essentially involved evaluating the function at a large number of points on the line (typically, 101) and then successively refining the grid. Convergence was usually achieved on each line in a handful of steps, and because of the unusual nature of the cost function, the line minimization scheme used was fairly efficient. Results obtained using our numerical method were compared to results obtained using a (100,000 step) binomial tree, and extremely good agreement was found after only a very small number of iterations.

For the initial step, it was necessary to supply both an initial form for the boundary and the direction in which to optimize. For all the runs presented here, we took the initial form to be just the linear term, with the c_n set equal to zero, so that the initial boundary was taken to be $S_T + (S^* - S_T)\,\xi$. For the initial step, we maximized in the direction

$$-\frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m \sin(2m-1)\pi\xi}{(2m-1)^2} = \begin{cases} \xi & 0<\xi<1/2 \\ 1-\xi & 1/2<\xi<1 \end{cases}.$$

In the initial step, therefore, we are assuming that the boundary is of the form

$$\begin{aligned} S_f &= S_T + (S^* - S_T)\,\xi - \frac{4c}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^m \sin(2m-1)\pi\xi}{(2m-1)^2} \\ &= \begin{cases} S_T + (S^* - S_T + c)\,\xi & 0<\xi<1/2 \\ S^* - (S^* - S_T - c)\,(1-\xi) & 1/2<\xi<1 \end{cases}, \end{aligned}$$

and finding the value of c which maximizes the value of the option. We should mention that although the exercise boundary lies between $0 \le \xi \le 1$, our simulation runs from $\xi = 1/2$ to $\xi = 0$, and so our simulation only sees the portion of the boundary between $0 \le \xi \le 1/2$. Because of this, during the initial step we are essentially assuming that the boundary is linear between $0 \leq \xi \leq 1/2$ and finding the slope that maximizes the value of the option.

One further point should be made: trying to find the location of a free boundary for a stochastic d.e. is inherently much harder than for a deterministic PDE, because the free boundary is constantly moving for the stochastic d.e.. In effect, we are trying to hit a moving target. By that we mean that the optimal boundary for one realization will differ from that for another realization. Using Monte Carlo, we average over a large number of realizations, so in some sense, our free boundary is tending towards the "real" free boundary; however, even averaging over a large number of realizations, such as the 1,000,000 used in our code, if the runs were to be repeated using a different seed for the random number generator, the results would be very slightly different: that difference might only be in the 10th significant figure for example, but there would still be a difference. As the number of realizations increases, the difference should decrease. The deterministic PDE (1.2) can be thought of as an average over the stochastic d.e. (1.1) when the number of realizations goes to infinity. One other problem that must be avoided is the generation of a "biased estimate": if the same paths are used at every step in the iteration process, we generate not the optimal boundary for the population as a whole, but rather that for the small number of paths we have repeatedly used; such a boundary is called a biased estimate.

2. Numerical results. In this section, we present some of our test results, obtained using the code described in § 1. In Tables 1–6, we show some sample results for the call, with similar results shown in 7–12 for the put. The parameters for each run are given in the captions to the tables: these are S_0, the initial stock price, E, the exercise price, r, the risk-free rate, D_0, the dividend yield, σ, the volatility, and $\tau_0 = T - t_0$ the tenor of the option. The values shown in the table are the values of the option from the Monte Carlo simulation using the current estimate of the free boundary. Along with these values, we also present the output of our Monte Carlo scheme after iterating in 10 directions, and also

τ_0	0.5	1	2.5	5	10	20
Euro.	0.023195	0.044423	0.086953	0.127347	0.161536	0.160888
Amer.	0.023201	0.044502	0.087941	0.132095	0.180189	0.219917
Monte.	0.023193	0.044507	0.087959	0.132031	0.179885	0.219269
% error	0.032543	0.011157	0.020411	0.048978	0.168709	0.294928

Table 1. Call: Run 1; $S_0 = 0.8, E = 0.9, r = 0.05, D_0 = 0.04, \sigma = 0.25$. "*Euro.*" and "*Amer.*" are values of European and American options computed using a 100,000 step binomial tree, "*Monte*" is the value returned by our Monte Carlo scheme, and "% error" is the percentage difference between *Amer.* and *Monte.*

τ_0	0.5	1	2.5	5	10	20
Euro.	0.338368	0.365258	0.398009	0.370437	0.250189	0.094601
Amer.	0.338368	0.365258	0.398447	0.400956	0.400956	0.400954
Monte.	0.338362	0.365240	0.398368	0.400818	0.400598	0.400292
% error	0.001800	0.005110	0.019803	0.034392	0.089220	0.164987

Table 2. Call: Run 2; as in Table 1 but $S_0 = 0.7, E = 0.4, r = 0.4, D_0 = 0.1, \sigma = 0.1$

τ_0	0.5	1	2.5	5	10	20
Euro.	0.057070	0.079965	0.121169	0.157195	0.183689	0.173111
Amer.	0.057107	0.080204	0.122987	0.164251	0.207832	0.242933
Monte.	0.057116	0.080210	0.122912	0.164192	0.207585	0.242328
% error	0.017089	0.007819	0.055885	0.035961	0.118953	0.249252

Table 3. Call: Run 3; as in Table 1 but $S_0 = 0.8, E = 0.8, r = 0.05, D_0 = 0.04, \sigma = 0.25$

τ_0	0.5	1	2.5	5	10	20
Euro.	0.197894	0.195609	0.188215	0.173993	0.141084	0.081668
Amer.	0.2	0.2	0.200445	0.202105	0.204394	0.205801
Monte.	0.2	0.2	0.200467	0.202061	0.204257	0.205710
% error	0.0	0.0	0.011020	0.022036	0.066897	0.044473

Table 4. Call: Run 4; as in Table 1 but $S_0 = 0.8, E = 0.6, r = 0.1, D_0 = 0.08, \sigma = 0.1$

S_0	80	90	100	110	120
Euro.	1.664384	4.494691	9.250614	15.79748	23.70618
Amer.	1.664384	4.494691	9.250615	15.79749	23.70620
Monte.	1.663624	4.499741	9.241533	15.80405	23.71288
% error	0.045668	0.112340	0.098170	0.041537	0.028176

Table 5. Call: Run 5; as in Table 1 but $E = 100, r = 0.07, D_0 = 0.03, \sigma = 0.3, \tau_0 = 0.5$

S_0	80	90	100	110	120
Euro.	12.13284	17.34267	23.30064	29.88174	36.97249
Amer.	12.14519	17.36829	23.34836	29.96346	37.10338
Monte.	12.15495	17.37829	23.31541	29.96723	37.11079
% error	0.080333	0.057547	0.141142	0.012584	0.019969

Table 6. Call: Run 6; as in Table 5 but $\tau_0 = 3$

for comparison purpose the values of European and American options obtained using a (100,000 step) binomial tree. In addition, for each run, we present the percentage difference between the Monte Carlo results and the American value found using the binomial tree. Scatter-plots of these errors against the tenor τ are shown in Fig. 1. For the call, in terms of a dollar metric, the results

τ_0	0.5	1	2.5	5	10	20
EURO.	0.118225	0.131894	0.150178	0.155113	0.139426	0.094915
AMER.	0.123108	0.140755	0.171396	0.197211	0.220503	0.236395
MONTE.	0.122902	0.140380	0.170856	0.196370	0.218977	0.233921
% ERROR	0.167317	0.265892	0.314922	0.426304	0.692309	1.046856

Table 7. Put: Run 1; $S_0 = 1, E = 1.1, r = 0.05, D_0 = 0.01, \sigma = 0.25$. Rows as in Table 1

τ_0	0.5	1	2.5	5	10	20
EURO.	0.054507	0.067462	0.078114	0.072078	0.048465	0.018024
AMER.	0.057784	0.074915	0.099378	0.115604	0.126328	0.130819
MONTE.	0.057624	0.074696	0.099077	0.115167	0.125311	0.130056
% ERROR	0.277470	0.293307	0.302315	0.378050	0.804711	0.583097

Table 8. Put: Run 2; as in Table 7 but $S_0 = 1, E = 1, r = 0.1, D_0 = 0.04, \sigma = 0.25$

τ_0	0.5	1	2.5	5	10	20
EURO.	0.222576	0.269822	0.285079	0.219015	0.098440	0.015569
AMER.	0.236077	0.303697	0.391449	0.438619	0.460432	0.465275
MONTE.	0.235644	0.302550	0.390739	0.436820	0.456430	0.464443
% ERROR	0.183236	0.377816	0.181294	0.410094	0.869300	0.178930

Table 9. Put: Run 3; as in Table 7 but $S_0 = 4, E = 4, r = 0.2, D_0 = 0.16, \sigma = 0.25$

τ_0	0.5	1	2.5	5	10	20
EURO.	0.088366	0.025296	0.000755	2.8×10^{-6}	5.4×10^{-11}	2.6×10^{-20}
AMER.	0.3	0.3	0.3	0.3	0.3	0.3
MONTE.	0.3	0.3	0.3	0.3	0.3	0.3
% ERROR	0.0	0.0	0.0	0.0	0.0	0.0

Table 10. Put: Run 4; as in Table 7 but $S_0 = 0.9, E = 1.2, r = 0.5, D_0 = 0.02, \sigma = 0.25$

E	80	90	100	110	120
EURO.	2.650647	5.622126	10.02104	15.76761	22.65020
AMER.	2.688789	5.722066	10.23865	16.18116	23.35970
MONTE.	2.677026	5.687059	10.19451	16.13835	23.33461
% ERROR	0.437452	0.611783	0.431103	0.264579	0.107408

Table 11. Put: Run 5; as in Table 7 but $S_0 = 100, r = 0.07, D_0 = 0.03, \sigma = 0.4, \tau_0 = 0.5$

E	80	90	100	110	120
EURO.	10.30938	14.16152	18.53213	23.36293	28.59839
AMER.	11.32567	15.72195	20.79330	26.49445	32.78102
MONTE.	11.28955	15.66149	20.71020	26.42345	32.68867
% ERROR	0.318934	0.384593	0.399663	0.267972	0.281705

Table 12. Put: Run 6; as in Table 11 but $\tau_0 = 3$

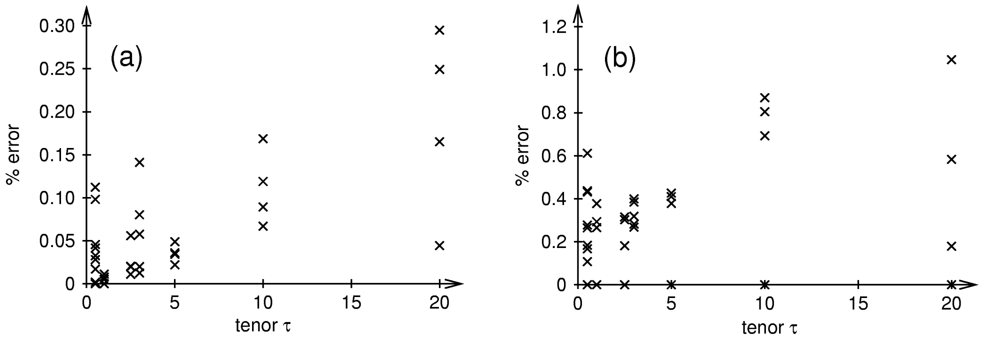

Fig. 1. Scatter-plots of the Monte Carlo error as a function of the life of the option. (a) call; (b) put

appear to be excellent: amongst the results presented here, the largest percentage error for the call was less than 0.3%. The error appears to increase with increasing tenor for the call, and this is at least partly due to the fact that we used the same number of grid points regardless of the value of τ_0, meaning that the step size, and consequently the error of the cost function, increases as τ_0 increases As a point of comparison for the accuracy of our results, in real life, option prices trade in discrete increments (the tick size). On the CBOE for example, the minimum tick size for DJIA options trading below \$300 is \$5, and \$10 for those above \$300, while for equity options, the minimum tick size for options trading below \$300 is \$6.25, and \$12.50 for those above \$300, so that for an equity option trading below \$300, the tick size is in excess of 2%, meaning that accuracy of our results is well within the tick size. For the put, the errors are a little larger: the largest error amongst the results presented here was 1.047%, which is still less than the tick size mentioned above. It is not entirely clear why the results for the put are not as good as those for the call, put presumably it is due in part to the well-known unpleasant behavior of the put boundary close to expiry [5, 35, 23, 3].

We should also mention that in some of the runs, immediate exercise was optimal, and our code was able to identify those cases and record them appropriately. This happened in Run 3 for the call, shown in Table 3, for $\tau_0 = 0.5$ and 1, and also for Run 4 for the put, shown in Table 10, for all the values of τ_0 considered. There were also some cases where there was very little difference between the value of the American and European options, meaning that the starting price S_0 was sufficiently far from the optimal exercise boundary that only a few outlying simulations would hit the boundary and therefore the option

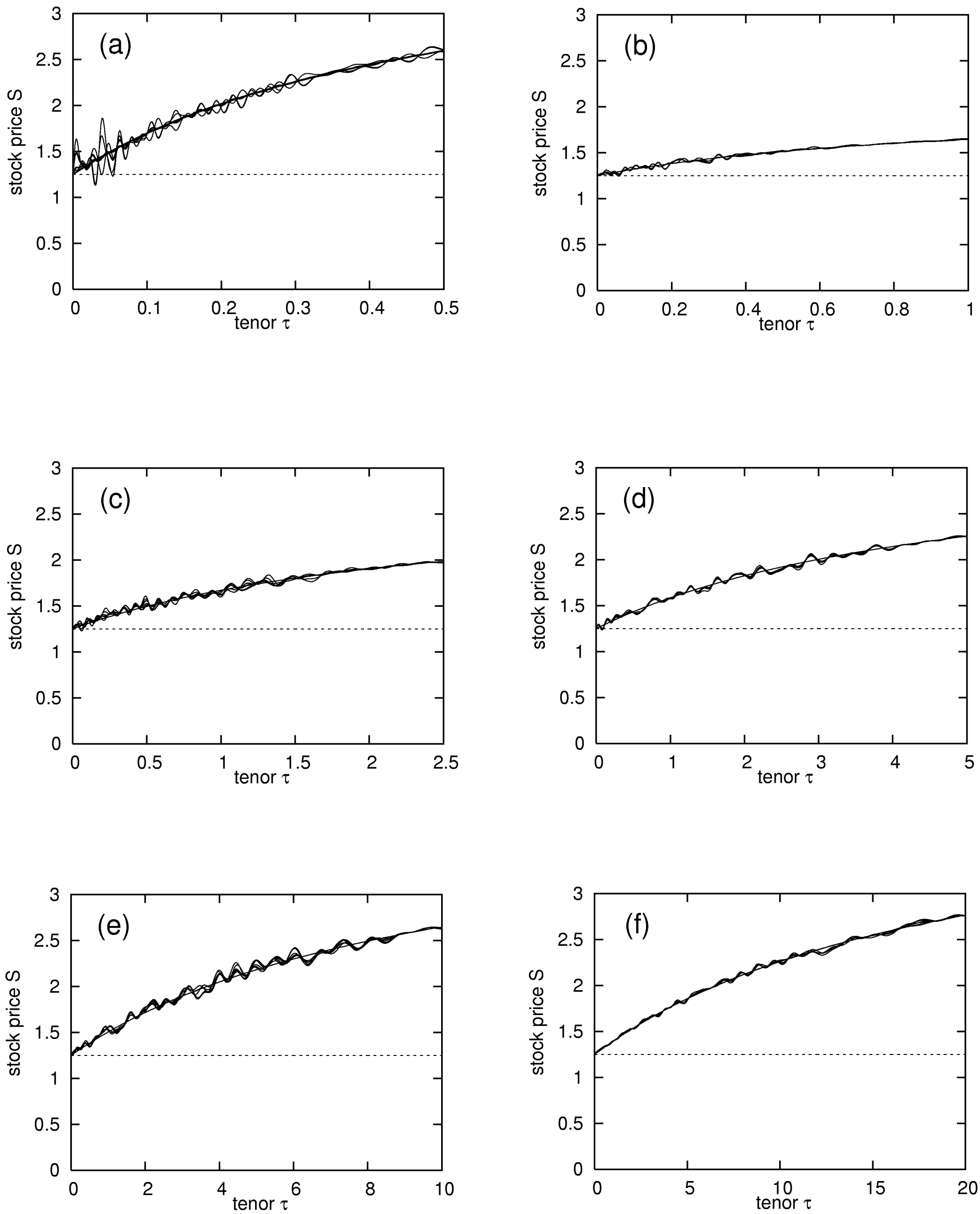

Fig. 2. Sample run for the call, corresponding to Run 1 in Table 1: $S_0 = 0.8$, $E = 0.9$, $r = 0.05$, $D_0 = 0.04$, $\sigma = 0.25$. (a) $\tau_0 = 0.5$; (b) $\tau_0 = 1$; (c) $\tau_0 = 2.5$; (d) $\tau_0 = 5$; (e) $\tau_0 = 10$; (f) $\tau_0 = 20$. Dashed line is location of boundary at expiry. Solid lines are successive iterations for the boundary.

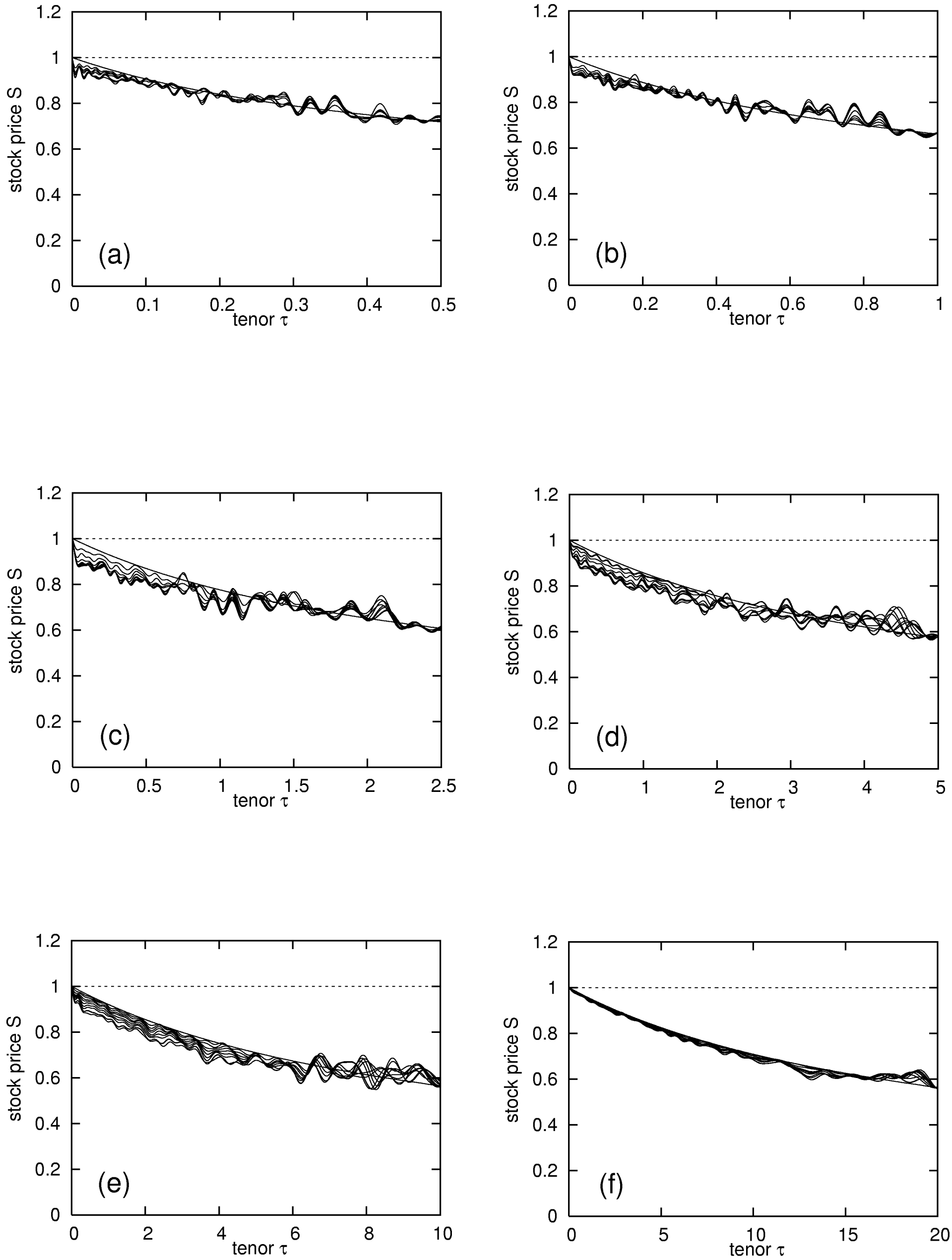

Fig. 3. Sample run for the put, corresponding to Run 1 in Table 7: $S_0 = 1$, $E = 1.1$, $r = 0.05$, $D_0 = 0.01$, $\sigma = 0.25$. (a) $\tau_0 = 0.5$; (b) $\tau_0 = 1$; (c) $\tau_0 = 2.5$; (d) $\tau_0 = 5$; (e) $\tau_0 = 10$; (f) $\tau_0 = 20$. Dashed line is location of boundary at expiry. Solid lines are successive iterations for the boundary.

would almost always be held to expiry. Examples of this include Run 1 for the call, shown in Table 1, with $\tau_0 = 0.5$. In these cases, although the percentage error between the true value and the Monte Carlo value remained very small, the code performed less well in terms of how much of the early exercise premium (meaning the difference between the European and the American options) was captured.

In Fig. 2, we plot the location of the exercise boundary after the first ten iterations for the call simulations shown in Table 1. Whereas we saw in Fig. 1 that the error under a dollar metric appears to increase with the tenor τ_0 for the call, under an "eyeball metric" it appears to decrease with increasing tenor. In Fig. 2(a) for example, there is a fairly large oscillation close to expiry for $\tau_0 = 0.5$, while the boundary for $\tau_0 = 20$ shown in In Fig. 2(f) is noticeably much smoother. Presumably if a larger number of basis functions were used, the boundary would be better resolved and the oscillations would be smaller.

Similar plots for the put are presented in Fig. 3, where we plot the location of the exercise boundary after the first ten iterations for the simulations shown in Table 7. For this run, the oscillation discussed above are actually largest for the intermediate values of τ_0, such as $\tau_0 = 2.5$, 5 and 10, than for either the very small or very large values of τ_0.

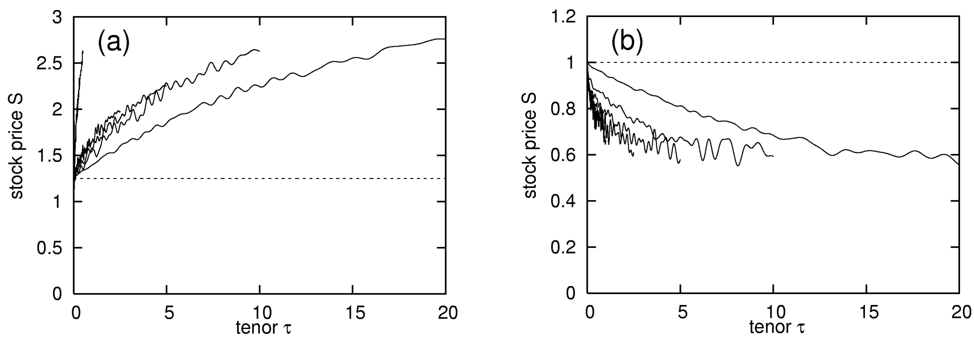

Fig. 4. Exercise boundaries for various values of τ_0 superimposed. (a) call, from Table 1 and Fig. 2; (b) put, from Table 7 and Fig. 3

In Fig. 4(a), we superimpose the results of Fig. 2 for the call, and do the same in Fig. 4(b) for the results of Fig. 3 for the call, superimposing the optimal exercise boundary after ten iterations for various values of the tenor τ_0. The oscillation seen in Fig. 3 is clearly visible here as well for intermediate values of τ_0. However, despite the fact that the exercise boundary appears dreadful under an eyeball metric, we would reiterate that it does very well under a dollar metric, meaning that an investor who used this boundary as his guide as to whether to hold an option or exercise it would do very well.

In closing this section, a few points should be made about our results. During the iteration process, we noticed that sometimes the value of an option went down from one iteration to the next: it should be remembered that we were using a different set of paths for each iteration, and so the value of the same estimate of the free boundary will differ from one iteration to the next. Similarly, since we were using a finite number of realizations, on some iterations, the value of the option will exceed the American value slightly. These two effects would presumably decrease if a larger number of paths were taken, and indeed some trial runs with more paths suggest that as the number of paths is increased, the variability is reduced but not eliminated. Along the same lines, it appears that although our results are highly accurate (indeed, extremely accurate for the call), it appears that as we take more and more iterations the value does not converge exactly but remains within a tight band around the true value, with this band becoming narrower as more paths are used. We will say a few more words about this in the final section, but we believe it is a generic problem with trying to fix a free boundary in a stochastic framework.

3. Discussion. In the preceding sections, we have proposed a new algorithm to allow Monte Carlo methods to be used for American options; this algorithm involves approximating the optimal exercise boundary as a linear term together with a finite sum of some basis functions, in our case sine functions on a transformed domain. In the sample results we have presented, it would appear that the method very quickly arrives at a very good approximation to the optimal exercise boundary where "*good*" means that if an investor used the approximate boundary as the basis of his exercise strategy, he would expect returns very close to the actual value of the option. However, the scheme does not pin down the free boundary exactly: this is less of a problem the more realizations are taken (and even with the 1,00,000 paths used in the results presented here, we do not consider it a "major" problem, since using the approximate boundary in that case would still enable an investor to capture almost all of the value of the option). We believe this is a problem inherent with trying to fix a free boundary in a stochastic framework: as we discussed in the Introduction, it occurs because we are trying to hit a moving target, and as we mentioned in the previous section, when we increased the number of paths the variability was reduced but not completely eliminated. In addition, the method appears to work poorly on an eyeball metric despite working extremely well on a dollar metric. This might be due to the non-analytic behavior of the boundary close to expiry, where it is thought to behave at best like $\sqrt{\tau}$ and at worst like $\sqrt{\tau \ln \tau}$ [5, 35, 23, 2, 3], and we would suggest that it might be worthwhile to try different basis functions which better capture the behavior close to expiry.

4. Acknowledgements. This work was partially funded by a grant from the Natural Sciences and Engineering Research Council of Canada, and the computations were performed at Sharcnet Canada. An earlier version of this research was presented at the CIMA2001 conference [37].

REFERENCES

[1] G. Allegretto, G. Barone-Adesi, R. J. Elliott. Numerical evaluation of the critical price and American options. *The European Journal of Finance* **1** (1995), 69–78.

[2] G. Alobaidi, R. Mallier. Asymptotic analysis of American call options. *Int. J. Math. Math. Sci.* **27** (2001), 177–188.

[3] G. Alobaidi, R. Mallier. On the optimal exercise boundary for an American put option. *J. Appl. Math.* **1** (2001), 39–45.

[4] G. Alobaidi, R. Mallier. Using Monte Carlo methods to evaluate sub-optimal exercise policies for American options. *Serdica Math. J.* **28** (2002), 207–218.

[5] G. Barles, J. Burdeau, M. Romano, N. Samsoen. Critical stock price near expiration. *Math. Finance* **5** (1995), 77–95.

[6] G. Barone-Adesi, R. E. Whaley. Efficient analytic approximation of American option values. *J. Finance* **41** (1987), 301-320.

[7] G. Barone-Adesi, R. Elliott. Approximations for the values of American options. *Stochastic Anal. Appl.* **9** (1991), 115–131.

[8] F. Black, M. Scholes. The pricing of options and corporate liabilities. *J. Political Economy* **81** (1973), 637–659.

[9] E. C. Blomeyer. An analytic approximation for the American put price on stocks with dividends. *J. Financial Quant. Anal.* **21** (1986), 229–233.

[10] P. Bossaert. Simulation estimators of optimal early exercise. Preprint, Carnegie Mellon University, 1989.

[11] P. Boyle. Options: a Monte-Carlo approach. *J. Financial Econom.* **4** (1977), 323–338.

[12] P. Boyle. A lattice framework for option pricing with two state variables. *J. Financial Quant. Anal.* **23** (1988), 1–12.

[13] P. Boyle, A. W. Kolkiewicz, K. S. Tan. Pricing American derivatives using simulation: a biased low approach. Monte Carlo and Quasi-Monte Carlo Methods 2000 (Eds K.-T. Fang et al.), Springer, Berlin, 181–200, 2002.

[14] M. J. Brennan, E. S. Schwartz. The valuation of the American put option. *J. Finance* **32** (1977), 449–462.

[15] M. Broadie, P. Glasserman. Pricing American-style securities using simulation. *J. Econom. Dynam. Control* **21** (1997), 1323–1352.

[16] M. Broadie, J. Detemple. American option valuation: new bounds, approximations, and a comparison of existing methods. *Rev. Financ. Stud.* **9** (1986), 1211–1250.

[17] D. Bunch, H. E. Johnson. A simple and numerically efficient valuation method for American puts using a modified Geske-Johnson approach. *J. Finance* 47 (1992), 809-816.

[18] J. F. Carrière. Valuation of the early-exercise price for options using simulation and non-parametric regression. *Insurance Math. Econom.* **19** (1996), 19–30.

[19] M. Chesney, R. Gibson. State space symmetry and two-factor option pricing models. *Advances in Futures and Options Research* **8** (1993), 85–112.

[20] G. Courtadon. A more accurate finite-difference approximation for the valuation of options. *J. Financial Quant. Anal.* **17** (1982), 697–705.

[21] J. Cox, S. Ross, M. Rubinstein. Option pricing: a simplified approach. *J. Financial Econom.* **7** (1979), 229–264.

[22] P. L'Ecuyer, S. Cote. Implementing a random number package with splitting facilities. *ACM Trans. Math. Software* **17** (1991), 98–111.

[23] J. D. Evans, R. E. Kuske, J. B. Keller. Americans options with dividends near expiry. *Math. Finance* **12** (2002), 219–237.

[24] M. Fu, S. Laprise, D. Madan, Y. Su, R. Wu. Pricing American options: a comparision of Monte Carlo approaches. *J. Comput. Finance* **4** (2001), 39–88.

[25] M. Fu, J. Q. Hu. Sensitivity Analysis for Monte Carlo Simulation of Option Pricing. *Probab. Engrg. Inform. Sci.* **9** (1995), 417–446.

[26] B. Gao, J. Huang, M. Subrahmanyam. The valuation of American barrier options using the decomposition technique. *J. Econom. Dynam. Control* **24** (2000), 1783–1827.

[27] R. Geske. A note on an analytical valuation formula for unprotected American call options on stocks with known dividends. *J. Financial Econom.* **7** (1979), 375–380.

[28] R. Geske. Comments on Whaley's note. *J. Financial Econom.* **9** (1981), 213–215.

[29] R. Geske, H. E. Johnson. The American put valued analytically. *J. Finance* **39** (1984), 1511–1524.

[30] D. Grant, G. Vora, D. Weeks. Path-dependent options: extending the Monte Carlo simulation approach. *Management Sci.* **43** (1997), 1589–1602.

[31] J. C. Hull. Options, Futures and Other Derivatives. Prentice-Hall, New York, 2000.

[32] A. Ibanez, F. Zapatero. Monte Carlo valuation of American options through computation of the optimal exercise frontier. Preprint **99-8**, Marshall School of Business, University of Southern California, 1999.

[33] G. R. Ierley, O. G. Ruehr. Analytic and numerical solutions of a nonlinear boundary-layer problem. *Stud. Appl. Math.* **75** (1986), 1–36.

[34] H. E. Johnson. An analytical approximation to the American put price. *J. Financial Quant. Anal.* **18** (1983), 141–148.

[35] R. E. Kuske, J. B. Keller. Optimal exercise boundary for an American put option. *Appl. Math. Finance* **5** (1998), 107–116.

[36] L. W. MacMillan. Analytic approximation for the American put option. *Adv. Futures and Options* **1A** (1986), 119–139.

[37] R. Mallier. Approximating the optimal exercise boundary for American options via Monte Carlo. Computational Intelligence: Methods and Applications (Eds L. I .Kuncheva et al.), ICSC Academic Press Canada, 2001, 365–371.

[38] R. Mallier. Evaluating approximations to the optimal exercise boundary for American options. *J. Appl. Math.* **2** (2002), 71–92.

[39] R. Mallier, G. Alobaidi. Laplace Transforms and American Options. *Appl. Math. Finance* **7** (2000), 241–256.

[40] R. McDonald, M. Scroder. A parity result for American options. *J. Comput. Finance* **1** (1998), 5–13.

[41] R. C. Merton. The theory of rational option pricing. *Bell J. Econom. and Management Sci.* **4** (1973), 141–183.

[42] R. C. Merton. On the problem of corporate debt: the risk structure of interest rates. *J. Finance* **29** (1974), 449–470.

[43] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery. Numerical Recipes in Fortran 77: the art of scientific computing, 2nd edition, Cambridge, New York, 1992.

[44] S. Raymar, M. Zwecher. A Monte Carlo valuation of American options on the maximum of several stocks. *J. Derivatives* **5** (1997), 7–23.

[45] R. Roll. An analytical formula for unprotected American call options on stocks with known dividends. *J. Financial Econom.* **5** (1977), 251–258.

[46] M. J. P. Selby, S. D. Hodges. On the evaluation of compound options. *Management Sci.* **33** (1987), 347–355.

[47] J. Tilley. Valuing American options in a path simulation model. *Trans. Soc. Actuaries* **45** (1983), 83–104.

[48] R. E. Whaley. On the valuation of American call options on stocks with known dividends. *J. Financial Econom.* **9** (1981), 207–211.

[49] P. Wilmott. Derivatives. The theory and Practice of Financial Engineering. Wiley, Chichester, 1998.

[50] L. Wu, Y.-K. Kwok. A front-fixing finite difference method for the valuation of American options. *J. Financial Engineering* **6** (1997), 83–97.

Roland Mallier
Department of Applied Mathematics
The University of Western Ontario
London ON N6A 5B7 Canada
e-mail: mallier@uwo.ca

Ghada Alobaidi
American University of Sharjah
Department of Mathematics and Statistics
P.O.Box 26666
Sharjah, United Arab Emirates
e-mail: galobaidi@aus.ac.ae

Received May 10, 2003