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APPLICATIONS OF THE FRÉCHET SUBDIFFERENTIAL

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Abstract. In this paper we prove two results of nonsmooth analysis involving the Fréchet subdifferential. One of these results provides a necessary optimality condition for an optimization problem which arise naturally from a class of wide studied problems. In the second result we establish a sufficient condition for the metric regularity of a set-valued map without continuity assumptions.

1. Preliminaries. Let X be a normed vector space and X^* its topological dual; we denote by B_X, U_X, S_X the open unit ball, the closed unit ball and the unit sphere of X, respectively. By w and w^* we mean the weak topology on X and the weak star topology on X^*. If S is a subset of X we denote by $\operatorname{cl} S$ the closure of S; if $x \in X$, we denote the distance from x to S by $d(x,S) = \inf_{y\in S} d(x,y)$ and by d_S the distance function with respect to S,

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$d_S(x) = d(x, S)$ for every $x \in X$ (by convention, $d(x, \emptyset) = \infty$); I_S will be the indicator function of S ($I_S(x) = 0$ if $x \in S$ and $I_S(x) = \infty$, if $x \notin S$). For $r > 0$ we note $B(S, r) := \{x \in X \mid d(x, S) < r\}$ and $D(S, r) := \{x \in X \mid d(x, S) \leq r\}$; of course, for an element $x \in X$, $B(x, r) = B(\{x\}, r)$ and $D(x, r) = D(\{x\}, r)$.

By Y, Z we denote another normed vector spaces and by $L(X, Y)$ the space of all continuous linear operators from X into Y. On the product space $X \times Y$ we consider the sum norm.

First, we recall the definitions of the Fréchet subdifferential. If $f : X \to \mathbb{R} \cup \{\infty\}$ is a function, we denote the domain of f by $\operatorname{Dom} f = \{x \in X \mid f(x) < \infty\}$.

Definition 1.1. *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous (lsc for short) function; we say that $x^* \in X^*$ belongs to the Fréchet canonical subdifferential of f at $x \in \operatorname{Dom} f$ (denoted $\partial^F f(x)$) if*

$$\liminf_{t \to 0} \left(\inf_{u \in U_X} t^{-1}(f(x + tu) - f(x)) - x^*(u) \right) \geq 0.$$

Definition 1.2. *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lsc function; f is called Fréchet smooth at $x \in \operatorname{Dom} f$ if $\nabla f(\cdot)$ (∇f denotes the Fréchet differential) exists on a neighbourhood U of x and is continuous on U from X with the norm topology to X^* with the norm topology.*

Definition 1.3. *Let $f : X \to \mathbb{R} \cup \{\infty\}$ a lsc function; we say that $x^* \in X^*$ belongs to the Fréchet subdifferential of viscosity of f at x (denoted $D^F f(x)$) if there exists a locally Lipschitz function g such that g is Fréchet smooth at x, $\nabla f(x) = x^*$ and $f - g$ attains a local minimum at x.*

It is proved in [3] that $\partial^F f = D^F f$ if the space X admits a C^1-smooth Lipschitz bump (i.e. with a nonempty bounded support) function. This will be the setting of our main results and for this reason we use only the notation ∂^F called in the sequel the Fréchet subdifferential. Using the Fréchet subdifferential we define the normal cone to a closed set $S \subset X$ at a point $x \in S$ in the following way:

$$N_{\partial^F}(S, x) := \partial^F I_S(x).$$

We also use the following notations:

1. $u \xrightarrow{f} x$ means that $u \to x$ and $f(u) \to f(x)$;

2. $x^* \in \|\cdot\|^* - \limsup\limits_{u \to x} \partial^F f(u)$ means that for every $\varepsilon > 0$ there exist x_ε and x_ε^* such that $x_\varepsilon^* \in \partial^F f(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x_\varepsilon^* - x^*\| < \varepsilon$; the notation $x^* \in \|\cdot\|^* - \limsup\limits_{u \xrightarrow{f} x} \partial^F f(u)$ has now a similar interpretation;

3. $x^* \in w^* - \limsup\limits_{u \to x} \partial^F f(u)$ means that for every $\varepsilon > 0$ and U a weak-star neighborhood of 0 in X^*, there exists $x_{\varepsilon,U}$ and $x_{\varepsilon,U}^*$ such that $x_{\varepsilon,U}^* \in \partial^F f(x_{\varepsilon,U})$ and $\|x_{\varepsilon,U} - x\| < \varepsilon$, $x_{\varepsilon,U}^* \in x^* + U$.

We list below the main properties of the Fréchet subdifferential which we shall use in the sequel (see [8], [5], [2], [10]). All the functions we consider in the next properties are lsc unless stated otherwise.

(P1) If f attains a local minimum at $x \in \operatorname{Dom} f$, then $0 \in \partial^F f(x)$.

(P2) If f is a convex function then $\partial^F f$ is the subdifferential in the sense of convex analysis. In particular, if S is convex and $x \in S$, then $N_{\partial^F}(S,x) = N(S,x)$, the normal cone to S at x in the sense of convex analysis.

(P3) If X is an Asplund space, $\varphi_1, \varphi_2, ..., \varphi_n : X \to \mathbb{R}$ is a family of convex Lipschitz functions and $x \in \operatorname{Dom} f$, then

$$\partial^F (f + \sum_{i=1}^{n} \varphi_i)(x) \subset \|\cdot\|^* - \limsup_{y \xrightarrow{f} x, z_i \to x} (\partial^F f(y) + \sum_{i=1}^{n} \partial^F \varphi_i(z_i)).$$

(P4) If X is an Asplund space then for every family $f_1, f_2, \ldots, f_n : X \to \mathbb{R} \cup \{\infty\}$ of lsc functions, $x \in \bigcap\limits_{i=1}^{n} \operatorname{Dom} f_i$ one has

$$\partial^F (\sum_{i=1}^{n} f_i)(x) \subset w^* - \limsup_{x_i \xrightarrow{f_i} x} \sum_{i=1}^{n} \partial^F f_i(x_i).$$

(P5) If X, Y are Banach spaces which admit C^1−smooth Lipschitz bump functions, $\varphi : X \times Y \to \mathbb{R} \cup \{\infty\}$ a lsc, proper function, bounded from below and $f(x) := \inf\limits_{y \in Y} \varphi(x,y)$ the marginal function associated with φ supposed to be lsc, then

$$x^* \in \partial^F f(x) \Rightarrow (x^*, 0) \in \|\cdot\|^* - \limsup_{x' \xrightarrow{f} x, \varphi(x',y') \to f(x)} \partial^F \varphi(x', y').$$

(P6) If $X = Y \times Z$ and $f(y,z) = g(y) + h(z)$ then $\partial^F f(y,z) = \partial^F g(y) \times \partial^F h(z)$. In particular, for all closed subsets A and B of X and for all $a \in A$,

$b \in B$ one has

$$N_{\partial^F}(A \times B, (a,b)) = N_{\partial^F}(A,a) \times N_{\partial^F}(B,b).$$

2. Application to an optimization problem. In the sequel we work with a set-valued map $F : X \rightrightarrows Y$ and we denote the domain and the graph of F by $\operatorname{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F = \{(x,y) \mid y \in F(x)\}$, respectively. $F^{-1} : Y \rightrightarrows X$ is the set-valued map given by the relation $(y,x) \in \operatorname{Gr} F^{-1}$ if and only if $(x,y) \in \operatorname{Gr} F$. If $A \subset X$, $F(A) := \bigcup\limits_{x \in A} F(x)$. If the graph of F is closed the Fréchet coderivative of F at a point $(x,y) \in \operatorname{Gr} F$ is the set-valued map $D^*_{\partial^F} F(x,y) : Y^* \rightrightarrows X^*$ given by:

$$D^*_{\partial^F} F(x,y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_{\partial^F}(\operatorname{Gr} F, (x,y))\}.$$

We consider Q a nonempty pointed closed convex cone in Y with nonempty interior (denoted $\operatorname{int} Q$) which introduces a partial order in Y by $y_1 \leq_Q y_2$ iff $y_2 - y_1 \in Q$. If A and B are subsets of X we denote $d(A,B) := \inf_{a \in A} d(a,B)$.

Let A and C be nonempty, closed subsets in X and Y, respectively. Consider the function $h : X \to \overline{\mathbb{R}}$, $h(u) := d(F(u), C)$. This section is devoted to the study of the optimization problem:

$$(\Pi_1) \begin{cases} \text{minimize } h(u) \\ \text{subject to } u \in A \end{cases}.$$

Besides its teoretical interest this problem is connected with some wide studied optimization problems involving vector-valued functions $g : X \to Y$ and set-valued maps $F : X \rightrightarrows Y$, like

$$(\Pi_2) \text{ minimize } g(u)$$

and

$$(\Pi_3) \text{ minimize } F(u),$$

where the minimum notions for these problems can be defined as follows.

Definition 2.1. *(a) An element $x \in X$ is called $Q-$strong local minimum for the problem (Π_2) if there exists a neighborhood U of x such that for all $x' \in U$, $g(x) - g(x') \in -Q$; (b) an element $x \in X$ is called $Q-$weak local minimum for* *the problem* $(\Pi 2)$ *if there exists a neighborhood* U *of* x *such that for all* $x' \in U$, $g(x) - g(x') \notin \operatorname{int} Q$.

Definition 2.2. *a) An element* $x \in X$ *is called* $Q-$*strong local minimum for the problem* (Π_3) *if there exists a neighborhood* U *of* x *and* $y \in F(x)$ *such that for all* $x' \in U$, $F(x') \subset y + Q$; *(b) an element* $x \in X$ *is called* $Q-$*weak local minimum for the problem* (Π_3) *if there exists a neighborhood* U *of* x *and* $y \in F(x)$ *such that for all* $x' \in U$, $F(x') \cap (y - \operatorname{int} Q) = \emptyset$.

For details on problems $(\Pi 2)$ and (Π_3) see, e.g., [11], [13] and the references therein.

The next two results establish the connections between the optimization problems considered above.

Proposition 2.1. *(i) If* $x \in X$ *is a* $Q-$*strong local minimum for the problem* (Π_2), *then* x *is a local minimum for the function* $h_1(u) = d(g(u), -Q)$; *(ii) if* x *is a local minimum for the function* h_1, *and* $g(x) \notin -Q$, *then* x *is* $Q-$*weak local minimum for the problem* (Π_2).

P r o o f. (i) From Definition 2.1 there exists a neighborhood U of x such that for all $x' \in U$, $g(x') - g(x) \in Q$; we can write

$$\begin{aligned} d(g(x'), -Q) &= d(0, -g(x') - Q) = \\ &= d(g(x), g(x) - g(x') - Q) \geq d(g(x), -Q) \end{aligned}$$

the last inequality being true because $-Q$ is a convex cone and so, $-Q + g(x) - g(x') \subset -Q$.

(ii) Since x is a local minimum for the function h_1 there exists a neighborhood U of x such that for all $x' \in U$, $h_1(x) \leq h_1(x')$. Suppose that there exists $x' \in U$ such that $g(x) - g(x') \in \operatorname{int} Q$. There exists $r > 0$ such that $B(g(x) - g(x'), r) \subset Q$ and this implies that $B(-Q + g(x') - g(x), r) \subset -Q$. Then, as above, $h_1(x) = d(g(x), -Q) = d(g(x'), -Q + g(x') - g(x))$. But

$$\begin{aligned} h_1(x') &= d(g(x'), -Q) \leq d(g(x'), B(-Q + g(x') - g(x), r)) = \\ &= \max\{d(g(x'), -Q + g(x') - g(x)) - r, 0\} = \max\{d(g(x), -Q) - r, 0\} = \\ &= \max\{d(g(x), -Q), r\} - r < d(g(x), -Q) = h_1(x) \end{aligned}$$

because $d(g(x), -Q) > 0$ taking into account that $g(x) \notin -Q$ and Q is a closed cone. So, we have that $h_1(x) > h_1(x')$, a contradiction. The proof is complete. $\square$

Proposition 2.2. *(i) If $x \in X$ is a $Q-$strong local minimum for the problem (Π_3), then x is a local minimum for the function $h_2(u) = d(F(u), -Q)$; (ii) if x is a local minimum for the function h_2, and $F(x)$ is compact and $F(x) \cap -Q = \emptyset$ then x is a $Q-$weak local minimum for the problem (Π_3).*

P r o o f. (i) From Definition 2.2 there exists a neighborhood U of x and $y \in F(x)$ such that for all $x' \in U$, $F(x') \subset y + Q$; in particular $F(x) \subset y + Q \subset F(x) + Q$, hence $F(x) + Q = y + Q$. Consequently,

$$h_2(x) = d(F(x), -Q) = d(0, F(x) + Q) = \\ = d(0, y + Q) = d(y, -Q).$$

The rest of the proof is similar with the proof of *(i)* in the above proposition.

(ii) Since $F(x)$ is compact, there exists $y \in F(x)$ such that $h_2(x) = d(y, -Q)$; in our assumptions, there exists a neighborhood U of x such that for all $x' \in U$, $h_2(x) \leq h_2(x')$. Suppose that there exists $x' \in U$ and $y' \in F(x')$ such that and $y - y' \in \operatorname{int} Q$. As above, $d(y, -Q) > d(y', -Q) \geq d(F(x'), -Q)$, a contradiction. $\square$

Remark 2.1. The function h_1 does not provide useful information on the problem (Π_2) if $g(X) \cap -Q \neq \emptyset$, where $g(X) = \{g(x) \mid x \in X\}$. However, even in this case, if g is bounded from below (i.e. there exists $y \in Y$ such that $y \leq g(x')$ for every $x' \in X$) taking $q \in Q\backslash\{0\}$ one can replace the function g with $\overline{g}(\cdot) = g(\cdot) - y + q$ to obtain a function h_1 with nonzero values: $d(\overline{g}(x'), -Q) = d(0, g(x') - y + q + Q) \geq d(0, q + Q) > 0$. A similar remark can be made on function h_2 and the problem (Π_3).

We are now able to prove a necessary optimality result for (Π_1). By Pr_Y we denote the projection operator.

Theorem 2.1. *Let X, Y be Banach spaces, which admit C^1-smooth Lipschitz bump functions. Suppose, with the above notations, that F has closed graph and is upper semicontinuous (usc for short). If $x \in A$ is a local minimimum for problem (Π_1), then for every $\varepsilon > 0$, U^* and V^* weak-star neighbourhoods of 0 in X^* and Y^*, there exist $u_\varepsilon \in A \cap B(x, \varepsilon), x_\varepsilon \in B(x, \varepsilon)$, $y_\varepsilon \in F(x_\varepsilon)$, $z_\varepsilon \in C$ s.t.*

$$0 \in D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(N_{\partial^F}(C, z_\varepsilon) + V^*) + N_{\partial^F}(A, u_\varepsilon) + U^*.$$

Moreover, $d(y_\varepsilon, C) < h(x) + \varepsilon$ and $d(z_\varepsilon, \operatorname{Pr}_Y(\operatorname{Gr} F \cap (B(x, \varepsilon) \times Y))) < h(x) + \varepsilon$.

P r o o f. Consider the function $h : X \to \mathbb{R} \cup \{\infty\}$, $h(u) := d(F(u), C)$. Using that, F is usc it can be proved that h is lsc. Indeed, consider $u \in X$, and

$0 < \lambda < h(u) = d(F(u), C)$; there is $\theta > 0$ such that $F(u) \cap D(C, \lambda + \theta) = \emptyset$. As F is usc at u we can find (see [9]) a neighborhood U of u such that for all $u' \in U$ the relation $F(u') \cap D(C, \lambda + \theta) = \emptyset$ is true. It implies $d(F(u'), C) \geq \lambda + \theta > \lambda$, for all $u' \in U$, so h is lsc. As x is a local minimum for the scalar problem:

$$(\Pi_1) \begin{cases} \text{minimize } h(x) \\ \text{subject to } x \in A \end{cases},$$

it follows that that x is a local minimum for the function $h + I_A$. So, using propety (P1),

$$0 \in \partial^F (h + I_A)(x).$$

But ∂^F satisfies property (P4) on X because every Banach space which admits a C^1-smooth Lipschitz bump function is Asplund, hence

$$0 \in w^* - \limsup\{\partial^F h(z) + \partial^F I_A(u); z \xrightarrow{h} x, u \xrightarrow{A} x\},$$

i.e.,

$$0 \in w^* - \limsup\{\partial^F h(z) + N_{\partial^F}(A, u); z \xrightarrow{h} x, u \xrightarrow{A} x\}. \tag{1}$$

The key of the proof is to express $\partial^F h(z)$. We can write $h(z) = \inf\{\|v - s\| + I_{\operatorname{Gr} F \times Y}(z, v, s) + I_{X \times Y \times C}(z, v, s); v \in Y, s \in Y\}$. Consider

$$\varphi_1(z, v, s) := \|v - s\|, \varphi_2(z, v, s) := I_{\operatorname{Gr} F \times Y}(z, v, s),$$
$$\varphi_3(z, v, s) := I_{X \times Y \times C}(z, v, s).$$

Let $x^* \in \partial^F h(z)$; applying property (P5) (in our setting its assumptions are verified), we have that

$$\begin{aligned}(x^*, 0, 0) \in \|\cdot\|^* - \limsup\{&\partial^F(\varphi_1 + \varphi_2 + \varphi_3)(z', v', s'); \\ &z' \to z, (z', v') \in \operatorname{Gr} F, s' \in C, \|v' - s'\| \to h(z)\} \subset \\ &\subset w^* - \limsup\{\partial^F(\varphi_1 + \varphi_2 + \varphi_3)(z', v', s'); \\ &z' \to z, (z', v') \in \operatorname{Gr} F, s' \in C, \|v' - s'\| \to h(z)\}.\end{aligned}$$

But, taking into account again property (P4) we have

$$\begin{aligned}\partial^F(\varphi_1 + \varphi_2 + \varphi_3)(z', v', s') \subset w^* - \limsup\{&\partial^F \varphi_1(z_1, v_1, s_1) + \\ &+ \partial^F \varphi_2(z_2, v_2, s_2) + \partial^F \varphi_3(z_3, v_3, s_3); \\ (z_i, v_i, s_i) \xrightarrow{\varphi_i} (z', v', s'), i = \overline{1, 3}\}.&\end{aligned}$$

From the relations above we have that

$$(x^*, 0, 0) \in w^* - \limsup\{\partial^F \varphi_1(z_1, v_1, s_1) + \partial^F \varphi_2(z_2, v_2, s_2) + \partial^F \varphi_3(z_3, v_3, s_3);\\ (z_i, v_i) \in \operatorname{Gr} F, s_i \in C, z_i \to z, \|v_i - s_i\| \to h(z), i = \overline{1,3}\}.$$

It is clear that φ_1 is a convex function and using property (P2),

$$\partial^F \varphi_1(z_1, v_1, s_1) \subset \{0\} \times \{(y^*, -y^*), \|y^*\| \leq 1\}.$$

On the other hand, from the property (P6),

$$\partial^F \varphi_2(z_2, v_2, s_2) = \operatorname{Gr} D^*_{\partial^F} F(z_2, v_2) \times \{0\}$$

and

$$\partial^F \varphi_3(z_3, v_3, s_3) = \{0\} \times \{0\} \times N_{\partial^F}(C, s_3).$$

Consider now $\varepsilon > 0$, U^* and V^* weak-star neighbourhoods of 0 in X^* and Y^* (arbitrary, but fixed). Take U_1^* and V_1^* symetric weak-star neighbourhoods of 0 in X^* and Y^* respectively with $U_1^* + U_1^* \subset U^*$ and $-V_1^* - V_1^* \subset V^*$. From relation (1) there exist $s_\varepsilon \in B(x, \varepsilon/2), u_\varepsilon \in A \cap B(x, \varepsilon)$ such that $|h(s_\varepsilon) - h(x)| < \varepsilon/2$ and

$$0 \in \partial^F h(s_\varepsilon) + N_{\partial^F}(A, u_\varepsilon) + U_1^*.$$

Take $s_\varepsilon^* \in \partial^F h(s_\varepsilon)$; then there exist $(x_\varepsilon, y_\varepsilon) \in \operatorname{Gr} F$ with $x_\varepsilon \in B(s_\varepsilon, \varepsilon/2)$, $(x_\varepsilon^*, y_\varepsilon^*) \in \operatorname{Gr} D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)$, $z_\varepsilon \in C$ with

$$|\|y_\varepsilon - z_\varepsilon\| - h(s_\varepsilon)| < \varepsilon/2$$

and $(x'_\varepsilon, y'_\varepsilon) \in \operatorname{Gr} F$ with $x'_\varepsilon \in B(s_\varepsilon, \varepsilon/2), z'_\varepsilon \in C$, $|\|y'_\varepsilon - z'_\varepsilon\| - h(s_\varepsilon)| < \varepsilon/2, z_\varepsilon^* \in N_{\partial^F}(C, z'_\varepsilon)$, $y^* \in U_{Y^*}$ such that

$$(s_\varepsilon^*, 0, 0) \in (0, y^*, -y^*) + (x_\varepsilon^*, y_\varepsilon^*, 0) + (0, 0, z_\varepsilon^*) + U_1^* \times V_1^* \times V_1^*.$$

Consequently,

$$\begin{aligned} s_\varepsilon^* &\in D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(-y_\varepsilon^*) + U_1^* \subset \\ &\subset D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(y^* - V_1^*) + U_1^* \subset \\ &\subset D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(z_\varepsilon^* - V_1^* - V_1^*) + U_1^* \subset \\ &\subset D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(N_{\partial^F}(C, z'_\varepsilon) + V^*) + U_1^*. \end{aligned}$$

We can write now

$$0 \in D^*_{\partial^F} F(x_\varepsilon, y_\varepsilon)(N^1_{\partial^F}(C, z'_\varepsilon) + V^*) + U^*_1 + N^1_{\partial^F}(A, u_\varepsilon) + U^*_1,$$

and the announced relations for $u_\varepsilon, x_\varepsilon, y_\varepsilon, z'_\varepsilon$, are also true, hence the conclusion. $\square$

Remark 2.2. If X and Y are finite dimensional and C is bounded the upper semicontinuity of F is not needed. In this case h is lsc: let $0 < \lambda < h(u)$; then $F(u) \cap D(C, \lambda) = \emptyset$, hence $\operatorname{Gr} F \cap (\{u\} \times D(C, \lambda)) = \emptyset$. Since $\operatorname{Gr} F$ is closed and $(\{u\} \times D(C, \lambda))$ is compact, there exists $\varepsilon > 0$ such that $\operatorname{Gr} F \cap (B(u, \varepsilon) \times B(C, \lambda + \varepsilon)) = \emptyset$. Then for all $u' \in B(u, \varepsilon)$, $d(F(u'), C) > \lambda$.

3. Application to metric regularity. The aim of this section is to give verifiable conditions in terms of Fréchet coderivative of a multifunction for satisfying a certain metric regularity property.

Let $f : X \times Y \to \mathbb{R} \cup \{\infty\}$. If the function $g(\cdot) = f(\cdot, y) : X \to \mathbb{R} \cup \{\infty\}$ with a fixed $y \in Y$ is lsc then we denote $\partial^F_x f(x, y) := \partial^F g(x)$. We also use the notations: $f_+(x, y) := \max(f(x, y), 0)$ and $S(y) := \{x \in X; f(x, y) \leq 0\}$. First, we observe that Theorem 2.4 from [6] remains true also if the abstract subdifferential considered satisfies a weaker sum principle. We work in the sequel with the Fréchet subdifferential but the next two results are true for an abstract subdifferential which satisfies certain properties. See also [1], [12], [14].

Theorem 3.1. *Let X be an Asplund space and $f : X \times Y \to \mathbb{R} \cup \{\infty\}$ an extended real-valued function such that for each $y \in Y$, $f(\cdot, y)$ is lsc. If there exists $a > 0$ such that for each $x \in X$ and $y \notin S^{-1}(x)$, $d(0, \partial^F_x f(x, y)) \geq a^{-1}$, then for every $x \in X$ and $y \in Y$, $d(x, S(y)) \leq a f_+(x, y)$.*

For the convenience of the reader we present the proof for the next local version of the previous result which we shall use in the sequel. A similar proof can be given for the above result.

Theorem 3.2. *Let X be an Asplund space and $f : X \times Y \to \mathbb{R} \cup \{\infty\}$ an extended real-valued function such that for each $y \in Y$, $f(\cdot, y)$ is lsc and let $x \in S(y)$. Suppose that there exists $a > 0$, $r > 0$ such that*

$$\forall u \in D(x, r), \forall v \in D(y, r) \backslash S^{-1}(u), d(0, \partial^F_x f(u, v)) \geq a^{-1}. \tag{2}$$

Then for every $u \in D(x, r/2)$ and $v \in D(y, r)$, $d(u, S(v)) \leq a f_+(u, v)$.

P r o o f. Suppose, by contradiction, that there exist $u \in D(x, r/2)$ and $v \in D(y, r)$ such that $d(u, S(v)) > af_+(u, v)$. As $d(u, S(v)) > 0$, we have $u \notin S(v)$. Take $\varepsilon = f_+(u, v)$, $\lambda = (a + \alpha)\varepsilon$ with $\alpha > 0$ such that $\lambda < d(u, S(v))$. It is clear that $f_+(u, v) \le \inf_{u' \in D(x, r/2)} f_+(u', v) + \varepsilon$, hence in our assumptions we can apply Ekeland's variational principle for the function $f_+(\cdot, v)$. Consequently, there exists $u' \in D(x, r/2)$ satisfying

$$\left\| u' - u \right\| \le \lambda$$

and

$$f_+(u', v) \le f_+(x', v) + \varepsilon\lambda^{-1} \left\| x' - u' \right\|, \forall x' \in D(x, r/2).$$

First, we have $\|u' - u\| \le \lambda < d(u, S(v))$. If $u' \in S(v)$ then $S(v)$ is nonempty, hence

$$d(u, S(v)) = \left| d(u', S(v)) - d(u, S(v)) \right| \le \left\| u' - u \right\| < d(u, S(v))$$

and this is a contradiction. So, $u' \notin S(v)$. Applying (P1) and (P3) to the function $f_+(\cdot, v) + \varepsilon\lambda^{-1} \|\cdot - u'\|$ we have that

$$0 \in \|\cdot\|^* - \limsup\{\partial_x^F f(u_1', y) + \partial^F(\varepsilon\lambda^{-1} \left\|\cdot - u'\right\|)(u_2'); (u_1', v) \xrightarrow{f} (u', v), u_2' \to u'\}.$$

Consider a positive real number θ with $\theta < f(u', v)$, $\theta < r/2$ and $\theta < a^{-1}(a + \alpha)^{-1}\alpha$. There exist u_1', u_2', x_1^*, x_2^* such that $\|u_1' - u'\| < \theta$, $\|u_2' - u'\| < \theta$, $|f_+(u_1', v) - f_+(u', v)| < \theta$, $x_1^* \in \partial_x^F f_+(u_1', v)$, $x_2^* \in X^*, \|x_2^*\| \le (a + \alpha)^{-1}$, $\|x_1^* + x_2^*\| < \theta$. Clearly $f_+(u_1', v) > 0$, hence $u_1' \notin S(v)$. Since $f(\cdot, v)$ is lsc, it coincides with $f_+(\cdot, v)$ on a neighborhood of u_1' and then we have, $\partial_x^F f_+(u_1', v) = \partial_x^F f(u_1', y)$. We also have $\|x_1^*\| < \|x_2^*\| + \theta < (a + \alpha)^{-1} + \theta < a^{-1}$, and $\|u_1' - x\| < \|u_1' - u'\| + \|u' - x\| < \theta + r/2 < r$, in contradiction with hypothesis. $\square$

In the sequel for a set $A \subset X$ and $x \in X$ we denote $\operatorname{pr}_\theta(x, A) := \{u \in A \mid \|x - u\| < d(x, A) + \theta\}$. We give now our main result of this section.

Theorem 3.3. *Let X and Y Banach spaces which admit a C^1−smooth Lipschitz bump function. Let $F : X \rightrightarrows Y$ be a multifunction with closed graph and $(x, y) \in \operatorname{Gr} F$; if there exist $\alpha > 0$, $r > 0$, such that for every $u \in D(x, r)$, $v \in D(y, r) \backslash F(u)$ there exists $\theta \in (0, 1)$, with*

$$\begin{aligned} &\inf\{\|x^*\|; x^* \in D^*_{\partial^F} F(u', v')(y^*), \|y^*\| = 1, \\ (3) \qquad &(u', v') \in \operatorname{pr}_\theta((u, v), \operatorname{Gr} F)\} > \alpha \end{aligned}$$

then there exists $a > 0$ *such that* $d(u, F^{-1}(v)) \le ad((u,v), \operatorname{Gr} F)$*, for every* $u \in D(x, r/2)$ *and* $v \in D(y, r)$.

Proof. There exists $a > 1$ such that $\alpha > a^{-1}$. It is enough to prove that our relation implies relation (2), for such an a, for the function $f : X \times Y \to \mathbb{R}$, $f(x,y) := d((x,y), \operatorname{Gr} F)$ in terms of ∂^F subdifferential and to use Theorem 3.2 to have the conclusion.

Consider $u \in D(x,r)$, $v \in D(y,r)$, $v \notin F(u)$ (which, for f as above is equivalent with $v \notin S^{-1}(u)$) and $x^* \in \partial_x^F f(u,v)$. Take $\gamma > 0$ and smaller than the difference between the inf from (3) and a^{-1}. If we take $h : X \to \mathbb{R}$, $h(\cdot) = f(\cdot, v)$ we have $x^* \in \partial^F h(u)$. But

$$h(u) = \inf\{\|u - x'\| + \|v - y'\| + I_{\operatorname{Gr} F}(x', y'); (x', y') \in X \times Y\},$$

so,

$$\begin{aligned}(x^*, 0, 0) \in \|\cdot\|^* - \limsup\{&\partial^F(\varphi_1 + \varphi_2 + \varphi_3)(x, x', y');\\ &x \to u, h(x) \to h(u), (\varphi_1 + \varphi_2 + \varphi_3)(x, x', y') \to h(u)\}\end{aligned}$$

where

$$\varphi_1(x, x', y') := \|x - x'\|, \varphi_2(x, x', y') := \|v - y'\|, \varphi_3(x, x', y') := I_{\operatorname{Gr} F}(x', y').$$

Consider $\varepsilon_1 > 0$ and smaller than

$$\lambda := \min(\gamma/(2 + a^{-1} + \gamma), \theta/4, d((u,v), \operatorname{Gr} F), 1 - a^{-1}).$$

For this ε_1 thre exist x, x', y' and u^*, u'^*, v'^* such that $(u^*, u'^*, v'^*) \in \partial^F(\varphi_1 + \varphi_2 + \varphi_3)(x, x', y')$, $\|x - u\| < \varepsilon_1, (x', y') \in \operatorname{Gr} F$,

$$\big|\|x - x'\| + \|v - y'\| - d((u,v), \operatorname{Gr} F)\big| < \varepsilon_1$$

and $\|(u^*, u'^*, v'^*) - (x^*, 0, 0)\| < \varepsilon_1$. Is implies that $(x', y') \in \operatorname{pr}_{\theta/2}((u,v), \operatorname{Gr} F)$ and also $\|x - x'\| + \|v - y'\| > 0$ i.e. $x \ne x'$ or $y' \ne v$. From property (P3), $\partial^F(\varphi_1 + \varphi_2 + \varphi_3)(x, x', y') \subset \|\cdot\|^* - \limsup\{\partial^F\varphi_1(x_1, x_1', y_1') + \partial^F\varphi_2(x_2, x_2', y_2') + \partial^F\varphi_3(x_3, x_3', y_3'); (x_i, x_i', y_i') \xrightarrow{\varphi_i} (x, x', y'), i = \overline{1,3}\}$. Using properties (P2) and (P6),

$$\partial^F\varphi_1(x_1, x_1', y_1') \subset \{(x_1^*, -x_1^*); \|x_1^*\| \le 1\} \times \{0\} \tag{4}$$

(5) $$\partial^F \varphi_2(x_2, x_2', y_2') \subset \{0\} \times \{0\} \times \{y_2^*; \|y_2^*\| \le 1\}$$

(6) $$\partial^F \varphi_3(x_3, x_3', y_3') = \{0\} \times N_{\partial^F}((x_3', y_3'); \operatorname{Gr} F).$$

First we consider the case $x \neq x'$; in this case we take $\varepsilon_2 > 0$ such that $\varepsilon_2 < \|x - x'\|/2$ and $\varepsilon_1 + \varepsilon_2 < \lambda$; there exist $x_1, x_1', \|x_1 - x\| < \varepsilon_2, \|x_1' - x'\| < \varepsilon_2$ and, from the choice of ε_2, we have $x_1 \neq x_1'$; it means that in relation (4) we can take "$= 1$" instead of "≤ 1". Hence, there exists $x_1^* \in U_{X^*}$ with $\|u^* - x_1^*\| < \varepsilon_2$. We can write

$$\|x^*\| > \|u^*\| - \varepsilon_1 > \|x_1^*\| - \|u^* - x_1^*\| - \varepsilon_1 > 1 - \varepsilon_2 - \varepsilon_1 > 1 - (1 - a^{-1}) = a^{-1}.$$

Consider now that $y' \neq v$; in this case we take $\varepsilon_2 > 0$ such that $\varepsilon_2 < \|y' - v\|$ and $\varepsilon_1 + \varepsilon_2 < \lambda$; there exist $x_1^*, x_3^*, y_2^*, y_3^*, (x_3', y_3') \in \operatorname{Gr} F$ such that $\|x_1^* - u^*\| < \varepsilon_2$, $\|-x_1^* + x_3^* - u'^*\| < \varepsilon_2, \|y_2^* + y_3^* - v'^*\| < \varepsilon_2$, $x_3^* \in D^*_{\partial^F} F(x_3', y_3')(-y_3^*)$, $\|y_2^*\| = 1$ (in the relation (5) we can take "$= 1$" instead of "≤ 1" because $\|y_2' - y'\| < \varepsilon_2$ implies that $y_2' \neq v$) and $\|(x_3', y_3') - (x', y')\| < \varepsilon_2$. Hence, as above,

$$\|x^*\| > \|x_1^*\| - \varepsilon_2 - \varepsilon_1 > \left\|x_3^* - u'^*\right\| - 2\varepsilon_2 - \varepsilon_1 > \|x_3^*\| - 2(\varepsilon_1 + \varepsilon_2).$$

We also have,

$$\|y_3^*\| > \left\|y_2^* - v'^*\right\| - \varepsilon_2 > 1 - \varepsilon_1 - \varepsilon_2.$$

Clearly, $(x_3', y_3') \in \operatorname{pr}_\theta((u, v), \operatorname{Gr} F)$. But we have

$$x_3^*/\|y_3^*\| \in D^*_{\partial^F} F(x_3', y_3')(-y_3^*/\|y_3^*\|)$$

and from hypothesis we obtain $\|x_3^*\|/\|y_3^*\| > a^{-1} + \gamma$, hence $\|x_3^*\| > (a^{-1} + \gamma)(1 - \varepsilon_1 - \varepsilon_2)$ and this ensures that $\|x^*\| > (a^{-1} + \gamma)(1 - \varepsilon_1 - \varepsilon_2) - 2(\varepsilon_1 + \varepsilon_2)$, i.e. $\|x^*\| > a^{-1}$. The proof is complete. □

Remark 3.1. Taking into account that $d((x, y), \operatorname{Gr} F) \le d(y, F(x))$, the preceding result contains Theorem 4.5. from [10] in two ways: the inequality of the conclusion is stronger and we did not suppose that F is usc as the quoted result did.

Let us to compare the above results with Theorems 1 and 1a from [4, Chapter 3]. In our results the hypotheses are stronger but we obtain a more accurate conclusion by means of the exact description of the neighborhoods involved: in this result it is indicated where the condition on the coderivative should take place and where the openess (regularity) holds. Taking into account these considerations, the results we mention are independent.

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