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STABILITY AND INSTABILITY OF SOLITARY WAVE SOLUTIONS OF A NONLINEAR DISPERSIVE SYSTEM OF BENJAMIN-BONA-MAHONY TYPE*

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Abstract. This paper concerns the orbital stability and instability of solitary waves of the system of coupling equations of Benjamin-Bona-Mahony type. By applying the abstract results of Grillakis, Shatah and Strauss and detailed spectral analysis, we obtain the existence and stability of the solitary waves.

1. Introduction. Considered herein is the stability and instability of solitary wave solutions for the system of nonlinear evolution equations

$$\left\{\begin{array}{l} u_t + f(u,v)_x - u_{xxt} = 0 \\ v_t + g(u,v)_x - v_{xxt} = 0 \\ u(x,0) = u_0(x), \;\; v(x,0) = v_0(x) \end{array}\right. \tag{1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions. This system can also be interpreted as a coupled nonlinear version of Benjamin-Bona-Mahony equations (BBM)

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$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.2}$$

A more general class of equations of BBM type of the form

$$Lu_t + (f(u))_x = 0 \tag{1.3}$$

was investigated by Souganidis and Strauss [16]. It is shown that for the generalized BBM equation, all solitary waves are stable when $p \le 4$ and when $p > 4$, there is a critical value of solitary wave speed $c_r > 1$ such that the solitary wave is stable for wave speed $c > c_r$ and unstable for $1 < c < c_r$. The stability and instability for a more general nonlinearity is investigated in [12], [13] and [14]. In particular for $f(u) = u^p$, stability is obtained for all $p > 0$.

For class of equations of Korteweg-de Vries (KdV) type

$$u_t + (f(u))_x - Mu_x = 0 \tag{1.4}$$

was investigated by Bona at el. [10]. In [4] the existence and stability of solitary wave solutions is investigated for a system of nonlinear evolution equation

$$\begin{cases} u_t + u_{xxx} + (u^p v^{p+1})_x = 0 \\ v_t + v_{xxx} + (u^{p+1} v^p)_x = 0 \end{cases} \tag{1.5}$$

which can be interpreted as a coupled nonlinear version of generalized KdV equations. It is shown that the solitary waves are stable for $p < 2$ and unstable for $p > 2$.

Our main purpose in this article will be to study orbital stability and instability of solitary wave solutions of system (1.1). We base our analysis on the invariants

$$E(u,v) = -\int_{-\infty}^{\infty} F(u,v)dx, \quad Q(u,v) = \frac{1}{2}\int_{-\infty}^{\infty} (u^2 + v^2 + u_x^2 + v_x^2)dx$$

$$I(u) = \int_{-\infty}^{\infty} u dx$$

where $F_u = f,\ \ F_v = g$ and $F(0,0) = 0$, following the proofs of [11], [12] and [16].

The plan of the paper is as follows. A discussion of system (1.1) is given in Section 2. The existence of solitary waves is developed in Section 3. The theory of stability and instability will be established in Section 4.

Notation

- The norm in $H^s(\mathbb{R})$ will be denoted by $\|\cdot\|_s$ and $\|\cdot\|$ will be denote the $L^2(\mathbb{R})$

norm.

• We denote $X^s = H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and $\|\mathbf{f}\|^2_{X^s} = \|f\|^2_s + \|g\|^2_s$ for $\mathbf{f} = (f, g)$. The norm in $L^p(\mathbb{R}) \times L^p(\mathbb{R})$ will be $\|\mathbf{f}\|_{L^p \times L^p} = \|f\|_{L^p} + \|g\|_{L^p}$.

• $D = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}$ and $M = \begin{pmatrix} 1 - \partial_x^2 & 0 \\ 0 & 1 - \partial_x^2 \end{pmatrix}$

2. The evolution equation. We begin with a basic theorem which guarantees the existence and uniqueness of solutions for the nonlinear dispersive system (1.1) in X^1.

Theorem 2.1. (a) *If* $\mathbf{u}_0 \in X^1$ *and* $f(u, v) = u + u^p v^{p+1}$, $g(u, v) = v + u^{p+1} v^p$, *then there is a unique global solution* $\mathbf{u}$ *of* (1.1) *in* $C([0, \infty); X^1)$
(b) *If* $\mathbf{u}_0 \in X^1$ *and* $f(u, v) = u^p v^{p+1}$, $g(u, v) = u^{p+1} v^p$, *then there is a unique global solution* $\mathbf{u}$ *of* (1.1) *in* $C([0, \infty); X^1)$

Proof. (a) To establish existence, we first write (1.1) as

$$\mathbf{u}_t = A\mathbf{u} + A\mathbf{f}\mathbf{u}$$

where $\mathbf{f} = (u^p v^{p+1}, u^{p+1} v^p)$ and $A = -DM^{-1}$, is bounded linear operator and is therefore the infinitesimal generator of a uniformly continuous semigroup of operators. When p is a positive integer, the map $\mathbf{f} : X^1 \to X^1$ is Lipschitz and in fact differentiable, so the composition $A\mathbf{f}$ is differentiable. By [15, Theorem 6.14], for any $\mathbf{u}_0 \in X^1$ there is some $T \in (0, \infty)$ so that a unique solution $\mathbf{u}(\cdot, t)$ with initial data $\mathbf{u}_0$ exists for $0 \leq t \leq T$.

Multiplying (1.1) by (u, v) yields

$$\frac{d}{dt}\|\mathbf{u}(t)\|^2_{X^1} = 0$$

This implies that $\mathbf{u}$ is bounded in X^1 and gives the global existence of a solution $\mathbf{u}$ for this case.

For the proof of (b), we write (1.1) as

$$\mathbf{u} = \mathbf{u}_0 - \int_0^t M^{-1} D\mathbf{f}(\mathbf{u}(\tau)) d\tau$$

We solve this integral equation by iteration in the space $C([0, T]; X^1)$. Because $X^1 \subset L^\infty \times L^\infty$ so that $\mathbf{u} \to \mathbf{f}(\mathbf{u})$ carries X^1 in X^1 in a locally Lipschitzian manner. Furthermore, the operator $M^{-1}\partial_x$ has order -1, so that the integral operator is locally Lipscitzian.

Lemma 2.1. *The unique solution* $\mathbf{u}$ *of* (1.1) *with initial data* $\mathbf{u}_0 = (u_0, v_0)$ *satisfies*

$$E(\mathbf{u}) = E(\mathbf{u}_0), \quad Q(\mathbf{u}) = Q(\mathbf{u}_0)$$

Moreover, if $\int_{-\infty}^{\infty} u_0(x)dx$ *and* $\int_{-\infty}^{\infty} v_0(x)dx$ *converge then* $I(u)$ *and* $I(v)$ *converges and are constants.*

Proof. The fact that E and Q are constant follows from the theorem of existence. Integration of the equations of (1.1) separately yields

$$\int_a^b u(x,t)dx - \int_a^b u(x,0)dx = \int_0^t \int_a^b M^{-1}\partial_x f(u,v)dxd\tau$$

and

$$\int_a^b v(x,t)dx - \int_a^b v(x,0)dx = \int_0^t \int_a^b M^{-1}\partial_x g(u,v)dxd\tau.$$

As $a \to -\infty, \quad b \to \infty$, both integrals on the right-hand side tend to zero. This complete the proof of Lemma 2.1.

Consider the linear initial value problem associated to Eq. (1.1);

$$\text{(2.1)} \qquad \begin{cases} u_t - u_{xxt} + u_x = 0 \\ v_t - v_{xxt} + v_x = 0 \\ (u(0), v(0)) = (u_0, v_0) \in X^1 \end{cases}$$

and the unitary group $V(t)$ which is defined by

$$\text{(2.2)} \qquad \mathbf{V}(t)f(x) = S_t * f(x)$$

where S_t is defined by the oscillatory integral

$$S_t(x) = \int_{-\infty}^{\infty} e^{it(\frac{\xi}{1+\xi^2} - x\xi)} d\xi.$$

Therefore the solution of Eq.(2.1) is given by the unitary group $\mathbf{W}(t)$ in X^s defined for $\mathbf{u}_0 = (u_0(x), v_0(x))$ by

$$\text{(2.3)} \qquad \mathbf{W}(t)\mathbf{u}_0 = (\mathbf{V}u_0(x), \mathbf{V}v_0(x)).$$

Theorem 2.2. *Let* $\mathbf{u} \in X^1 \cap (L^1(\mathbb{R}) \times L^1(\mathbb{R}))$ *and let* $\mathbf{u}(x,t)$ *be the solution of (1.1) with initial data* $\mathbf{u}_0$. *Then there exists* $0 < \eta < 1$ *such that*

$$\text{(2.4)} \qquad \sup_{-\infty<z<\infty} \left| \int_{-\infty}^{z} \mathbf{u}(x,t)dx \right| \le c(1 + t^\eta)$$

where the constant c depends only on $\mathbf{u}_0$ and f and g.

To proof of Theorem 2.2 we need of the following lemma, which is proved in [16].

Lemmma 2.2. *Let $S(t)$ be the evolution operator for the linear equation*

$$((1-\partial_x^2)\partial_t+\partial_x)w=0$$

$(S(t)w(0)=w(t))$. Then $S(t):H^1\cap L^1\to L^\infty$ for all $t>0$. Moreover, there exists $c>0$ such that

$$|S(t)w_0|\le ct^{-\theta}(|w_0|_1+\|w_0\|_1),\;\;\theta=\frac{1}{4}$$

From Lemma 2.2 we have

$$|\mathbf{W}(t)\mathbf{u}_0|_\infty\le ct^{-\theta}(|\mathbf{u}_0|_{L^1\times L^1}+\|\mathbf{u}_0\|_{X^1}) \tag{2.5}$$

where c is a constant.

P r o o f o f T h e o r e m 2.2. Following the ideas of Souganidis and Strauss [16], we let $G(\mathbf{u})=(f(u,v)-u,g(u,v)-v)$, so that the equation takes the form $M\mathbf{u}_t+D\mathbf{u}=-DG(\mathbf{u})$

Let $\mathbf{z}(t)=\mathbf{W}(t)\mathbf{u}_0$, that is

$$(M\partial_t+D)\mathbf{z}=0,\;\;\mathbf{z}(0)=\mathbf{u}_0.$$

Then

$$\mathbf{u}(t)=\mathbf{z}(t)-\int_0^t\mathbf{W}(t-\tau)M^{-1}DG(\mathbf{u})d\tau=$$

$$\mathbf{z}(t)-D\int_0^t\mathbf{W}(t-\tau)M^{-1}G(\mathbf{u}(\tau))d\tau.$$

Let

$$U(x,t)=\int_{-\infty}^x\mathbf{u}(y,t)dy\;\text{ and }\;Z(x,t)=\int_{-\infty}^x\mathbf{z}(y,t)dy.$$

Then

$$U(t)=Z(t)-\int_0^t\mathbf{W}(t-\tau)M^{-1}G(\mathbf{u})(\tau)d\tau. \tag{2.6}$$

We estimate the two terms on the right-hand side of (2.6) separately. First, we write, using the equation for $\mathbf{z}(x,t)$

$$\mathbf{z}(t) = \mathbf{u}_0 - D\int_0^t M^{-1}\mathbf{z}(\tau)d\tau$$

so that

$$Z(t) = U_0 - \int_0^t M^{-1}\mathbf{z}(\tau)d\tau = U_0 - \int_0^t \mathbf{W}(\tau)M^{-1}\mathbf{u}_0 d\tau$$

with $U_0(x) = \int_{-\infty}^x \mathbf{u}_0(y)dy$. Using (2.5), we have

$$|Z(x,t)| \leq |\mathbf{u}_0|_{L^1\times L^1} + c\int_0^t (1+\tau)^{-\theta}d\tau(|M^{-1}\mathbf{u}_0|_{L1\times L^1} + \|M^{-1}\mathbf{u}_0\|_{X^1}) \leq$$

$$c(1+t)^{\eta}(|\mathbf{u}_0|_{L^1\times L^1} + \|M^{-1}\mathbf{u}_0\|_{X^1})$$

where $\eta = 1-\theta$. Noting that $\|M^{-1}\mathbf{u}_0\|_{X^1} \leq c\|\mathbf{u}_0\|_1$, then

$$|Z(x,t)| \leq c(1+t)^{\eta}(|\mathbf{u}_0|_{L^1\times L^1} + \|\mathbf{u}_0\|_{X^1}). \tag{2.7}$$

Let $P(x,t)$ denote the second term on the right-hand side of Eq. (2.6), then by (2.5)

$$|P(x,t)| \leq \int_0^t |\mathbf{W}(t-\tau)M^{-1}G(\mathbf{u})|d\tau \leq$$

$$c\int_0^t (1+t-\tau)^{-\theta}d\tau(|M^{-1}G(\mathbf{u})|_{L^1\times L^1} + \|M^{-1}G(\mathbf{u})\|_{X^1}).$$

Since $H^1 \subset L^\infty$, then $|M^{-1}G(\mathbf{u}(\tau))|_{L^1\times L^1}$ is bounded uniformly in τ by a constant which depends only on $\mathbf{u}_0$.

Next observe that

$$\|M^{-1}G(\mathbf{u}(\tau))\|_{X^1} \leq \left|\frac{G(\mathbf{u})}{\mathbf{u}}\right|_\infty \|\mathbf{u}\|_{X^1}.$$

Thus

$$|P(x,t)| \leq c\int_0^t (1+t-\tau)^{-\theta}d\tau \leq c(1+t)^{\eta}$$

where again $\eta = 1-\theta = \dfrac{3}{4}$.

3. Solitary waves. We consider a smooth solution of (1.1) that vanishes at infinity of the form $(u(x,t), v(x,t)) = (\varphi(x-ct), \psi(x-ct)) = \Phi_c$. Substituting Φ_c in (1.1) and assuming that $\varphi, \psi, \varphi'', \psi'' \to 0$ as $|\xi| \to \infty$ we obtain

$$\left\{ \begin{array}{l} -c\varphi + c\varphi'' + f(\varphi,\psi) = 0 \\ -c\psi + c\psi'' + g(\varphi,\psi) = 0 \end{array} \right. \tag{3.1}$$

As we will see below, the system (3.1) has an explicit smooth solution of the form $\Phi_c = (\phi_c, \phi_c)$.

We observe from (3.1) that if E', Q' represent the Frechet derivatives of E, Q then

$$E'(\varphi_c, \psi_c) + cQ'(\varphi_c, \psi_c) = 0 \tag{3.2}$$

Moreover, if H_c is the linearized operator of $E' + cQ'$ around Φ_c, namely

$$H_c = E''(\Phi_c) + cQ''(\Phi_c) = \tag{3.3}$$

$$\left(\begin{array}{cc} -c\partial_x^2 + c - F_{u^2}(\Phi_c) & -F_{uv}(\Phi_c) \\ -F_{uv}(\Phi_c) & -c\partial_x^2 + c - F_{v^2}(\Phi_c) \end{array} \right)$$

then $H_c(\partial_x\varphi_c, \partial_x\psi_c) = 0$.
We now establish some assumptions on Φ_c and H_c necessary for the problem of stability and instability. They are as follows.

Assumption 1 (Existence of solitary waves). There is an interval $(c_1, c_2) \subset \mathbb{R}$ such that for every $c \in (c_1, c_2)$ there exists a solution Φ_c of (3.2) in X^3. The curve $c \to \Phi_c$ is C^1 with values in X^2. Moreover, $(1+|\xi|)^{\frac{1}{2}}\dfrac{d\Phi_c}{dc}$, Φ_c, $M\Phi_c \in L^1(\mathbb{R}) \times L^1(\mathbb{R})$.

Assumption 2 (Spectrum of the linearized operator). The zero eigenvalue of the operator H_c (with eigenfunction $(\partial_x\varphi_c, \partial_x\psi_c)$, $\varphi_c > 0$, $\psi_c > 0$) is simple. H_c has a unique negative simple eigenvalue with eigenfunction χ_c. Besides the negative and zero eigenvalue, the rest of the spectrum of H_c is positive and bounded away from zero. Moreover, the mapping $c \to \chi_c$ is continuous with values in X^3 and $(1+|\xi|)^{\frac{1}{2}}\chi_c$ is $L^1 \times L^1$, $\chi_{1c} > 0$, $\chi_{2c} > 0$.

Denote $L_1 = -c\partial_x^2 + c - (F_{u^2} + F_{uv})$, $L_2 = -c\partial_x^2 + c + (F_{uv} - F_{u^2})$

Lemma 3.1. *Assume* $F_{uv} = F_{vu}$, $F_{u^2} = F_{v^2}$, $c + F_{uv} - F_{u^2} > 0$ *and* $dimKerL_1 = 1$ *and* $h(x)$ *with an unique zero* ($h(x) = 0 \Leftrightarrow x = 0$)*, where* $h \in KerL_1$*. Then* H_c *has a unique negative simple eigenvalue, zero is the simple eigenvalue.*

P r o o f. Suppose that $H_c(f,g) = 0$, for $(f,g) \in X^1$, then from (3.3) we obtain

$$\begin{cases} -c\partial_x^2 f + cf - F_{u^2} f - F_{uv} g = 0 \\ -c\partial_x^2 g + cg - F_{v^2} g - F_{uv} f = 0 \end{cases} \tag{3.4}$$

Then

$$\begin{cases} -c\partial_x^2(f+g) + c(f+g) - (F_{u^2} + F_{uv})(f+g) = 0 \\ -c\partial_x^2(f-g) + c(f-g) + (F_{uv} - F_{u^2})(f-g) = 0 \end{cases} \tag{3.5}$$

Thus, since $(f+g) \to 0, \quad |x| \to \infty$ and the unique solution of $L_2 h_1(x) = 0$ is $h_1 \equiv 0$, we obtain $f = g$ and $(f,g) = \frac{k}{2}(h,h)$.

Now let $H_c(f,g) = \lambda(f,g)$, for $\lambda < 0$. Then

$$\begin{cases} -c\partial_x^2(f+g) + c(f+g) - (F_{u^2} + F_{uv})(f+g) = \lambda(f+g) \\ -c\partial_x^2(f-g) + c(f-g) + (F_{uv} - F_{u^2})(f-g) = \lambda(f-g) \end{cases} \tag{3.6}$$

Since $h(x)$ has a unique zero, by Sturm-Lioville theory, we see that L_1 a unique negative simple eigenvalue β, with eigenfunction χ_c. Moreover, it follows from equation $L_1\chi_c = \beta\chi_c$ and a simple bootstrap argument, that $\chi_c \in H^\infty(\mathbb{R})$. Hence from Eq.(3.6) we obtain that $\lambda = \beta, \quad f + g = k_1\chi_c$ and $f = g$ and $H_c(\chi_c, \chi_c) = \lambda(\chi_c, \chi_c)$.
Finally, using the fact that $\inf\{\gamma|\gamma \in \sigma(L_1) \setminus \{0, \beta\}\} > \eta > 0$ we conclude the proof of the Lemma.

Lemma 3.2. *Let $f(u,v) = u + u^p v^{p+1}, \;\; g(u,v) = v + u^{p+1}v^p$. Assumptions* 1 *and* 2 *are true for the case of $\Phi_c = (\phi_c, \phi_c)$, $c > 1$.*

P r o o f. Substituting $\Phi_c = (\phi_c, \phi_c)$ in Eq. (3.1), we obtain the nonlinear elliptic equation

$$-c\phi_c'' + (c-1)\phi_c - \phi_c^{2p+1} = 0. \tag{3.7}$$

If $c > 1$, there exists a unique (up to translation) solution ϕ_c of (3.7) which is given explicitly by

$$\phi_c(\xi) = (p+1)^{\frac{1}{2p}}(c-1)^{\frac{1}{2p}} \sec h^{\frac{1}{p}}\left(p(\frac{c-1}{c})^{\frac{1}{2}}\xi\right). \tag{3.8}$$

Thus Assumption 1 is true. □

We have $F(u,v) = \dfrac{u^2+v^2}{2} + \dfrac{u^{p+1}v^{p+1}}{p+1}, \;\; F_{u^2} = 1 + pu^{p-1}v^{p+1}, \;\; F_{v^2} = 1 + pu^{p+1}v^{p-1}, \;\; F_{uv} = (p+1)u^p v^p, \;\; L_1 = -c\partial_x^2 + (c-1) - (2p+1)\phi_c^{2p}$. From

(3.7) and (3.8), $\partial_x \phi_c \in \operatorname{Ker} L_1$, $\partial_x \phi_c = 0 \Leftrightarrow x = 0$ and $\dim \operatorname{Ker} L_1 = 1$. Thus from Lemma 3.1 Assumption 2 is true.

Lemma 3.3. *Let $f(u,v) = u^p v^{p+1}$, $g(u,v) = u^{p+1} v^p$. Assumptions 1 and 2 are true for the case of $\Phi_c = (\phi_c, \phi_c)$, $c > 0$.*

P r o o f. Substituting Φ_c in Eq. (3.1), we obtain

$$-c\phi_c'' + c\phi_c - \phi_c^{2p+1} = 0. \tag{3.9}$$

For $c > 0$, there exists a unique solution ϕ_c of (3.9) and as in Lemma 3.2, we obtain that Assumptions 1 and 2 are true. □

4. Orbital stability and instability. We begin by specifying the precise form in which the stability and instability is to be interpreted. Denoting by τ_s, $s \in \mathbb{R}$, the translation operator $\tau_s f = f(x+s)$, for all $x \in \mathbb{R}$, we define $T(s)\mathbf{f} = (\tau_s f, \tau_s g)$ for $\mathbf{f} = (f, g)$. Thus, for $\varepsilon > 0$ consider the tabular neighborhood

$$U_\varepsilon = \{\mathbf{g} \in X^1 \mid \inf_{s\in\mathbf{R}} \|\mathbf{g} - T(s)\Phi\|_{X^1} < \varepsilon\}$$

Definition 4.1. *The solitary wave Φ_c is X^1-stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathbf{u}_0 \in U_\delta$, then (1.1) has a unique solution $\mathbf{u}(t) \in C([0,\infty); X^1)$, with $\mathbf{u}(0) = \mathbf{u}_0$, such that $\mathbf{u}(t) \in U_\varepsilon$ for all $t \in \mathbb{R}$.*
The solitary wave Φ_c is X^1-unstable if Φ_c is not X^1-stable.
Denote

$$d(c) = E(\Phi_c) + cQ(\Phi_c)$$

after differentiation with respect to c, we have

$$d'(c) = E'(\Phi_c) + cQ'(\Phi_c) + Q(\Phi_c) = Q(\Phi_c) \tag{4.1}$$

$$d''(c) = \left\langle Q'(\Phi_c), \frac{d\Phi_c}{dc} \right\rangle. \tag{4.2}$$

For instability we will need a series of preliminary results which are similar those used by Souganidis and Strauss [16].

Lemma 4.1. *There is $\varepsilon > 0$ and a unique C^1 map $\alpha : U_\varepsilon \to \mathbb{R}$, such that, for $\mathbf{u} \in U_\varepsilon$ and $r \in \mathbb{R}$*

$$\langle \mathbf{u}(\cdot + \alpha(\mathbf{u})), D\Phi_c \rangle = 0\,, \quad \alpha(\Phi_c) = 0$$

$$\mathbf{u}(\cdot + r) = \alpha(\mathbf{u}) - r$$

$$\alpha'(\mathbf{u}) = \frac{D\Phi_c(\cdot - \alpha(\mathbf{u}))}{\langle \mathbf{u}, M\Phi_c(\cdot - \alpha(\mathbf{u})) \rangle}.$$

P r o o f. Consider the functional

$$F(\mathbf{u},\alpha)=\int_{-\infty}^{\infty}[u(x+\alpha)\partial_x\varphi_c(x)+v(x+\alpha)\partial_x\psi_c(x)]dx$$

defined on pairs $\mathbf{u}=(u,v)\in L^2(\mathbb{R})\times L^2(\mathbb{R})$ and $\alpha\in\mathbf{R}$. Since $F(\Phi_c,0)=0$ and $\frac{\partial}{\partial\alpha}F(\Phi_c,0)=\|\partial_x\Phi_c\|^2_{X^1}\neq 0$, the implicit function theorem can be applied to obtain the lemma. □

Theorem 4.1. *Let $c>0$ be fixed. If $d''(c)<0$, there is a curve $w\to\Psi_w$ which satisfies $Q(\Phi_c)=Q(\Psi_w)$, $\Phi_c=\Psi_c$ and on which $E(\mathbf{u})$ has a strict local maximum at $\mathbf{u}=\Phi_c$.*

P r o o f. Following the ideas of Souganidis and Strauss [16] we note that for $G(w,s)=Q(\Phi_w+s\chi_c)$, $G(c,0)=Q(\Phi_c)$ and

$$\frac{\partial}{\partial_s}Q(\Phi_w+s\chi_c)(c,0)=\langle Q'(\Phi_c),\chi_c\rangle=\langle M\Phi_c,\chi_c\rangle=$$

$$\frac{1}{c}\left[(f(\varphi_c,\psi_c),\chi_{1c})+(g(\varphi_c,\psi_c),\chi_{2c})\right]\neq 0.$$

Therefore, it follows from the implicit function theorem that there is C^1 function $s(w)$ for w near c, such that $s(c)=0$ and $G(w,s(w))=Q(\Phi_c)$ for w near c. Next we define $\Psi_w=\Phi_w+s(w)\chi_c$. It is easily seen that $\frac{d}{dw}E(\Psi_w)_{|w=c}=0$ and

$$\frac{d^2}{dw^2}E(\Psi_w)_{|w=c}=\langle H_c\mathbf{y},\mathbf{y}\rangle$$

where $\mathbf{y}=\frac{d\Psi_w}{dw}_{|w=c}=\frac{d}{dc}\Phi_c+s'(c)\chi_c$. So it suffices to show that $\langle H_c\mathbf{y},\mathbf{y}\rangle<0$. We have

$$0=\frac{d}{dw}Q(\Psi_w)_{|w=c}=\left\langle Q'(\Phi_c),\frac{d}{dw}\Psi_{w|w=c}\right\rangle=$$

$$\langle M\Phi_c,\mathbf{y}\rangle=\left\langle M\Phi_c,\frac{d\Phi_c}{dc}\rangle+s'(c)\langle M\Phi_c,\chi_c\right\rangle.$$

From (4.2), $d''(c)=\left\langle Q'(\Phi_c),\frac{d\Phi_c}{dc}\right\rangle=-s'(c)\langle M\Phi_c,\chi_c\rangle$. But also $H_c\mathbf{y}=H_c\frac{d\Phi_c}{dc}+s'(c)H_c\chi_c=-M\Phi_c+s'(c)H_c\chi_c$.
Therefore

$$\langle H_c\mathbf{y},\mathbf{y}\rangle=s'(c)\langle H_c\chi_c,\mathbf{y}\rangle-\langle M\Phi_c,\mathbf{y}\rangle=$$

$$-s'(c)\langle M\Phi_c, \chi_c\rangle + s^{\partial 2}(c)\langle H_c\chi_c, \chi_c\rangle = d''(c) + s^{\partial 2}(c)\langle H_c\chi_c, \chi_c\rangle < 0$$

This proves the theorem. □

Next we define an auxiliary operator B which will play a critical role in the proof of instability. It follows from the calculation given in the proof of Theorem 4.1 that $\langle H_c\mathbf{y}, \mathbf{y}\rangle < 0$ and $\langle M\Phi_c, \mathbf{y}\rangle = 0$.

Definition 4.2. *For $\mathbf{u} \in U_\varepsilon$, define $B(\mathbf{u})$ by the formula*

$$B(\mathbf{u}) = \mathbf{y}(\cdot - \alpha(\mathbf{u})) - \frac{\langle M\mathbf{u}, \mathbf{y}(\cdot - \alpha(\mathbf{u}))\rangle M^{-1}\partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u}))}{\langle \mathbf{u}, \partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u}))\rangle}. \tag{4.3}$$

The next lemma summarizes the properties of B.

Lemma 4.2. *B is a C^1 function from U_ε into X^1. Moreover, B commutes with translations, $B(\Phi_c) = \mathbf{y}$ and $\langle B(\mathbf{u}, M\mathbf{u}\rangle = 0$ for every $\mathbf{u} \in U_\varepsilon$.*

Consider now the equation (determining a curve Γ in U_ε)

$$\frac{d\mathbf{u}_\lambda}{d\lambda} = B(\mathbf{u}_\lambda), \quad \mathbf{u}(0) = \mathbf{v}. \tag{4.4}$$

We denote $\mathbf{u}(\lambda, x) = \rho(\lambda, \mathbf{v})$ the solution of (4.4), which exists in a neighborhood of $\lambda = 0$, since B is of class C^1. Further, we have $Q(\mathbf{u}(\lambda, x)) = Q(\mathbf{v})$, since

$$\frac{dQ(\mathbf{u}(\lambda, x))}{d\lambda}_{|\lambda=0} = \langle Q'(\mathbf{u}), \frac{d\mathbf{u}}{d\lambda}\rangle = \langle M\mathbf{u}, B\mathbf{u}\rangle = 0$$

and also

$$\frac{d\rho(\lambda, \Phi_c)}{d\lambda} = B(\Phi_c) = \mathbf{y}.$$

Lemma 4.3. *There is a C^1 functional $\Lambda : D_\varepsilon \to \mathbb{R}$, where $D_\varepsilon = \{\mathbf{v} \in U_\varepsilon | Q(\mathbf{v}) = Q(\Phi_c)\}$, such that if $\mathbf{v} \in D_\varepsilon$ and $\mathbf{v}$ is not a translate of Φ_c, then*

$$E(\Phi_c) < E(\mathbf{v}) + \Lambda(\mathbf{v})\langle E'(\mathbf{v}), B(\mathbf{v})\rangle. \tag{4.5}$$

Lemma 4.4. *The curve $w \to \Psi_w$ constructed in Theorem 4.1 satisfies $E(\Psi_w) < E(\Phi_c)$ for $w \neq c$, $Q(\Psi_w) = Q(\Phi_c)$ and $\langle E'(\Psi_w), B(\Psi_w)\rangle$ changes sign as w passes through c.*

Theorem 4.2. *Assume Assumptions 1 and 2. If $d''(c) < 0$, then the solitary wave Φ_c is X^1-unstable.*

P r o o f. Let $\varepsilon > 0$ be given small enough such that Lemma 4.1 and its consequences apply with U_ε. To prove instability of Φ_c, it suffices to show that there are some elements $\mathbf{u}_0 \in X^1$ which are arbitrary close to Φ_c but for which the solution $\mathbf{u}$ of Eq. (1.1) with initial data $\mathbf{u}_0$ exists from U_ε in finite time. By Lemma 4.4 we can find $\mathbf{u}_0 \in X^3 \subset X^1$, near to Φ_c and which satisfies $Q(\mathbf{u}_0) = Q(\Phi_c)$, $E(\mathbf{u}_0) < E(\Phi_c)$ and $\langle E'(\mathbf{u}_0), B(\mathbf{u}_0)\rangle > 0$. For a fixed $\mathbf{u}_0$, let $[0, t_1)$ denote the maximal interval for which $\mathbf{u}(\cdot, t)$ lies continuously in U_ε. Let T be the maximum existence time of the solution $\mathbf{u}$ in Eq. 1.1. If T is finite, then we have the X^1-instability for Φ_c by definition. So we may assume that $T = \infty$ and it suffices to show that $t_1 < \infty$.

In view of Theorems 2.1 and 2.2 $\mathbf{u}$ has the following properties

$$\mathbf{u} \in C([0, t_1); X^1), \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x)$$

$$\sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{x} \mathbf{u}(z, t)dz \right| \le c_0(1 + t^{\frac{3}{4}}), \quad t \in [0, t_1)$$

$$\sup_{t\in[0,t_1)} \|\mathbf{u}(t)\|_{X^1} \le c_1$$

where c_1 depends only on Φ_c and ε, and c_0 depends only on c_1 and $\|\mathbf{u}_0\|_{L^1\times L^1} + \|\mathbf{u}_0\|_{X^1}$.

Let $\beta(t) = \alpha(\mathbf{u}(t)), \mathbf{Y}(x) = \int_{-\infty}^{x} M\mathbf{y}(\rho)d\rho = \int_{-\infty}^{x} \mathbf{y}(\rho)d\rho + L\mathbf{y}(x)$, where $L = \begin{pmatrix} -i|\xi| & 0 \\ 0 & -i|\xi| \end{pmatrix}$ and define

$$A(t) = \int_{-\infty}^{\infty} \mathbf{Y}(x - \beta(t))\mathbf{u}(x, t)dx, \quad 0 \le t < t_1. \tag{4.6}$$

The next step is to show that the integral in Eq. (4.6) converges. Indeed, if $\mathcal{H}$ is the Heaviside function and $\gamma = \displaystyle\int_{-\infty}^{\infty} \mathbf{u}_0(x)dx$ (we note that by Assumptions 1 and 2 $\displaystyle\int_{-\infty}^{\infty} (1 + |x|)^{\frac{1}{2}}|\mathbf{y}(x)|dx < \infty$ and function $R(x) = \displaystyle\int_{-\infty}^{x} \mathbf{y}(\rho)d\rho - \gamma\mathcal{H}(x)$ belong $L^2 \times L^2$), we obtain from Eq. (4.6) that

$$A(t) = \int_{-\infty}^{\infty} R(x - \beta(t))\mathbf{u}(x, t)dx + \gamma \int_{\beta(t)}^{\infty} \mathbf{u}(x, t)dx +$$

$$\int_{-\infty}^{\infty} L\mathbf{y}(x - \beta(t))\mathbf{u}(x, t)dx.$$

Hence

$$|A(t) \leq |R|_2 \|\mathbf{u}\|_{X^1} + c_0(1 + t^{\frac{3}{4}}) + \|L\mathbf{u}\|_{X^1}\|\mathbf{u}\|_{X^1} \leq c_2(1 + t^{\frac{3}{4}}). \tag{4.7}$$

Now

$$\frac{dA}{dt} = -\langle \alpha'(\mathbf{u}), \frac{d\mathbf{u}}{dt}\rangle\langle M\mathbf{y}, \mathbf{u}\rangle + \langle \mathbf{Y}(\cdot - \beta), \frac{d\mathbf{u}}{dt}\rangle =$$
$$\langle -\langle \mathbf{y}(\cdot - \beta), M\mathbf{u}\rangle\alpha'(\mathbf{u}) + \mathbf{Y}(\cdot - \beta), \frac{d\mathbf{u}}{dt}\rangle =$$
$$\langle -\langle \mathbf{y}(\cdot - \beta), M\mathbf{u}\rangle\alpha'(\mathbf{u}) + \mathbf{Y}(\cdot - \beta), M^{-1}\partial_x \mathbf{f}\rangle =$$
$$\langle -\langle \mathbf{y}(\cdot - \beta), M\mathbf{u}\rangle\partial_x M^{-1}\alpha'(\mathbf{u}) + \mathbf{y}(\cdot - \beta), \mathbf{f}(\mathbf{u})\rangle =$$
$$= -\langle B(\mathbf{u}), E'(\mathbf{u})\rangle.$$

Because $0 < E(\Phi_c) - E(\mathbf{u}_0) = E(\Phi_c) - E(\mathbf{u})$, Lemma 4.3 implies that $0 < \Lambda(\mathbf{u}(t))\langle E'(\mathbf{u}(t)), B(\mathbf{u}(t))\rangle$. Moreover, since $\mathbf{u}(t) \in U_\varepsilon$ and $\Lambda(\Phi_c) = 0$, we may assume that $\Lambda(\mathbf{u}(t)) < 1$ by choosing ε small if necessary. Therefore for all $t \in [0, t_1)$, $\langle E'(\mathbf{u}(t)), B(\mathbf{u}(t))\rangle > E(\Phi_c) - E(\mathbf{u}_0) > 0$. Hence for $0 \leq t < t_1$

$$-\frac{dA}{dt} \geq E(\Phi_c) - E(\mathbf{u}_0) > 0.$$

Comparing this with (4.7), we conclude that $t_1 < \infty$.

Now we will consider the stability theory. The stability of solitary wave solutions of (1.1) is an immediate consequence of the fact that $d''(c) > 0$ implies that Φ_c is a local minimum of E subject to the constancy of Q. This is a general fact, not special to the equations under consideration in this paper.

Lemma 4.5. *Let $d''(c) > 0$. If $\langle M\Phi_c, \mathbf{y}_1\rangle = \langle \partial_x\Phi_c, \mathbf{y}_1\rangle = 0$, then there is $K > 0$ such that $\langle H_c\mathbf{y}_1, \mathbf{y}_1\rangle > K\|\mathbf{y}_1\|^2$.*

Proof. Putting $\mathbf{y}_1 = a_1\chi_c + \mathbf{p}_1$, $a_1 = (\mathbf{y}_1, \chi_c)$, we have

$$\langle H_c\mathbf{y}_1, \mathbf{y}_1\rangle = -a_1^2\lambda_0^2 + \langle H_c\mathbf{p}_1, \mathbf{p}_1\rangle \tag{4.8}$$

where $-\lambda_0^2$ is a negative eigenvalue of H_c. Write $\frac{d\Phi_c}{dc} = a_0\chi_c + b_0\partial_c\Phi_c + \mathbf{p}_0$, where $\mathbf{p}_0$ is in the positive subspace of H_c. Therefore

$$\left\langle H_c\frac{d\Phi_c}{dc}, \frac{d\Phi_c}{dc}\right\rangle = -\lambda_0^2 a_0^2 + \langle H_c\mathbf{p}_0, \mathbf{p}_0\rangle \tag{4.9}$$

and

$$\langle H_c\mathbf{p}_0, \mathbf{p}_0\rangle = -d''(c) + \lambda_0^2 a_0^2.$$

We have

$$(4.10)\qquad 0=-(M\Phi_c,\mathbf{y}_1)=\left\langle H_c\frac{d\Phi_c}{dc},\mathbf{y}_1\right\rangle=-a_0a_1\lambda_0^2+\langle H_c\mathbf{p}_0,\mathbf{p}_1\rangle.$$

In combination with (4.8)-(4.10) this implies

$$(4.11)\qquad \langle H_c\mathbf{y}_1,\mathbf{y}_1\rangle\ge -a_1^2\lambda_0^2+\frac{\langle H_c\mathbf{p}_0,\mathbf{p}_1\rangle^2}{\langle H_c\mathbf{p}_0,\mathbf{p}_0\rangle}=-a_1^2\lambda_0^2+\frac{a_0^2a_1^2\lambda_0^4}{-d''(c)+a_0^2\lambda_0^2}=$$

$$=\frac{d''(c)a_1^2\lambda_0^2}{\langle H_c\mathbf{p}_0,\mathbf{p}_0\rangle}=a_1^2K>0.$$

Denote by $\mathcal{N}$ the set of all $\mathbf{y}_1$, which satisfy, the conditions and $\|\mathbf{y}_1\|=1$. We claim that there exists a constant $K_1>0$ such that

$$(4.12)\qquad \langle H_c\mathbf{y}_1,\mathbf{y}_1\rangle\ge K_1,\quad \mathbf{y}_1\in\mathcal{N}$$

To verify the claim, let us assume the contrary: there exists a sequence $\mathbf{y}_n\in\mathcal{N},\ \mathbf{y}_n=a_n\chi_c+\mathbf{p}_n$ for which

$$(4.13)\qquad \langle H_c\mathbf{y}_n,\mathbf{y}_n\rangle=-a_n^2\lambda_0^2+\langle H_c\mathbf{p}_n,\mathbf{p}_n\rangle\to 0.$$

Then (4.11) implies $0<a_n^2K=-a_n^2\lambda_0^2+\langle H_c\mathbf{p}_n,\mathbf{p}_n\rangle\to 0$. Using (4.13) we conclude that also $\langle H_c\mathbf{p}_n,\mathbf{p}_n\rangle\to 0$, which is equivalent to $\mathbf{p}_n\to 0$, which contradicts $\|\mathbf{p}_n\|\to 1$.

Lemma 4.6. *Let $d''(c)>0$. Then there is positive constants K and ε, such that for every $\mathbf{u}\in U_\varepsilon$ with $Q(\mathbf{u})=Q(\Phi_c)$*

$$(4.14)\qquad E(\mathbf{u})-E(\Phi_c)\ge K\|\mathbf{u}(\cdot-\alpha(\mathbf{u}))-\Phi_c\|_{X^1}^2$$

where $\alpha(\mathbf{u})$ is defined by Lemma 4.1.

Proof. Denote $\Psi=\mathbf{u}(x+\alpha(\mathbf{u}))-\Phi_c(x)=\mu M\Phi_c+\mathbf{y}$, where $(M\Phi_c,\mathbf{y})=0$. One has

$$Q(\Phi_c)=Q(\mathbf{u})=Q(\mathbf{u}(x+\alpha(\mathbf{u})))=Q(\Psi_c+\Phi_c)=$$
$$Q(\Phi_c)+\langle Q'(\Phi_c),\Psi_c\rangle+O(\|\Psi\|^2)=Q(\Phi_c)+\mu\langle Q'(\Phi_c),M\Phi_c\rangle+$$
$$\langle Q'(\Phi_c),\mathbf{y}\rangle+O(\|\Psi\|^2)=Q(\Phi_c)+\mu\|M\Phi_c\|^2+O(\|\Psi\|^2).$$

This implies

$$(4.15)\qquad \mu=O(\|\Psi\|^2).$$

Let $L(\mathbf{u}) = E(\mathbf{u}) + cQ(\mathbf{u})$. Another Taylor expansion gives

$$L(\mathbf{u}) = L(\mathbf{u}(\cdot + \alpha(\mathbf{u}))) = L(\Phi_c) + \langle L'(\Phi_c), \Psi\rangle + \frac{1}{2}\langle L''(\Phi_c)\Psi, \Psi\rangle + o(\|\Psi\|^2_{X^1}).$$

Since $Q(\mathbf{u}) = Q(\Phi_c)$, $L'(\Phi_c) = 0$ and $L''(\Phi_c) = H_c$, this can be written as

$$E(\mathbf{u}) - E(\Phi_c) = \frac{1}{2}\langle H_c\Psi, \Psi\rangle + O(\|\Psi\|^3_{X^1}). \tag{4.16}$$

Applying (4.15), we obtain

$$\langle H_c\Psi, \Psi\rangle = \langle H_c\mathbf{y}, \mathbf{y}\rangle + O(\|\Psi\|^3). \tag{4.17}$$

Now write $\mathbf{y}$ as $\mathbf{y} = \mu_1\Phi_c + \Omega$, $(\partial\Phi, \Omega) = 0$. Then

$$\langle H_c\mathbf{y}, \mathbf{y}\rangle = \langle H_c\Omega, \Omega\rangle.$$

Applying Lemma 4.5, we obtain

$$\langle H_c\mathbf{y}, \mathbf{y}\rangle = \langle H_c\Omega, \Omega\rangle \geq \|\Omega\|^2.$$

By Lemma 4.1 $0 = \langle\Psi, \partial\Phi\rangle = \mu_1\langle\partial_x\Phi, \partial_x\Phi\rangle + \langle\Omega, \partial_x\Phi\rangle$ and $|\mu_1| \leq \|\Omega\|^2$. Therefore $\|\mathbf{y}\|^2 \leq m\|\Omega\|^2$, where $m = 1 + \|\partial_x\Phi\|$. Then

$$\langle H_c\mathbf{y}, \mathbf{y}\rangle \geq K_1\|\mathbf{y}\|.$$

This estimate in combination with

$$\|\mathbf{y}\| = \|\Psi - \mu M\Phi_c\| \geq \|\Psi\| - |\mu|.\|M\Phi_c\| = \|\Psi\| + O(\|\Psi\|^2)$$

and (4.17) gives the inequality

$$\langle H_c\Psi, \Psi\rangle \geq K_1\|\Psi\|^2 + O(\|\Psi\|^3). \tag{4.18}$$

Now we estimate directly $\langle H_c\Psi, \Psi\rangle$ from bellow. One has

$$\langle H_c\Psi, \Psi\rangle = c_0\|\Psi\|^2_{X^1} - c_1\|\Psi\|^2. \tag{4.19}$$

Combining (4.16), (4.18) and (4.19), we have

$$E(\mathbf{u}) - E(\Phi_c) \geq K\|\Psi\|^2_{X^1} - O(\|\Psi\|^3_{X^1}).$$

Therefore for $\|\Psi\|_{X^1}$ sufficiently small (which can be ensured by choosing ε small enough), the last inequality implies (4.14). The proof is complete.

Theorem 4.3. *Let $d''(c)>0$. Then Φ_c is X^1-stable.*

P r o o f. If Φ_c is unstable, there exists a sequence of initial data $\mathbf{u}_n(0)$ and $\varepsilon>0$ such that $\inf_{s\in\mathbb{R}}\|\mathbf{u}_n(0)-\Phi_c(\cdot+s)\|_{X^1}\to 0$ as $n\to\infty$ but

$$\sup_{t>0}\inf_{s\in\mathbb{R}}\|\mathbf{u}_n(t)-\Phi_c(\cdot+s)\|_{X^1}\geq\varepsilon$$

where $\mathbf{u}_n(t)$ is the unique solution of (1.1) with initial data $\mathbf{u}_n(0)$. By continuity in t, we can pick the first time t_n so that

$$\inf_{s\in\mathbb{R}}\|\mathbf{u}_n(t_n)-\Phi_c(\cdot+s)\|_{X^1}=\varepsilon \tag{4.20}$$

because E and Q are continuous on X^1 and translation invariant,

$$E(\mathbf{u}_n(\cdot,t_n))=E(\mathbf{u}_n(0))\to E(\Phi_c),$$

$$Q(\mathbf{u}_n(\cdot,t_n))=Q(\mathbf{u}_n(0))\to Q(\Phi_c).$$

Next choose $\mathbf{w}_n\in U_\varepsilon$ so that $Q(\mathbf{w}_n)=Q(\Phi_c)$ and $\|\mathbf{w}_n-\mathbf{u}_n(\cdot,t_n)\|_{X^1}\to 0$ By Lemma 4.6

$$0\leftarrow E(\mathbf{w}_n)-E(\Phi_c)\geq\|\mathbf{w}_n(\cdot+\alpha(\mathbf{w}_n))-\Phi_c\|^2_{X^1}.$$

Hence $\|\mathbf{u}_n(t_n)-\Phi_c(\cdot-\alpha(\mathbf{w}_n))\|_{X^1}\to 0$, which contradicts (4.20). This means that the orbit Φ_c is stable, thus Theorem 4.3 is established. □

Theorem 4.4. *Let $f(u,v)=u+u^pv^{p+1}$, $g(u,v)=v+u^{p+1}v^p$, $p\geq 1$, $\Phi_c=(\phi_c,\phi_c)$ and $c>1$*

(i) *if $p\leq 2$, then the solitary wave Φ_c is X^1-stable for all $c>1$*

(ii) *if $p>2$, there is $c_0>1$, such that the solitary wave Φ_c is X^1-stable for $c>c_0$ and unstable for $1<c<c_0$.*

P r o o f. By Lemma 3.2 and Theorems 4.2 and 4.3 it is sufficient to compute $d''(c)$. In fact from (3.8) and (4.1) and using that $\displaystyle\int sech^r\xi d\xi=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{r}{2})}{\Gamma(\frac{r+1}{2})}$ and $\Gamma(a+1)=a\Gamma(a)$, we obtain

$$d'(c)=\int_{-\infty}^{\infty}\phi^2+\phi^{\partial 2}dx=\alpha^2(\frac{1}{\beta}+\frac{\beta}{p^2})\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p}+\frac{1}{2})}-\frac{\alpha^2\beta}{p^2}\frac{1}{p+2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p}+\frac{1}{2})}=$$

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p}+\frac{1}{2})}\left[\frac{\alpha^2}{\beta}+\frac{\alpha^2\beta}{p^2}\frac{p+1}{p+2}\right]$$

where $\alpha(c) = (p+1)^{\frac{1}{2p}}(c-1)^{\frac{1}{2p}}$, $\beta(c) = p\left(\dfrac{c-1}{c}\right)^{\frac{1}{2}}$

Hence

$$d''(c) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{p})}{\Gamma(\frac{1}{p}+\frac{1}{2})}\frac{1}{2p}\left[(p^2+4p+6)c^2 - (2p^2+3p+2)c - p\right].$$

Denote by $h(c) = (p^2+4p+6)c^2 - (2p^2+3p+2)c - p$, which is an increasing function of c. Since $h(1) = (2-p)(2+p)$, then if $p \le 2$, $d''(c) > 0$ for all $c > 1$. If $p > 2$, there is $c_0 > 1$, such that $d''(c) > 0$ for $c > c_0$ and $d''(c) < 0$ for $1 < c < c_0$. □

Theorem 4.5. *Let* $f(u,v) = u^p v^{p+1}$, $g(u,v) = u^{p+1}v^p$, $\Phi_c = (\phi_c, \phi_c)$, $c > 0$. *The* Φ_c *is* X^1*-stable for all* $p \ge 1$ *and* $c > 0$.

P r o o f. Let $\Phi_c = c^{\frac{1}{2p}}\Psi$. Then Ψ is independent of c and solve the equation

$$-M\Psi + \Psi^{2p+1} = 0.$$

But

$$d'(c) = Q(\Phi_c) = \frac{1}{2}\langle M\Phi_c\Phi_c\rangle = \frac{1}{2}c^{\frac{1}{p}}\langle M\Psi, \Psi\rangle$$

and

$$d''(c) = \frac{1}{2}\frac{1}{p}c^{\frac{1-p}{p}}\langle M\Psi, \Psi\rangle > 0. \qquad \square$$

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