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ON NONADAPTIVE SEARCH PROBLEM

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Abstract. We consider nonadaptive search problem for an unknown element x from the set $A = \{1, 2, 3, \ldots, 2^n\}$, $n \geq 3$. For fixed integer S the questions are of the form: Does x belong to a subset B of A, where the sum of the elements of B is equal to S? We wish to find all integers S for which nonadaptive search with n questions finds x. We continue our investigation from [4] and solve the last remaining case $n = 2^k$, $k \geq 2$.

1. Introduction. We start with the general description of a search problem. Given a set A and let $x \in A$ be an unknown element. We want to find x by asking questions whether x belongs to a subset B of A, such that B satisfies given conditions. By imposing different restrictions on B we obtain different search problems. Also, if every question is stated after the answer of the previous one we say that this is an *adaptive search* [5], [6]. In this case one can make use of the information given by the answers so far. If all questions are asked simultaneously we say that this is a *nonadaptive search* [1], [2], [7].

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Consider the following nonadaptive search for the unknown element x in the set $A = \{1, 2, 3, \ldots, 2^n\}$, $n \geq 3$. For a given natural number S we are allowed to ask whether x belongs to a subset B of A if the sum of the elements of B equals S (see [3]). In this case we say that B is a *question set of weight* S or, when S is clear from the context, just a question set. Since $|A| = 2^n$, the minimum number of question sets of weight S, needed to find the unknown element, is greater or equal to n.

Call a natural number S *good* if for some m there exists a collection $B_1, B_2, \ldots, B_m$ of question sets of weight S which determines x. If $m = n$, i.e. x can be found by n question sets of weight S, then S is called *proper*. It has been shown in [4] that S is good if and only if

$$S \in [2^n - 1; 2^{2n-1} - 2^{n-1} + 1]$$

and, when $n \neq 2^k$, then S is proper if and only if

$$S \in \left[2^{2n-2} + 2^{n-2} - \frac{1}{2}\binom{2n-1}{n-1}; 2^{2n-2} + 2^{n-2} + \frac{1}{2}\binom{2n-1}{n-1}\right].$$

In this paper we consider the nonadaptive search problem for question sets of weight S and find all proper numbers S for the case left $n = 2^k$, $k \geq 2$. For obtaining our results we use combined approach including knowledge from Algebra, Combinatorics and Coding Theory.

2. Preliminary results. We start with some notations (see [4]). We say that a vector $(v_1, v_2, \ldots, v_{2^n})$ is a *characteristic vector* for a subset B of A if $v_i = 1$ when $i \in B$ and $v_i = 0$ otherwise. It is clear that $\sum\limits_{y \in B} y = \sum\limits_{i=1}^{2^n} i.v_i$. The *Hamming weight* of the vector $V = (v_1, v_2, \ldots, v_n)$ is defined by $\mathrm{wt}(V) = |\{i | v_i \neq 0\}|$. A $n \times 2^n$ matrix G is called *characteristic matrix* for a collection $B_1, B_2, \ldots, B_n$ of subsets if the rows of G are the characteristic vectors of $B_1, B_2, \ldots, B_n$. The *weight* of a characteristic matrix G with vector columns $V_1, V_2, \ldots, V_{2^n}$ is defined by

$$\mathrm{wt}(G) = \frac{1}{n} \sum_{i=1}^{2^n} i.\mathrm{wt}(V_i).$$

Consider a collection $B_1, B_2, \ldots, B_n$ of question sets of weight S. By asking whether x belongs to B_i for $i = 1, 2, \ldots, n$ we obtain as answers a sequence of "yes" and "no" of length n. In order to find x, every element from A should get a unique sequence of "yes" and "no". Note also that if the vector V_i is the i-th column of the characteristic matrix for this collection, then the element i gets as answer the transpose of V_i (1 meaning "yes" and 0 meaning "no"). Therefore, if the unknown element can be found by the collection $B_1, B_2, \ldots, B_n$ then the columns of the corresponding characteristic matrix are all binary vectors of length n. Thus, our problem is equivalent to finding a binary $n \times 2^n$ matrix G having as columns all binary vectors of length n and the scalar product of every row of G with $(1, 2, 3, \ldots, 2^n)$ equals S. Call such a matrix *proper*. It is clear that if a matrix G with vector columns $V_1, V_2, \ldots, V_{2^n}$ is proper then $\mathrm{wt}(G) = S$.

Denote by $\overline{G}$ the matrix obtained from G by interchanging 0 and 1. It is easy to see that $\overline{G}$ is proper matrix and $\mathrm{wt}\left(\overline{G}\right) = 2^{2n-1} + 2^{n-1} - \mathrm{wt}(G)$.

To make this paper self-contained recall a theorem from [4].

Theorem 1. *If a natural number S is proper then*

$$S \in \left[2^{2n-2} + 2^{n-2} - \frac{1}{2}\binom{2n-1}{n-1}; 2^{2n-2} + 2^{n-2} + \frac{1}{2}\binom{2n-1}{n-1}\right].$$

P r o o f. Let S be a proper number and G be a proper matrix of weight $\mathrm{wt}(G) = S$. We show first that $S \geq 2^{2n-2} + 2^{n-2} - \frac{1}{2}\binom{2n-1}{n-1}$. Label the columns of G by $1, 2, \ldots, 2^n$ and denote by S_i, $i = 0, 1, \ldots, n$ the sum of the labels of the vector columns of G having weight i. Note that $n.S = n.\mathrm{wt}(G) = \sum_{i=0}^{n} i.S_i$. Further, since there are $\binom{n}{i}$ vector columns of weight i we obtain

$$S_n \geq 1, \qquad S_n + S_{n-1} \geq 1 + 2 + \cdots + \left(\binom{n}{n} + \binom{n}{n-1}\right),$$

$$S_n + S_{n-1} + S_{n-2} \geq 1 + 2 + \cdots + \left(\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2}\right)$$

and so on, up to

$$S_n + S_{n-1} + \cdots + S_1 \geq 1 + 2 + \cdots + \left(\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{1}\right).$$

Adding the above inequalities gives

$$\begin{aligned}\sum_{i=0}^{n} i.S_i \quad \geq \quad & 1+\frac{\left(\binom{n}{n}+\binom{n}{n-1}\right)\left(\binom{n}{n}+\binom{n}{n-1}+1\right)}{2} \\ & +\frac{\left(\binom{n}{n}+\binom{n}{n-1}+\binom{n}{n-2}\right)\left(\binom{n}{n}+\binom{n}{n-1}+\binom{n}{n-2}+1\right)}{2} \\ & +\cdots+\frac{\left(\binom{n}{n}+\binom{n}{n-1}+\cdots+\binom{n}{1}\right)\left(\binom{n}{n}+\binom{n}{n-1}+\cdots+\binom{n}{1}+1\right)}{2}.\end{aligned}$$

Simple calculations show that the latter inequality is equivalent to

$$\mathrm{wt}(G) \geq 2^{2n-2}+2^{n-2}-\frac{1}{2}\binom{2n-1}{n-1}.$$

Since $\mathrm{wt}(G) = S$ we get our assertion. To prove the inequality $S \leq 2^{2n-2} + 2^{n-2} + \frac{1}{2}\binom{2n-1}{n-1}$ recall that $\mathrm{wt}(G) = 2^{2n-1} + 2^{n-1} - \mathrm{wt}\left(\overline{G}\right)$ and use that $\mathrm{wt}\left(\overline{G}\right) \geq 2^{2n-2} + 2^{n-2} - \frac{1}{2}\binom{2n-1}{n-1}$. □

Remark 1. It is not difficult to prove that the term $\frac{1}{2}\binom{2n-1}{n-1}$ is an integer if and only if n is not a power of 2.

We continue with the notation concerning our results. Let $V = (v_1, v_2, \ldots, v_n)^t$ be a binary vector column of length n. Denote by π the cyclic shift of V by one position, i.e. $\pi(V) = (v_2, v_3, \ldots, v_n, v_1)^t$. It is well known that π partitions the set of all binary vectors of length n into orbits and the length of each orbit is a divisor of n. Also, the elements in the same orbit have equal weights. If the length of the orbit containing V where $\mathrm{wt}(V) = w$ equals l then call the matrix with columns $V, \pi(V), \pi^2(V), \ldots, \pi^{l-1}(V)$ an *orbit matrix of weight w and length l*. Denote such a matrix by $C_{w,l}$. It is easy to be seen that n divides lw and there are $\frac{lw}{n}$ ones in every row of $C_{w,l}$. Note also that $\overline{C_{w,l}}$ is an orbit matrix of weight $n - w$. If there are more than one orbit matrix of given weight w and length l we label them as $C^1_{w,l}, C^2_{w,l}, \ldots$ and so on.

Example 1. Let $n = 4$. There is a single orbit matrix for every weight

$w = 4, 3, 1$ and 0, namely

$$C_{4,1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ C_{3,4} = \begin{pmatrix} 0\,1\,1\,1 \\ 1\,0\,1\,1 \\ 1\,1\,0\,1 \\ 1\,1\,1\,0 \end{pmatrix}, \ C_{1,4} = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}, \text{ and } \ C_{0,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

There are two orbit matrices of weight 2, namely

$$C_{2,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } C_{2,4} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

For our further considerations it is appropriate to consider the matrix

$$C_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Note that C_2 is obtained by arranging in special order all vector columns of weight 2. The interval from Theorem 1 for $n = 2^2$ is $[50.5; 85.5]$. It turns out that for all $S \in [51; 85]$ there exists a proper matrix of weight S. Moreover, to construct a proper matrix of certain weight we make use of the matrices $C_{4,1}$, $C_{3,4}$, C_2, $C_{1,4}$ and $C_{0,1}$. For example, consider the matrix $G = (C_{4,1}C_{3,4}C_2C_{1,4}C_{0,1})$. It is a characteristic matrix for the collection of subsets $B_1 = \{1, 3, 4, 5, 7, 9, 10, 12\}$, $B_2 = \{1, 2, 4, 5, 7, 8, 11, 13\}$, $B_3 = \{1, 2, 3, 5, 6, 9, 11, 14\}$ and $B_4 = \{1, 2, 3, 4, 6, 8, 10, 15\}$. Note that

$$\sum_{y \in B_1} y = \sum_{y \in B_2} y = \sum_{y \in B_3} y = 51 \text{ and } \sum_{y \in B_4} y = 49$$

Applying the transposition $(10, 12)$ over the columns of G we obtain a proper matrix of weight 51, namely

$$\begin{pmatrix} 1&0&1&1&1&0&1&0&1&1&0&1&0&0&0&0\\ 1&1&0&1&1&0&1&1&0&0&1&0&1&0&0&0\\ 1&1&1&0&1&1&0&0&1&0&1&0&0&1&0&0\\ 1&1&1&1&0&1&0&1&0&0&0&1&0&0&1&0 \end{pmatrix}.$$

One can find in similar manner proper matrices of weight S for any $S \in [51; 85]$. It turns out that this is the case for any $n = 2^k$, $k \geq 2$, i.e. by manipulating the orbit matrices one can find a matrix having special property (as G above) and after single transposition of two columns of this matrix one can find a proper matrix of weight S for any S in the interval from Theorem 1.

In what follows $n = 2^k$ for $k \geq 3$.

Definition 1. *A matrix G is called special of type S if it is the characteristic matrix for a collection $B_1, B_2, \ldots, B_n$ and $\sum\limits_{y \in B_i} y = S$ for $i = 1, 2, \ldots, n-1$ and $\sum\limits_{y \in B_n} y = S - 2^{k-1}$. Equivalently, the scalar product of all but the last rows of G with the vector $(1, 2, \ldots, 2^n)$ equals S and the scalar product of the last row with the same vector equals $S - 2^{k-1}$. For a special matrix G of type S we write $t(G) = S$.*

The connection between special and proper matrices is revealed in the following lemma.

Lemma 1. *If in a special matrix G of type S there exist two vector columns $V_i = (v_1, v_2, \ldots, v_{n-1}, 1)$ and $V_j = (v_1, v_2, \ldots, v_{n-1}, 0)$ such that $j - i = 2^{k-1}$, then there exists a proper matrix of weight S.*

Proof. It suffices to interchange i-th and j-th columns of G. $\square$

Let H_1 be submatrix of a matrix G. If H_2 is a matrix having the same dimensions as H_1 then denote by $G(H_1 \to H_2)$ the matrix obtained from G by replacing H_1 by H_2. The next lemmas show how, given a special matrix one can obtain new special matrices by transformations of the type $H_1 \to H_2$.

Lemma 2. *Consider a special matrix G and vector columns V and W. If $A = V\overline{V}$ and $B = W\overline{W}$ are submatrices of G then, changing the places of A and B, we obtain a special matrix of the same type.*

Proof. Let (a_1, a_2) and (b_1, b_2) be the intersection pairs of A and B with i-th row of G. Since $(a_1, a_2), (b_1, b_2) \in \{(0,1), (1,0)\}$ we have that $(a_1, a_2) = (b_1, b_2)$ or $(a_1, a_2) = (\overline{b_1}, \overline{b_2})$. It is easy to see that when changing the places of A and B then the scalar product of i-th row with $(1, 2, 3, \ldots, 2^n)$ does not change. $\square$

The following lemma is fairly obvious.

Lemma 3. *Let G be a special matrix and V be a vector column. Denote by $C_{n,1}$ the vector column of weight n. Then:*

a) $G\left(V\overline{V\,C_{n,1}} \to \overline{C_{n,1}}V\overline{V}\right)$ *is a special matrix of type* $\mathrm{t}(G) + 1$*;*

b) $G\left(C_{n,1}V\overline{V} \to V\overline{V}\,C_{n,1}\right)$ *is a special matrix of type* $\mathrm{t}(G) + 1$*;*

c) $G\left(C_{n,1}\overline{C_{n,1}} \to \overline{C_{n,1}}C_{n,1}\right)$ *is a special matrix of type* $\mathrm{t}(G) + 1$*.*

Lemma 4. *Let G be a special matrix. If a vector column V and an orbit matrix $C_{w,l}$ are such that $V\overline{V\,C_{w,l}}$ is a submatrix of G then $G\left(V\overline{V C_{w,l}} \to \overline{C_{w,l}}V\overline{V}\right)$ is a special matrix of type* $\mathrm{t}(G) + (2w-n)\dfrac{l}{n}$.

P r o o f. For every row the transformation means that $\dfrac{(n-w)l}{n}$ ones (recall that there are $\dfrac{(n-w)l}{n}$ ones in every row of $\overline{C_{w,l}}$) are moved two positions backwards and one pair $(0,1)$ (or $(1,0)$) is moved l positions forward. Therefore the change in the scalar product of i-th row for $i = 1, 2, \ldots, n$ of G with $(1, 2, 3, \ldots, 2^n)$ is equal to

$$-\frac{2(n-w)l}{n} + l = (2w-n)\frac{l}{n}. \qquad \square$$

Lemma 5. *Consider an orbit matrix of weight w and length l $C_{w,l} = \left(V\pi(V)\pi^2(V)\ldots\pi^{l-1}(V)\right)$, where V is a vector-column of weight w. Also, set*

$$T_w = \left(V\overline{V}\pi\left(V\right)\pi\left(\overline{V}\right)\ldots\pi^{l-1}\left(V\right)\pi^{l-1}\left(\overline{V}\right)\right)$$

and $T_{n-w} = \overline{T_w}$.

a) If G is a special matrix having $C_{w,l}\overline{C_{w,l}}$ as a submarix then $G\left(C_{w,l}\overline{C_{w,l}} \to T_w\right)$ *is a special matrix of type* $\mathrm{t}(G) + (2w-n)\dfrac{l(l-1)}{2n}$*;*

b) If G is a special matrix having T_w as a submatrix then $G\left(T_w \to T_{n-w}\right)$ is a special matrix of type $\mathrm{t}(G) + (2w-n)\dfrac{l}{n}$.

P r o o f. a) Without loss of generality we may assume that the first column of $C_{w,l}$ coincides wih the first column of G. Then the contribution of $C_{w,l}\overline{C_{w,l}}$ to the scalar product of the i-th row of G with $(1, 2, 3, \ldots, 2^n)$ is $\dfrac{l(l+1)}{2}+\dfrac{(n-w)l^2}{n}$.

Further, let $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots \alpha_l, \beta_l)$ be a row of T_w. Amongst the pairs (α_k, β_k) for $k = 1, 2, \ldots, l$, there are $\dfrac{wl}{n}$ pairs $(1, 0)$ and $\dfrac{(n-w)l}{n}$ pairs $(0, 1)$. If all pairs were $(1, 0)$, then the scalar product of the i-th row of T_w with $(1, 2, 3, \ldots, 2l)$ would be $1+3+\cdots+(2l-1)$. Since we have to change $\dfrac{(n-w)l}{n}$ pairs from $(1, 0)$ to $(0, 1)$ and each change increases the scalar product by 1, we obtain that the contribution of T_w to the scalar product of the i-th row of G with $(1, 2, 3, \ldots, 2^n)$ is equal to $1 + 3 + \cdots + (2l - 1) + \dfrac{(n-w)l}{n} = l^2 + \dfrac{(n-w)l}{n}$.

Thus, the change of the scalar product for each row equals

$$l^2 + \frac{(n-w)l}{n} - \frac{l(l+1)}{2} + \frac{(n-w)l^2}{n} = (2w-n)\frac{l(l-1)}{2n}.$$

b) As in a) the contribution of T_w to the scalar product of the i-th row of G with $(1, 2, 3, \ldots, 2^n)$ is equal to $l^2 + \dfrac{(n-w)l}{n}$. The same arguments applied to T_{n-w} give that the corresponding contribution is equal to $l^2 + \dfrac{wl}{n}$. Thus, the change is $(2w-n)\dfrac{l}{n}$. $\square$

Theorem 2. *There exists a matrix G with 2^k rows and $\binom{2^k}{2^{k-1}}$ columns of the form $(V_1\overline{V_1}, V_2\overline{V_2}, \ldots, V_t\overline{V_t})$ where:*

– $V_1, \overline{V_1}, V_2, \overline{V_2}, \ldots, V_t, \overline{V_t}$ are all binary vectors of length 2^k and weight 2^{k-1} and

– the scalar product of the first $2^k - 1$ rows with $\left(1, 2, \ldots, \binom{2^k}{2^{k-1}}\right)$ equals $S = \dfrac{1}{4}\left(\binom{2^k}{2^{k-1}}\left(\binom{2^k}{2^{k-1}}+1\right)+2\right)$ and the scalar product of the last row with the same vector equals $S - 2^{k-1}$.

P r o o f. There are $\binom{2^k}{2^{k-1}}$ vectors of weight 2^{k-1} and therefore there are that many columns in our matrix. Moreover, all such vectors partition into orbits which lengths divide 2^k. Since there is only one orbit of length 2 (consisting of $(1010\dots10)^t$ and $\pi(1010\dots10)^t)$ and only one orbit of length 4 (consisting of $(11001100\dots1100)^t$ and $\pi^l(11001100\dots1100)^t$ for $l=1,2,3$) it follows that $\binom{2^k}{2^{k-1}} = 8s+6$. Therefore $t = 4s+3$ which implies that $S = t^2 + \frac{t+1}{2}$. Since $\frac{t+1}{2} = 2s+2$ is even we have that S is odd.

We prove now that for a given matrix of the form $G = (V_1\overline{V_1}, V_2\overline{V_2}, \dots, V_t\overline{V_t})$ the scalar products of the rows with $\left(1,2,\dots,\binom{2^k}{2^{k-1}}\right)$ have one and the same parity. Without loss of generality (see Lemma 2) any two rows of this matrix can be written in the form $ABCD$ where

$$A = \begin{pmatrix} 0101\dots0101 \\ 0101\dots0101 \end{pmatrix}, \qquad B = \begin{pmatrix} 0101\dots0101 \\ 1010\dots1010 \end{pmatrix},$$

$$C = \begin{pmatrix} 1010\dots1010 \\ 0101\dots0101 \end{pmatrix}, \qquad D = \begin{pmatrix} 1010\dots1010 \\ 1010\dots1010 \end{pmatrix}.$$

Denote the number of columns of A, B, C and D by $2a$, $2b$, $2c$ and $2d$ respectively. It is clear that if b and c have the same parity then the scalar products also have the same parity. The number of vector columns in G having two fixed entries 01 or 10 equals $2\binom{2^k-2}{2^{k-1}-1}$. Therefore $b+c = \binom{2^k-2}{2^{k-1}-1}$ and since $\binom{2^k-2}{2^{k-1}-1}$ is divisible by 2^{k-1} (recall that $k \geq 2$) we obtain that $b+c$ is even. Thus, b and c have the same parity and we get our assertion.

If $b > c$ then the scalar product of the first row with the vector $\left(1,2,\dots,\binom{2^k}{2^{k-1}}\right)$ is greater than the scalar product of the second row with the same vector. We consider all vector columns from G which intersect the first row of B in 1. There are $\binom{2^k-2}{2^{k-1}-1}$ possibilities for such a vector column and since $b > \frac{1}{2}\binom{2^k-2}{2^{k-1}-1}$ we have that there are two complementary vectors. Therefore

$$G = \begin{pmatrix} \dots & 0\,1 & \dots & 0\,1 & \dots \\ \dots & 1\,0 & \dots & 1\,0 & \dots \\ \dots & v_1\overline{v_1} & \dots & \overline{v_1}v_1 & \dots \\ \dots & v_2\overline{v_2} & \dots & \overline{v_2}v_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & v_{2^k}\overline{v_{2^k}} & \dots & \overline{v_{2^k}}v_{2^k} & \dots \end{pmatrix}$$

It is clear now that the columns of the matrix

$$G_1 = \begin{pmatrix} \dots & 10 & \dots & 10 & \dots \\ \dots & 01 & \dots & 01 & \dots \\ \dots & v_1\overline{v_1} & \dots & \overline{v_1}v_1 & \dots \\ \dots & v_2\overline{v_2} & \dots & \overline{v_2}v_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & v_{2^k}\overline{v_{2^k}} & \dots & \overline{v_{2^k}}v_{2^k} & \dots \end{pmatrix}$$

are all vectors of weight 2^{k-1}, the scalar product of the first row with the vector $\left(1,2,3,\dots,\binom{2^k}{2^{k-1}}\right)$ dicreases by 2, the scalar product of the second row with the same vector increases by 2, and all other scalar products do not change.

Therefore if the scalar product of two rows with $\left(1,2,3,\dots,\binom{2^k}{2^{k-1}}\right)$ are S_1 and S_2, $S_2 > S_1$ then we can obtain a matrix of desired form for which the corresponding scalar products are S_1+2 and S_2-2 and all other products do not change.

Note that, since $G = (V_1\overline{V_1}, V_2\overline{V_2}, \dots, V_t\overline{V_t})$, each row of G is formed by pairs $v_{2s-1}v_{2s}$, where $v_{2s} = \overline{v_{2s-1}}$ for $s = 1,2,\dots,t$. It is clear that we can arrange the vector columns of G in a way that all pairs $v_{2s-1}v_{2s}$ in the last row are such that $v_{2s-1} = 1$ and $v_{2s} = 0$. Then the scalar product of this row with $\left(1,2,3,\dots,\binom{2^k}{2^{k-1}}\right)$ equals $1+3+\dots+\left(\binom{2^k}{2^{k-1}}-1\right) = t^2$. It follows now that the scalar products of all rows have the parity of S, i.e. they are odd.

Since all vector columns of G are of weight 2^{k-1} we have that the sum of the scalar products of all rows of G with $\left(1,2,3,\dots,\binom{2^k}{2^{k-1}}\right)$ equals

$$2^{k-1}\sum_{i=1}^{\binom{2^k}{2^{k-1}}} i = 2^kS - 2^{k-1}.$$

Therefore if the the scalar products of the first 2^k-1 rows with $\left(1,2,3,\dots,\binom{2^k}{2^{k-1}}\right)$ is equal to S then the scalar product of the last row is equal to $S-2^{k-1}$.

Consider one of the first (2^k-1)-st rows of G. The number of pairs $v_{2s-1}v_{2s} = 01$ for $s = 1,2,\dots,t$ is equal to $\binom{2^k-2}{2^{k-1}-1}$. Therefore the scalar product of this row is equal to $t^2 + \binom{2^k-2}{2^{k-1}-1} > t^2 + \frac{t+1}{2} = S$. We can apply the described procedure to get the scalar product of this row to be equal to $t^2 + \binom{2^k-2}{2^{k-1}-1} - 2$ and the scalar product of the last row equal to $t^2 + 2$. Continuing this way we obtain a matrix of desired form with the property: the scalar product of the choosen row is equal to $t^2 + \frac{t+1}{2} = S$, the scalar product of the last row is equal to $t^2 + \binom{2^k-2}{2^{k-1}-1} - \frac{t+1}{2}$ and the scalar products of the remaining rows does not change. Note that the scalar product of the last row never becomes bigger than S. By repeating the above with the remaining $2^k - 2$ rows we obtain a matrix with the desired property. □

Denote the matrix from Theorem 2 by $C_{2^{k-1}}$.

Lemma 6. *The matrix $G = C_1C_2 \dots C_m$, where $C_1, C_2, \dots, C_m$ is a permutation of all orbit matrices of weights $2^k, 2^k-1, \dots, 2^{k-1}+1$, their compliments and $C_{2^{k-1}}$, is special.*

P r o o f. An orbit matrix $C_{w,l}$ and its compliment $\overline{C_{w,l}}$ add one and the same amount to the scalar product of every row with $(1, 2, 3, \dots, 2^n)$ (see Lemma 5). Note that the above is true no matter how the vector columns of $C_{w,l}$ are ordered. □

Remark 2. Let V, where $\mathrm{wt}(V) = w$ be a vector column and $(V, \pi(V), \pi^2(V), \dots, \pi^{l-1}(V))$ be the orbit of V. It follows from the proof of Lemma 6 that the columns of the corresponding orbit matrix $C_{w,l}$ can be taken as any permutation of the above vectors. It follows also that the simultaneous change $C^p_{w,l} \leftrightarrow C^q_{w,l}$ and $\overline{C^p_{w,l}} \leftrightarrow \overline{C^q_{w,l}}$ gives special matrix of the same type.

When w is odd then $\gcd(w, 2^k) = 1$. Therefore all orbit matrices of odd weight are of length $n = 2^k$.

Lemma 7. *If one of the following conditions is satisfied for a special matrix G of type S then there exists a proper matrix of weight S.*

a) $C_{2^k,1}C_{2^k-1,2^k}$ is a submatrix of G;

b) There exist two pairs $V\overline{V}$ and $W\overline{W}$ such that $V = (v_1, v_2, \dots, v_{n-1}, v_n)$ and $W = (v_1, v_2, \dots, v_{n-1}, \overline{v_n})$ and two arbitrary pairs of complimentary vectors with difference of their positions equal to 2^{k-1}.

P r o o f. We make use of Lemma 1 and Remark 2. a) By Remark 2 we may assume that the 2^{k-1}-th (recall that $n = 2^k$) column of $C_{2^k-1,2^k}$ can be chosen as $(1, 1, 1, \ldots, 1, 0)^t$. Now Lemma 1 implies that there exists a proper matrix of weight S.

b) Follows from Lemma 2 and Lemma 1. □

3. Main theorem.

Example 1 (Continued)**.** When $n = 4$ we construct proper matrices for all $S \in [51; 85]$. For simplisity write $C_4 = C_{4,1}$, $C_3 = C_{3,4}$, $C_1 = C_{1,4}$ and $C_0 = C_{0,1}$. Also, if $V_1 = (0, 0, 1, 1)^t$, $V_2 = (0, 1, 0, 1)^t$ and $V_3 = (1, 0, 0, 1)^t$, then $C_2 = (V_1\overline{V_1}\,V_2\overline{V_2}\,V_3\overline{V_3}\,)$. The following table gives a list of special matrices of type $S \in [51; 61]$.

special matrix G	type
$C_4C_3C_2C_1C_0$	51
$C_4C_3C_2C_0C_1$	52
$C_4\ \ C_3V_1\overline{V_1}\,V_2\overline{V_2}\,C_1V_3\overline{V_3}\,C_0$	53
$C_4C_3V_1\overline{V_1}\,V_2\overline{V_2}\,C_1C_0V_3\overline{V_3}$	54
$C_4C_3V_1\overline{V_1}\,C_1V_2\overline{V_2}\,V_3\overline{V_3}\,C_0$	55
$C_4C_3V_1\overline{V_1}\,C_1V_2\overline{V_2}\,C_0V_3\overline{V_3}$	56
$C_4C_3V_1C_1\overline{V_1}\,V_2\overline{V_2}\,V_3\overline{V_3}\,C_0$	57
$C_4C_3C_1V_1\overline{V_1}\,V_2\overline{V_2}\,C_0V_3\overline{V_3}$	58
$C_4C_3C_1V_1\overline{V_1}\,C_0V_2\overline{V_2}\,V_3\overline{V_3}$	59
$C_4C_3C_1C_0V_1\overline{V_1}\,V_2\overline{V_2}\,V_3\overline{V_3}$	60
$C_4C_3C_0C_1V_1\overline{V_1}\,V_2\overline{V_2}\,V_3\overline{V_3}$	61

Note that, by Lemma 7, if C_4C_3 is a submatrix of a special matrix of type S, then there exists a proper matrix of weight S. Thus, for each $S \in [51; 61]$ there exists a proper matrix of weight S. Since $\mathrm{wt}(\overline{G}) = 136 - \mathrm{wt}(G)$, there also exists a proper matrix for $S \in [75; 85]$.

The matrix

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} = \left(W_1\overline{W_1}W_2\overline{W_2}W_3\overline{W_3}W_4\overline{W_4}\right)$$

is obtained by the transformations $C_3C_1 \longrightarrow T_3 \longrightarrow T_1$. The following table gives a list of special matrices of type $S \in [62; 68]$.

special matrix G	type
$C_4T_1C_2C_0$	62
$C_4T_1V_1\overline{V_1}\,V_2\overline{V_2}\,C_0V_3\overline{V_3}$	63
$C_4T_1V_1\overline{V_1}\,C_0V_2\overline{V_2}\,V_3\overline{V_3}$	64
$C_4T_1C_0C_2$	65
$C_4W_1\overline{W_1}\,W_2\overline{W_2}\,C_0W_3\overline{W_3}\,C_2$	66
$C_4W_1\overline{W_1}\,C_0W_2\overline{W_2}\,W_3\overline{W_3}\,C_2$	67
$C_4C_0T_1C_2$	68

Note that any of the above matrices consisits of two vector columns (C_4 and C_0) and 7 pairs of complementary vectors (4 pairs from T_1 and 3 pairs from C_2). Moreover, there always exist two consecutive pairs of complementary vectors. Without loss of generality we may assume (see Lemma 2) that if $W\overline{W}V\overline{V}$ are two consecutive pairs then $V = (1,0,0,0)^t$ and $W = (1,0,0,1)^t$. Lemma 1 now implies that there exists a proper matrix of weight $S \in [62; 68]$. Since $\mathrm{wt}(\overline{G}) = 136 - \mathrm{wt}(G)$, there exists a proper matrix for $S \in [69; 74]$ as well.

The following theorem is the main result of our paper.

Theorem 3. *Let $n = 2^k$, $k \geq 3$. Then S is proper if and only if*

$$S \in \left[2^{2n-2} + 2^{n-2} - \frac{1}{2}\left(\binom{2n-1}{n-1} - 1\right); 2^{2n-2} + 2^{n-2} + \frac{1}{2}\left(\binom{2n-1}{n-1} - 1\right)\right].$$

P r o o f. Since $\binom{2n-1}{n-1}$ is odd, Theorem 1 gives that all proper integers belong to the interval given in the theorem. We have to show that if

$$S \in \left[2^{2n-2} + 2^{n-2} - \frac{1}{2}\left(\binom{2n-1}{n-1} - 1\right); 2^{2n-2} + 2^{n-2} + \frac{1}{2}\left(\binom{2n-1}{n-1} - 1\right)\right]$$

then S is proper. The proof follows the steps from Example 1. We show that for any S in the given interval there exists a special matrix of weight S for which one of the conditions of Lemma 7 is satysfied. Note also that since $\mathrm{wt}(\overline{G}) + \mathrm{wt}(G) = 2^{n-1}(2^n+1)$ it sufices to prove the result for the first half of the given interval, i.e. up to $2^{2n-2} + 2^{n-2}$.

It is clear that there is a single orbit matrix for each $w = 2^k, 2^k - 1$. Denote by $C_{2^k,1}$ the only orbit matrix of weight 2^k and by $C_{2^k-1,2^k}$ the only orbit matrix of weight $2^k - 1$. Also, let $C_{2^{k-1}}$ be the matrix with columns all vectors of weight 2^{k-1} having the property given in Theorem 2.

Consider a special matrix G of type S. Call a transformation $H_1 \longrightarrow H_2$ *admissible* if for each $S \in [t(G); t(G(H_1 \longrightarrow H_2))]$ there exists a special matrix of type S for which one of the conditions of Lemma 7 is satysfied. If G and $G_1 = G(H_1 \longrightarrow H_2)$ are special matrices and $w = \mathrm{t}(G_1) - \mathrm{t}(G)$ then we write $t^+(H_1 \longrightarrow H_2) = w$.

Consider the following matrix:

$$G_1 = C_{2^k,1}C_{2^k-1,2^k}C^1_{2^k-2,l_1}\ldots C^p_{2^{k-1}+1,2^k}C_{2^{k-1}}\overline{C^p_{2^{k-1}+1,2^k}C^1_{2^k-2,l_1}\ldots C_{2^k-1,2^k}C_{2^k,1}}.$$

This matrix is obtained by ordering the orbit matrices $C_{w,l}$ in decreasing order of their weights. Also, any two matrices symmetric with respect to $C_{2^{k-1}}$ are complimentary to each other. By Lemma 6 the matrix G_1 is special. It follows from the proof of Theorem 1 and from Lemma 7 (the matrix $C_{2^k,1}C_{2^k-1,2^k}$ is a sumbmatrix of G_1) that there exists a proper matrix of weight $S = 2^{2n-2} + 2^{n-2} - \frac{1}{2}\left(\binom{2n-1}{n-1} - 1\right)$.

Starting from G_1, move one by one all pairs of complementary columns from $C_{2^{k-1}}$ by skipping one by one the matrices $\overline{C^t_{w,l}}$ for $2^{k-1}+1 \le w \le 2^k$ to the left of $\overline{C_{2^k,1}}$. By Lemma 4 we have $t^+(\overline{C_{2^k-1,2^k}C_{2^k,1}} \longrightarrow \overline{C_{2^k,1}C_{2^k-1,2^k}}) = 1$, $t^+(V\overline{V}\overline{C_{w,l}} \longrightarrow \overline{C_{w,l}}V\overline{V}) = (2w - 2^k)\frac{l}{2^k}$ (in particular $t^+(V\overline{V}\overline{C_{2^{k-1}+1,2^k}} \longrightarrow \overline{C_{2^{k-1}+1,2^k}}V\overline{V}) = 2$). Recall that $C_{2^{k-1}}$ consists of $\frac{1}{2}\binom{2^k}{2^{k-1}}$ pairs of the form $V\overline{V}$. Also, the matrix $C_{2^k,1}C_{2^k-1,2^k}$ is a sumbmatrix of G_1.

It is not difficult to be seen that all such transformations are admissible. We obtain the matrix

$$G_2 = C_{2^k,1}C_{2^k-1,2^k}C^1_{2^k-2,l_1}\ldots C^s_{2^{k-1}+1,2^k}\overline{C^s_{2^{k-1}+1,2^k}C^1_{2^k-2,l_1}C_{2^k-1,2^k}}C_{2^{k-1}}\overline{C_{2^k,1}}.$$

Lemma 5a) applied for $w = 2^{k-1}+1$ and $l = 2^k$ gives that $t^+(C^s_{2^{k-1}+1,2^k}\overline{C^s_{2^{k-1}+1,2^k}} \to T^s_{2^{k-1}+1}) = 2^k - 1$ and Lemma 3a) shows that $t^+(V\overline{V}\overline{C_{2^k,1}} \to \overline{C_{2^k,1}}V\overline{V}) = 1$. Thus, since there are $\frac{1}{2}\binom{2^k}{2^{k-1}}$ pairs of the form $V\overline{V}$ in $C_{2^{k-1}}$ and $\frac{1}{2}\binom{2^k}{2^{k-1}} > 2^k - 1$ we have that $C^s_{2^{k-1}+1,2^k}\overline{C^s_{2^{k-1}+1,2^k}} \to T^s_{2^{k-1}+1}$ is admissible transformation. Next, move one by one the pairs of complements from $T^s_{2^{k-1}+1}$ by skipping one by one the matrices $\overline{C^t_{w,l}}$ for $2^{k-1}+1 \leq w \leq 2^k - 1$ to the left of $C_{2^{k-1}}$. Repeat the above for all pairs $C^t_{2^{k-1}+1,n}\overline{C^t_{2^{k-1}+1}}$ for $t = s-1, s-2, \ldots, 1$. Denote the resulting matrix by G_3.

$$G_3 = C_{2^k,1}C_{2^k-1,2^k}C^1_{2^k-2,l_1}\ldots C^s_{2^k-2,l_s}\overline{C^s_{2^k-2,l_s}C^1_{2^k-2,l_1}C_{2^k-1,2^k}}T^1_{2^{k-1}+1}\cdots$$

$$T^s_{2^{k-1}+1}C_{2^{k-1}}\overline{C_{2^k,1}}.$$

It is easy to see that the above transformations are admissible. Note that if $T^s_{2^{k-1}+1}C_{2^{k-1}}$ is a submatrix of a special matrix of type S then by Lemma 7b) there exists a proper matrix of weight S. Therefore, continuing this way we obtain by sequence of admissible transformations the following matrix

$$G_4 = C_{2^k,1}T_{2^k-1}T^1_{2^k-2}\ldots T^1_{2^{k-1}+1}\ldots T^s_{2^{k-1}+1}C_{2^{k-1}}\overline{C_{2^k,1}}.$$

It is clear now that by admissible transformations of the type $T_w \to T_{n-w}$ (which is equivalent to $T_w \to \overline{T_w}$), $V\overline{V}\overline{C_{2^k,1}} \to \overline{C_{2^k,1}}V\overline{V}$ and $C_{2^k,1}V\overline{V} \to V\overline{V}\,C_{2^k,1}$ one can obtain

$$G_5 = \overline{C_{2^k,1}T_{2^k-1}T^1_{2^k-2}\ldots T^1_{2^{k-1}+1}\ldots T^s_{2^{k-1}+1}}C_{2^{k-1}}C_{2^k,1}.$$

Let $(v_1, v_2, \ldots v_n)$ be the first row of G_5 (note that $v_1 = 0$ and $v_n = 1$). Each pair $v_{2s}v_{2s+1}$ for $s = 1, 2, \ldots, 2^{n-1} - 1$ is such that $v_{2s} + v_{2s+1} = 1$. Therefore the minimal possible value of the scalar product of such row with $(1, 2, \ldots, 2^n)$ is achieved when $v_{2s} = 1$ for all $s = 1, 2, \ldots, 2^{n-1} - 1$. Since the scalar product of $(0, 1, 0, 1, 0, \ldots, 1, 0, 1, 0, 1)$ with $(1, 2, \ldots, 2^n)$ is equal to $2 + 4 + 6 + \cdots + 2^n = 2^{2n-2} + 2^{n-1}$ we have that $t(G_5) > 2^{2n-2} + 2^{n-2}$. This completes the proof. $\square$

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