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THE VARIETY OF LEIBNIZ ALGEBRAS DEFINED BY THE IDENTITY $x(y(zt)) \equiv 0$*

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Abstract. Let F be a field of characteristic zero. In this paper we study the variety of Leibniz algebras ${}_3\mathbf{N}$ determined by the identity $x(y(zt)) \equiv 0$. The algebras of this variety are left nilpotent of class not more than 3. We give a complete description of the vector space of multilinear identities in the language of representation theory of the symmetric group S_n and Young diagrams. We also show that the variety ${}_3\mathbf{N}$ is generated by an abelian extension of the Heisenberg Lie algebra. It has turned out that ${}_3\mathbf{N}$ has many properties which are similar to the properties of the variety of the abelian-by-nilpotent of class 2 Lie algebras. It has overexponential growth of the codimension sequence and subexponential growth of the colength sequence.

1. Introduction. We study varieties of Leibniz algebras over a field F of zero characteristic. It is well known that in characteristic zero all polynomial identities are completely determined by the multilinear ones. One of the most

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important numerical characteristics of polynomial identities of a variety of algebras are the codimension, the cocharacter and the colength sequences. There are a lot of papers about the codimension growth of associative and Lie algebras. Recently the systematic study of polynomial identities of Leibniz algebras has been also started (see for example [3]).

A *Leibniz algebra* L over a field F is a nonassociative algebra with multiplication

$$(-,-) : L \times L \longrightarrow L$$

satisfying the Leibniz identity

$$(x,(y,z)) = ((x,y),z) - ((x,z),y). \tag{1}$$

In other words, the operator of right multiplication $(-,z)$ is a derivation of the algebra. Notice that this identity is equivalent to the classical Jacobi identity when $(-,-)$ is skew-symmetric. The Leibniz identity allows us to express any product as a linear combination of left-normed products. We will omit the Leibniz parentheses and use the left-normed notation $a_1a_2\cdots a_n = ((a_1,\ldots,a_{n-1}),a_n)$. Identities of Leibniz algebras are very close to the defining identities of Lie algebras. In particular, the following relations follow from (1):

$$x(yz) \equiv xyz - xzy, x(yy) \equiv 0, x(yz) \equiv -x(zy),$$

$$x(yzt) + x(zty) + x(tyz) \equiv 0.$$

In this paper we study the variety of Leibniz algebras determined by the identity

$$x(y(zt)) \equiv 0. \tag{2}$$

Denote this variety by ${}_3\mathbf{N}$. Our main purpose is to give a complete description of the space of multilinear identities of ${}_3\mathbf{N}$ in the language of representation theory of the symmetric group S_n and Young diagrams.

We recall all essential notions. Their definitions are similar to those for varieties of associative and Lie algebras.

Let $\mathbf{V}$ be a variety of Leibniz algebras over a field F. Denote by $F(X,\mathbf{V})$ the relatively free algebra of the variety $\mathbf{V}$ with a countable set of generators $X = \{x_1, x_2, \ldots\}$. Denote also by $P_n = P_n(\mathbf{V})$ the set of all multilinear Leibniz polynomials in $x_1,\ldots,x_n$ in $F(X,\mathbf{V})$. The left action of the symmetric group S_n defined by $\sigma(x_i) = x_{\sigma(i)}$, $\sigma \in S_n$, can be naturally extended to the vector space P_n. The structure of P_n as an S_n-module, $n = 1,2,\ldots$, is an important characterization of $\mathbf{V}$ and gives very useful information about $\mathbf{V}$.

Denote by χ_λ the irreducible character of the symmetric group S_n corresponding to the partition λ of n and, for a variety $\mathbf{V}$, consider the decomposition of the S_n-character $\chi(P_n(\mathbf{V}))$ as a sum of irreducible components

$$\chi_n(\mathbf{V}) = \chi(P_n(\mathbf{V})) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \tag{3}$$

The character $\chi_n(\mathbf{V})$ is called the *n-th cocharacter* of $\mathbf{V}$ and the integer $c_n(\mathbf{V}) = \dim P_n(\mathbf{V})$ is the *n-th codimension* of $\mathbf{V}$. Important numerical characteristics of $\mathbf{V}$ are also the multiplicities m_λ in (3). The total number of summands

$$l_n(\mathbf{V}) = \sum_{\lambda \vdash n,} m_\lambda$$

in the sum (3) is called the *n-th colength* of the variety $\mathbf{V}$.

Denote by d_λ the dimension of the irreducible S_n-module corresponding to λ. The following relation

$$c_n(\mathbf{V}) = \dim P_n(\mathbf{V}) = \sum_{\lambda \vdash n,} m_\lambda d_\lambda$$

holds for the above introduced numerical characteristics. It is well known that for any nontrivial variety of associative algebras $\mathbf{V}$, the sequence of codimensions is exponentially bounded [7] and the colength function $l_n(\mathbf{V})$ is polynomially bounded [2].

The variety ${}_3\mathbf{N}$ is similar to the variety $\mathbf{AN}_2$ of all abelian-by-nilpotent of class 2 Lie algebras determined by the Lie identity $(x_1x_2x_3)(x_4x_5x_6) \equiv 0$. The Lie variety $\mathbf{AN}_2$ was investigated in many papers (see for example [8], [4], [9], [6]). Both varieties ${}_3\mathbf{N}$ and $\mathbf{AN}_2$ have overexponential growth of their codimension sequences and non-polynomial but subexponential growth of the colength sequences.

As by-products of the proofs of our main results we give an explicit basis of $P_n({}_3\mathbf{N})$ indexed with the involutions of the symmetric group S_{n-1}. We also establish that the variety ${}_3\mathbf{N}$ is generated by a Leibniz algebra which is an abelian extension of the infinitely dimensional Heisenberg Lie algebra.

2. Main results. First we give examples of Leibniz algebras from the variety ${}_3\mathbf{N}$. We need these algebras in the proof of our main theorem. We will show that they generate the variety ${}_3\mathbf{N}$.

Let $T_k = F[t_1, \dots, t_k]$ be the algebra of polynomials in k commuting variables $t_1, \dots, t_k$ and let H_k be the Lie algebra with basis $\{a_1, \dots, a_k, b_1, \dots, b_k, c\}$ and multiplication table

$$a_ib_j = \delta_{ij}c, \ a_ia_j = b_ib_j = a_ic = b_jc = 0,$$

where δ_{ij} is the Kronecker delta. The algebra H_k is called the Heisenberg algebra and satisfies the Lie identity $x_1x_2x_3 \equiv 0$. The vector space T_k becomes a right H_k-module if we define the action of the basis elements of H_k on the polynomial $f \in T_k$ by

$$fc = f, fa_s = f'_s, fb_s = t_s f,$$

where f'_s is the partial derivative of f with respect to t_s. We also define the trivial left action of H_k on T_k by $a_s f = b_s f = cf = 0$, $f \in T_k$.
The algebra we need is a direct sum of the vector spaces T_k and H_k with multiplication determined by the rule

$$(f + x)(g + y) = fy + xy,$$

where f, g are polynomials from T_k and x, y are elements from H_k. Let us denote this algebra by H^k. It is easy to see that for any k the algebra H^k is a Leibniz algebra.

Lemma 1. *The algebra H^k satisfies the identity* (2), *i.e.* $H^k \in {}_3\mathbf{N}$ *for any* $k = 1, 2, \ldots$.

Proof. If $f_i \in T_k$ and $x_i \in H_k$, $i = 1, 2, 3, 4$, then

$$(f_1 + x_1)((f_2 + x_2)((f_3 + x_3)(f_4 + x_4)))$$

$$= (f_1 + x_1)((f_2 + x_2)(f_3x_4 + x_3x_4))$$

$$= (f_1 + x_1)(f_2(x_3x_4) + x_2(x_3x_4)) = 0. \qquad \square$$

Lemma 2. *The vector space $P_n({}_3\mathbf{N})$ is spanned by the multilinear products*

$$(4) \qquad \Theta_{(i,i_1,\ldots,i_m,j_1,\ldots,j_m)} = x_i(x_{i_1}x_{j_1})(x_{i_2}x_{j_2})\cdots(x_{i_m}x_{j_m})x_{k_1}\cdots x_{k_{n-2m-1}},$$

with $i_s < j_s, s = 1, \ldots, m, i_1 < i_2 < \cdots < i_m, k_1 < k_2 < \cdots < k_{n-2m-1}$.

Proof. Note that the identities (1) and (2) allow us to transform the polynomial elements from $P_n({}_3\mathbf{N})$ as follows. First,

$$(5) \qquad xy_2y_1 = xy_1y_2 + x(y_2y_1)$$

and we can rearrange the positions of the generators in the left-normed products adding summands with products in parentheses. Second,

$$(6) \qquad xy(zt) = x(zt)y$$

and we can move any product $(x_{i_s}x_{j_s})$ if it does not stand at the most left position. Third, we can permute two generators inside brackets, namely

$$(7) \qquad x(y_2y_1) = -x(y_1y_2).$$

Hence we can change the position of any pair of generators y_k and y_{k+1} using the identity (5):

$$y_1 \cdots y_k y_{k+1} = y_1 \cdots y_{k+1} y_k + y_1 \cdots y_{k-1}(y_k y_{k+1}), \quad k > 1.$$

Then by (7) we can rearrange the order of y_k and y_{k+1} in the product $(y_k y_{k+1})$ if necessary and move it to the appropriate position using the identity (6)

$$y_1 \cdots y_{k-1}(y_k y_{k+1}) = y_1 \cdots (y_k y_{k+1}) y_{k-1} = \cdots = y_1(y_k y_{k+1}) \cdots y_{k-1}$$

and (6) again allows to change the places of the products $(y_i y_j)$:

$$x(y_{\sigma(1)} z_{\sigma(1)}) \cdots (y_{\sigma(k)} z_{\sigma(k)}) = x(y_1 z_1) \cdots (y_k z_k), \quad \sigma \in S_k. \qquad \square$$

The following proposition gives a set of algebras which generates the variety ${}_3\mathbf{N}$ as well as a basis for the vector space $P_n({}_3\mathbf{N})$.

Proposition 3. (i) *The algebras $H^k = T_k + H_k$, $k = 1, 2, \ldots$, generate the variety ${}_3\mathbf{N}$.*

(ii) *The set of all elements* (4) *is a linear basis of $P_n({}_3\mathbf{N})$.*

Proof. By Lemma 2 every element of $P_n({}_3\mathbf{N})$ is a linear combination of the elements of the type (4). Suppose that (4) enjoy the equality in $P_n({}_3\mathbf{N})$

$$\sum_{(i,i_1,\ldots,i_m,j_1,\ldots,j_m)} \alpha_{(i,i_1,\ldots,i_m,j_1,\ldots,j_m)} \Theta_{(i,i_1,\ldots,i_m,j_1,\ldots,j_m)} = 0 \tag{8}$$

for some $\alpha_{(i,i_1,\ldots,i_m,j_1,\ldots,j_m)} \in F$. Hence (8) is a polynomial identity for ${}_3\mathbf{N}$ and vanishes on the algebras H^k. We will prove both parts of the proposition if we find a $k > 0$ such that (8) is different from 0 for some elements in H^k. Each element $\Theta_{(i,i_1,\ldots,i_m,j_1,\ldots,j_m)}$ is defined by the number m of products $(x_{i_s} x_{j_s})$, a fixed generator x_i and the $2m$-tuple $(i_1, \ldots, i_m,\ j_1, \ldots, j_m)$ (satisfying $i_1 < i_2 < \ldots < i_m$, $i_1 < j_1, \ldots, i_m < j_m$). Pick the element Θ_I, $I = (i,\ i_1, \ldots, i_m, j_1, \ldots, j_m)$ with nonzero coefficient α_I which contains the least number m of products. Replace the variables x_s, $s = 1, 2, \ldots, n$, with the following elements of the algebra H^m: $x_i = f$, $x_{i_s} = a_s$, $x_{j_s} = b_s$, $s = 1, \ldots, m$, (the rest of the elements x_α are replaced by c). Let us check that the value of all other Θ_J, $J \neq I$, of the type (4) after this substitution is zero. Recall that $xf = 0$ for any x from H_k. Hence any element Θ_J will be zero, if its left factor is not equal to x_i. If Θ_J has more than m products of degree 2 then it will be 0 since the element c from the center of algebra H_m will be placed into some $(x_{i_s} x_{j_s})$. The multiplication rules imply that Θ_J takes zero value as soon as $J \neq I$.

So, the result of the substitution is equal to $\alpha_I \cdot f c^{n-m-1} = \alpha_I \cdot f \neq 0$ and all elements (4) are linearly independent. This completes the proof of the proposition. $\square$

The following theorem describes the codimension sequence of ${}_3\mathbf{N}$.

Theorem 4. *The codimension sequence of* ${}_3\mathbf{N}$ *satisfies*

$$c_n({}_3\mathbf{N}) = n \cdot \mathrm{inv}(n-1) = n \cdot \sum_{\lambda \vdash (n-1)} d_\lambda,$$

where $\mathrm{inv}(m)$ *is the number of involutions (permutations of order two) in* S_m.

P r o o f. There exists an obvious one-to-one correspondence between the elements (4) and the ordered pairs

$$(i, \{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\})$$

involving $2m+1$ pairwise different elements $i, i_1, \dots, i_m, j_1, \dots, j_m$ such that $i_1 < j_1, i_2 < j_2, \dots, i_m < j_m$. We may identify the sets $\{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\}$ with the involutions of the symmetric group S_{n-1} acting on $\{1, \dots, i-1, i+1, \dots, n\}$ because any permutation $\sigma \in S_{n-1}$ of order two can be written as a product of independent transpositions $\sigma = \tau_1 \cdots \tau_m$, where any transposition τ_s has the form (i_s, j_s), $s = 1, \dots, m$.

Hence by Proposition 3 we conclude that

$$c_n({}_3\mathbf{N}) = \dim P_n({}_3\mathbf{N}) = n \cdot \mathrm{inv}(n-1),$$

where $\mathrm{inv}(n-1)$ is the number of involutions in the symmetric group S_{n-1}. The second equality

$$n \cdot \mathrm{inv}(n-1) = n \cdot \sum_{\lambda \vdash (n-1)} d_\lambda$$

follows from the well known equality $\mathrm{inv}(m) = \sum_{\lambda \vdash m} d_\lambda$ (which can be found e.g. as Proposition 2 from [8]). $\square$

Now we will show that the equality $c_n({}_3\mathbf{N}) = n \cdot \sum_{\lambda \vdash (n-1)} d_\lambda$ reflects the structure of some S_{n-1}-submodules of $P_n({}_3\mathbf{N})$. We consider the subspace $Q_n^{(i)}$ of $P_n({}_3\mathbf{N})$ spanned by all monomials starting with x_i:

$$Q_n^{(i)} = \mathrm{span}\{x_i x_{j_1} \dots x_{j_{(n-1)}} \mid \{j_1, \dots, j_{n-1}\} = \mathbf{N}_n \setminus \{i\}\},$$

where $i = 1, \dots, n$ and $\mathbf{N}_n = \{1, 2, \dots, n\}$.

All subspaces $Q_n^{(i)}$, $i = 1, \dots, n$, have the same S_{n-1}-module structure, where S_{n-1} acts on $\mathbf{N}_n \setminus \{i\}$. For convenience we will investigate the S_{n-1}-module $Q_n^{(n)}$.

Proposition 5. *The character* $\chi(Q_n^{(n)})$ *of the* S_{n-1}*-module* $Q_n^{(n)}$ *is*

$$\chi(Q_n^{(n)}) = \sum_{\lambda \vdash (n-1)} \chi_\lambda,$$

i.e. it is a sum of all irreducible S_{n-1}-characters, all participating with multiplicity 1.

Proof. Denote by λ'_i the i-th column of the Young diagram corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$.

Consider the associative polynomial

$$S_{\lambda'_i} = S_{\lambda'_i}(X_1, \ldots, X_{\lambda'_i}) = \sum_{\sigma \in S_{\lambda'_i}} (-1)^\sigma X_{\sigma(1)} \cdots X_{\sigma(\lambda'_i)}$$

where X_i is the operator of right multiplication by x_i, $i = 1, \ldots, n$, i.e. $wX_i = wx_i$ and $w(X_iX_j) = ((wx_i)x_j)$. We relate with the partition λ the multihomogeneous element

$$x_n S_{\lambda'_1} S_{\lambda'_2} \cdots S_{\lambda'_m}. \tag{9}$$

If the polynomial (9) is a non-zero element of the free algebra of the variety ${}_3\mathbf{N}$, then its complete linearization generates an irreducible S_{n-1}-module of $Q_n^{(n)}$ corresponding to λ. We will prove that for any partition $\lambda \vdash (n-1)$ the polynomial (9) is not an identity for some algebra H^k.

We start with the case when the diagram of λ has only one column. Using the properties of our variety and the identity (1), we obtain

$$x_n S_{n-1}(X_1, \ldots, X_{n-1}) = \frac{1}{2^k} \sum_{\sigma \in S_{n-1}} (-1)^\sigma x_n (x_{\sigma(1)} x_{\sigma(2)}) \ldots (x_{\sigma(2k-1)} x_{\sigma(2k)})$$

for odd $n = 2k + 1$ and

$$x_n S_{n-1}(X_1, \ldots, X_{n-1}) = \frac{1}{2^k} \sum_{\sigma \in S_{n-1}} (-1)^\sigma x_n (x_{\sigma(1)} x_{\sigma(2)}) \cdots (x_{\sigma(2k-1)} x_{\sigma(2k)}) x_{\sigma(n-1)}$$

for even $n = 2k + 2$.

Let us replace x_n with some $f \in T_n$ and substitute a_1, b_1, a_2, $b_2, \ldots$ from H_n instead of x_1, x_2, x_3, $x_4, \ldots, x_{n-1}$ respectively. The result of the substitution is equal to $\alpha \cdot (fc^k) = \alpha \cdot f$ when $n = 2k + 1$ or $\alpha \cdot (fc^k a_{k+1}) = \alpha \cdot f'_{k+1}$ where $\alpha = \frac{k!}{2^k}$ and f'_{k+1} is the partial derivative with respect to t_{k+1}. All terms in the sum will equal zero except the case when for every $s = 1, 2, \ldots, k$ the pair a_s, b_s is within the same parentheses $(a_s b_s)$. Thus, for a suitable f, we have a non-zero result of the substitution.

Clearly, similar reasons work in the general situation.

So, for any partition $\lambda \vdash (n - 1)$ the element (9) is not equal to zero in $P_n({}_3\mathbf{N})$. In this way, the decomposition of the character $\chi(Q_n^{(n)})$ as a sum of irreducible characters of the symmetric group S_{n-1} has the form

$$\chi(Q_n^{(i)}) = \sum_{\lambda\vdash(n-1),} p_\lambda \chi_\lambda,$$

where $p_\lambda \geq 1$ for all $\lambda \vdash n-1$. This implies that

$$\dim Q_n^{(i)} = \sum_{\lambda\vdash(n-1)} p_\lambda d_\lambda \geq \sum_{\lambda\vdash(n-1)} d_\lambda, \quad i = 1, \ldots, n.$$

Since the vector space $P_n({}_3\mathbf{N})$ is the direct sum of the subspaces $Q_n^{(i)}$ for $i = 1, 2, \ldots, n$, we have

$$c_n({}_3\mathbf{N}) = \dim P_n({}_3\mathbf{N}) = \sum_{i=1}^{n} \dim Q_n^{(i)} \geq n \cdot \sum_{\lambda\vdash(n-1)} d_\lambda.$$

Hence, by Theorem 4, $p_\lambda = 1$ for all $\lambda \vdash n-1$ and this completes the proof. □

Now we will investigate the multiplicities of the variety ${}_3\mathbf{N}$.

The box of a Young diagram is called an "inner corner" if after removing this box, we also get a Young diagram. For example, the number of inner corners for the diagram of the partition $(3, 2, 2, 1)$ equals to 3 and the diagram of the partition (4,2,2) has two inner corners.

Denote by $r(\lambda)$ the number of inner corners of the diagram of the partition $\lambda \vdash n$. Clearly, $r(\lambda)$ is equal to the number of distinct lengths of the rows of the Young diagram. Hence we have the restriction $1 + 2 + \ldots + r(\lambda) \leq n$.

The following observation is obvious.

Remark 6. $r(\lambda) < \sqrt{2n}$.

Theorem 7. *The n-th cocharacter of the variety ${}_3\mathbf{N}$ has the form*

$$\chi_n({}_3\mathbf{N}) = \chi(P_n({}_3\mathbf{N})) = \sum_{\lambda\vdash n} r(\lambda)\chi_\lambda,$$

i.e. the multiplicity m_λ is equal to the number $r(\lambda)$ of the inner corners of the diagram of λ.

P r o o f. Fix some partition $\lambda \vdash n$. Recall (see for example [5]) that the G-module V is *induced* from the H-module W, where H is a subgroup G, (and the representation of G in V is induced by the representation of H in W) if W is a subspace of V and the following conditions hold:

1) W is a submodule of V considered as an H-module;

2) $V = \bigoplus_{s\in G/H} sW$.

So, from the definition of induced module we have that, as an S_n-module, $P_n({}_3\mathbf{N})$ is induced by the S_{n-1}-module $Q_n^{(n)}$. Since, by Proposition 5, the character of $Q_n^{(n)}$ is the sum of all irreducible S_{n-1}-characters, by the branching rule for representations of symmetric groups we have that the multiplicity m_λ in $\chi_n({}_3\mathbf{N})$ equals the number of inner corners of the diagram of the partition $\lambda \vdash n$. The proof of Theorem 7 is completed. □

By Remark 6 and Theorem 7 the multiplicities m_λ of the variety ${}_3\mathbf{N}$, $\lambda \vdash n$, are bounded by $\sqrt{2n}$. On the other hand, by the well known result about the number of different partitions, the colength of the variety ${}_3\mathbf{N}$ cannot be restricted by any polynomial function and has intermediate growth. We will obtain more precise asymptotics of the colength of ${}_3\mathbf{N}$. Recall the asymptotic formula for the number $p(n)$ of partitions of n (see [1]):

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

Corollary 8. *The colength* $l_n({}_3\mathbf{N})$ *satisfies the following inequalities*

$$p(n) \leq l_n({}_3\mathbf{N}) < \sqrt{2n} \cdot p(n),$$

where $p(n)$ *is the number of partitions of* n*.*

Proof. From Theorem 7 we have

$$l_n({}_3\mathbf{N}) = \sum_{\lambda \vdash n} m_\lambda = \sum_{\lambda \vdash n} r(\lambda).$$

Using Remark 6 we obtain $1 \leq m_\lambda < \sqrt{2n}$. Hence we have the inequalities

$$p(n) \leq l_n({}_3\mathbf{N}) < \sqrt{2n} \cdot p(n).$$

The equality $p(n) = l_n({}_3\mathbf{N})$ holds if and only if $r(\lambda) = 1$ for all $\lambda \vdash n$, i.e. for $n = 1, 2$. □

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