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WEIERSTRASS POINTS WITH FIRST NON-GAP FOUR ON A DOUBLE COVERING OF A HYPERELLIPTIC CURVE

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Abstract. Let H be a 4-semigroup, i.e., a numerical semigroup whose minimum positive element is four. We denote by $4r(H)+2$ the minimum element of H which is congruent to 2 modulo 4. If the genus g of H is larger than $3r(H)-1$, then there is a cyclic covering $\pi : C \longrightarrow \mathbb{P}^1$ of curves with degree 4 and its ramification point P such that the Weierstrass semigroup $H(P)$ of P is H (Komeda [1]). In this paper it is showed that we can construct a double covering of a hyperelliptic curve and its ramification point P such that $H(P)$ is equal to H even if $g \leq 3r(H)-1$.

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1. Introduction. Let $\mathbb{Z}_{\geq 0}$ be the additive semigroup of non-negative integers. A subsemigroup of $\mathbb{Z}_{\geq 0}$ is called a *numerical semigroup* if the complement of H in $\mathbb{Z}_{\geq 0}$ is finite. The cardinality of $\mathbb{Z}_{\geq 0}\backslash H$ is said to be the *genus* of H, which is denoted by $g(H)$. For a 4-semigroup H let $S(H) = \{4, s_1, s_2, s_3\}$ be the standard basis for H, i.e., $s_i = \mathrm{Min}\{h \in H | h \equiv i \bmod 4\}$ for $i = 1, 2, 3$. We set $s_2 = 4r(H) + 2$. On the other hand, let C be a complete non-singular irreducible curve over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. For any point P of C we define the *Weierstrass semigroup* $H(P)$ of P as follows:

$$H(P) = \{n \in \mathbb{Z}_{\geq 0} | \text{there exists } f \in \mathbb{K}(C) \text{ with } (f)_\infty = nP\}$$

where $\mathbb{K}(C)$ denotes the function field of C. It is known that $H(P)$ is a numerical semigroup whose genus is equal to the genus of the curve C. In the case where $g(H) \leq 3r(H) - 1$ it was only shown that the moduli space $\mathcal{M}_H$ of pointed curves (C, P) with $H(P) = H$ is non-empty (Komeda [1] Corollary 4.13). We did not give a curve C and its point P with $H(P) = H$. In this paper even if $g(H) \leq 3r(H) - 1$, it will be shown that we can find a double covering C of a hyperelliptic curve with its ramification point P such that $H(P) = H$. We note that such a curve C is not a cyclic covering of $\mathbb{P}^1$ with degree 4.

2. On double coverings of a hyperelliptic curve. In this section we construct a double covering of a hyperelliptic curve using the method of Mumford [2] and investigate the Weierstrass semigroup of a ramification point of the covering. Let C be a curve. For any even number t let $P_1, \ldots, P_t$ be distinct points of C. Let us take an invertible sheaf $\mathcal{L}$ and an isomorphism ϕ such that

$$\phi : \mathcal{L}^{\otimes 2} \cong \mathcal{O}_C\left(-\sum_{i=1}^{t} P_i\right) \subset \mathcal{O}_C.$$

Let $\mathcal{S}$ be a sheaf of $\mathcal{O}_C$−algebras of the form $\mathcal{S} \cong \mathcal{O}_C \oplus \mathcal{L}$ where multiplication is given by

$$(a, l) \cdot (b, m) = (a \cdot b + \phi(l \otimes m), a \cdot m + b \cdot l).$$

Then the canonical morphism $\pi : \tilde{C} = \mathrm{Spec}\, \mathcal{S} \longrightarrow C$ is a double covering of curves whose branch locus is $\sum_{i=1}^{t} P_i$ (Mumford [2]). Hence, if r is the genus of C, then by Riemann-Hurwitz formula the genus of $\tilde{C}$ is $2r - 1 + \frac{t}{2}$. For any i let $\tilde{P}_i \in \tilde{C}$ such that $\pi(\tilde{P}_i) = P_i$.

Proposition 2.1. *For any i and any positive integer n we have*

$$h^0(C, \pi_* \mathcal{O}_{\tilde{C}}(2n\tilde{P}_i)) = h^0(C, \mathcal{O}_C(nP_i)) + h^0(C, \mathcal{L} \otimes \mathcal{O}_C(nP_i)).$$

Proof. First we note that $\pi_* \mathcal{O}_{\tilde{C}} \cong \mathcal{O}_C \oplus \mathcal{L}$. Hence for any point P of C we get

$$\pi_* \mathcal{O}_{\tilde{C}}(n\pi^* P) \cong \pi_*(\mathcal{O}_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}}(n\pi^* P)) \cong \pi_*(\mathcal{O}_{\tilde{C}} \otimes \pi^* \mathcal{O}_C(nP))$$

$$\cong \pi_* \mathcal{O}_{\tilde{C}} \otimes \mathcal{O}_C(nP) \cong (\mathcal{O}_C \oplus \mathcal{L}) \otimes \mathcal{O}_C(nP) \cong \mathcal{O}_C(nP) \oplus (\mathcal{L} \otimes \mathcal{O}_C(nP)).$$

Since we have $\pi_* \mathcal{O}_{\tilde{C}}(n\pi^* P_i) = \pi_* \mathcal{O}_{\tilde{C}}(2n\tilde{P}_i)$, we get the desired equality. □

From now on we consider the case where C is a hyperelliptic curve of genus $r \geq 2$.

Lemma 2.2. *Let P_i be a Weierstrass point on C. Then $\tilde{C}$ is non-hyperelliptic. Moreover, $H(\tilde{P}_i)$ is a 4-semigroup.*

Proof. Since the curve C is not rational, $H(\tilde{P}_i) \not\ni 2$ follows from Proposition 2.1. Next we will show that $H(\tilde{P}_i) \not\ni 3$. Assume that $H(\tilde{P}_i) \ni 3$. We know that $H(\tilde{P}_i)$ also contains 4, because P_i is a Weierstrass point on a hyperelliptic curve C. Hence, we obtain $g(H(\tilde{P}_i)) \leq 3$. But

$$3 \geq g(H(\tilde{P}_i)) = 2r - 1 + \frac{t}{2} \geq 2 \times 2 - 1 + 1 = 4,$$

which is a contradiction. Therefore, $H(\tilde{P}_i)$ is a 4-semigroup. Assume that $\tilde{C}$ were hyperelliptic. Since $H(\tilde{P}_i)$ is a 4-semigroup, $\tilde{P}_i$ is not a Weierstrass point. Therefore, we get

$$\mathbb{Z}_{\geq 0} \backslash H(\tilde{P}_i) = \{1, \ldots, n\}$$

for some n. But $H(\tilde{P}_i)$ is a 4-semigroup, which implies that $n + 1 = 4$. Hence, we get $\mathbb{Z}_{\geq 0} \backslash H(\tilde{P}_i) = \{1, 2, 3\}$. Thus, we see that $H(\tilde{P}_i)$ is generated by $4, 5, 6$ and 7, which implies that there is $f \in \mathbb{K}(\tilde{C})$ such that $(f)_\infty = 6\tilde{P}_i$. Let us take a local parameter t at $\tilde{P}_i$ such that $\sigma^* t = -t$ where σ is the involution on $\tilde{C}$ such that $\tilde{C}/ < \sigma > \cong C$. Then f is written by

$$f = \frac{a}{t^6} + (\text{higher order})$$

locally at $\tilde{P}_i$ where a is a non-zero element of k. Moreover, we have

$$\sigma^* f = \frac{a}{t^6} + (\text{higher order})$$

locally at $\tilde{P}_i$. Hence $f + \sigma^* f$ is a non-zero function on $\tilde{C}$ such that $(f + \sigma^* f)_\infty = 6\tilde{P}_i$ on $\tilde{C}$, which implies that $(f + \sigma^* f)_\infty = 3P_i$ on C. Hence we get $H(P_i) \ni 3$. Since P_i is a Weierstrass point on a hyperelliptic curve of genus $r \geq 2$, we get $H(P_i) \ni 2$. Thus, $2 \leq r = g(H(P_i)) \leq 1$, which is a contradiction. Hence $\tilde{C}$ is non-hyperelliptic. □

Lemma 2.3. *Let the notation be as in Lemma* 2.2. *If $r \geq 3$, then $\tilde{C}$ is not bielliptic.*

P r o o f. We note that

$$g(\tilde{C}) = 2r - 1 + \frac{t}{2} \geq 2 \times 3 - 1 + \frac{2}{2} = 6.$$

If $H(\tilde{P}_i) \ni 6$, then $H(P_i) \ni 3$, which is a contradiction. Thus, $H(\tilde{P}_i) \not\ni 6$. Since by Lemma 2.2 $H(\tilde{P}_i)$ is a 4-semigroup, $\tilde{C}$ is not bielliptic (Komeda [1] Lemma 2.8.) □

Proposition 2.4. *Let P_i be a Weierstrass point on a hyperelliptic curve C of genus $r \geq 5$. There exists an odd number s with $1 \leq s \leq t - 1$ such that*

$$S(H(\tilde{P}_i)) = \{4, 2r + s, 2r + 2t - s, 4r + 2\}.$$

P r o o f. In view of $r \geq 5$ the genus of $\tilde{C}$ is at least 10. By Lemmas 2.2 and 2.3 we must have

$$S(H(\tilde{P}_i)) = \{4, 4r + 2, 4m + 1, 4n + 3\}$$

(Komeda [1] Proposition 3.1). We get

$$\text{Min}\{h \in H(\tilde{P}_i) | h \text{ is odd}\} \geq 2r + 1,$$

because $4r+2 \in S(H(\tilde{P}_i))$. We set

$$\mathrm{Min}\ \{h \in H(\tilde{P}_i)) | h \text{ is odd}\} = 2r + s$$

with odd $s \geq 1$. If $s > t-1$, then we obtain

$$2r + \frac{t}{2} - 1 = g(H(\tilde{P}_i)) \geq r + \left[\frac{2r+t+1}{4}\right] + \left[\frac{2r+t+3}{4}\right] = 2r + \frac{t}{2}$$

where for any real number x the symbol $[x]$ denotes the largest integer less than or equal to x. This is a contradiction. Thus, $s \leq t-1$. Let $S(H(\tilde{P}_i)) = \{4, 2r+s, h, 4r+2\}$. Then we must have

$$\left[\frac{h}{4}\right] = r - 1 + \frac{t}{2} - \left[\frac{2r+s}{4}\right].$$

Since h is an odd number such that $h \not\equiv 2r+s \bmod 4$, we obtain $h = 2r+2t-s$. □

Example 2.5. Let the notaion be as in Proposition 2.4. If $t = 2$, then

$$S(H(\tilde{P}_i)) = \{4, 2r+1, 2r+3, 4r+2\}.$$

In this case the semigroup $H(\tilde{P}_i)$ is generated by $4, 2r+1$ and $2r+3$.

Combining Proposition 2.4 with Proposition 2.1 we get the following:

Theorem 2.6. *Let P_i be a Weierstrass point on a hyperelliptic curve C of genus $r \geq 5$. Let $t \leq 2r$ and s an odd number with $1 \leq s \leq t-1$. Then the following conditions are equivalent:*
i) $S(H(\tilde{P}_i)) = \{4, 2r+s, 2r+2t-s, 4r+2\}$.
ii) $h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s+1}{2}\right) P_i\right)\right) = 1$ *and* $h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s-1}{2}\right) P_i\right)\right) = 0$.

P r o o f. By Proposition 2.1 we have

$$h^0\left(C, \pi_* \mathcal{O}_{\tilde{C}}\left((2r+s+1)\tilde{P}_i\right)\right) = h^0\left(C, \pi_* \mathcal{O}_{\tilde{C}}\left(2\left(r + \frac{s+1}{2}\right)\tilde{P}_i\right)\right)$$

$$= h^0\left(C, \mathcal{O}_C\left(\left(r + \frac{s+1}{2}\right) P_i\right)\right) + h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s+1}{2}\right) P_i\right)\right).$$

Since P_i is a Weierstrass point on a hyperelliptic curve and we have $t \leq 2r$ and $s \leq t-1$, we get

$$h^0(C, \pi_* \mathcal{O}_{\tilde{C}}((2r+s+1)\tilde{P}_i)) = \left[\frac{2r+s+1}{4}\right] + 1 + h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s+1}{2}\right)P_i\right)\right).$$

First we show that i) implies ii). Since $2r+s \in H(\tilde{P}_i)$, we have

$$h^0(\mathcal{O}_{\tilde{C}}((2r+s)\tilde{P}_i)) = \left[\frac{2r+s}{4}\right] + 2.$$

Hence, we get

$$h^0\left(\mathcal{O}_{\tilde{C}}\left((2r+s+1)\tilde{P}_i\right)\right) = \left[\frac{2r+s+1}{4}\right] + 2.$$

By the above formula we obtain

$$h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s+1}{2}\right)P_i\right)\right) = 1.$$

Since we have

$$h^0\left(\mathcal{O}_{\tilde{C}}\left((2r+s-1)\tilde{P}_i\right)\right) = \left[\frac{2r+s-1}{4}\right] + 1,$$

we get

$$h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s-1}{2}\right)P_i\right)\right) = 0.$$

Assume that ii) holds. By Proposition 2.4 there exists an odd number s' with $1 \leq s' \leq t-1$ such that

$$S(H(\tilde{P}_i)) = \{4, 2r+s', 2r+2t-s', 4r+2\}.$$

If $s' \leq s-2$, we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((2r+s'+1)\tilde{P}_i))$$

$$= h^0\left(C, \mathcal{O}_C\left(\left(r + \frac{s'+1}{2}\right)P_i\right)\right) + h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s'+1}{2}\right)P_i\right)\right)$$

$$= \left[\frac{2r+s'+1}{4}\right]+1$$

because of

$$h^0\left(C, \mathcal{L}\otimes\mathcal{O}_C\left(\left(r+\frac{s-1}{2}\right)P_i\right)\right)=0.$$

Hence $2r+s' \notin H(\tilde{P}_i)$, which is a contradiction. Assume that $s' \geq s+2$. Since we have

$$h^0(C, \mathcal{L}\otimes\mathcal{O}_C((r+\frac{s+1}{2})P_i))=1,$$

we know that

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((2r+s+1)\tilde{P}_i)) = \left[\frac{2r+s+1}{4}\right]+2.$$

Therefore there exists an odd number h with $h \leq 2r+s$ such that $h \in H(\tilde{P}_i)$. Then $h < 2r+s'$, which is a contradiction. Hence $s' = s$. □

Since for a 4-semigroup H with $g(H) \geq 3r(H)$ there exist a cyclic covering of the projective line $\mathbb{P}^1$ with degree 4 and its total ramification point P such that $H(P) = H$ (Komeda [1] §4), we want to investigate 4-semigroups H with $g(H) \leq 3r(H)-1$.

Proposition 2.7. *Let H be a 4-semigroup with $g(H) \leq 3r(H)-1$. Then there exist $2 \leq t \leq 2r$ and an odd number s with $1 \leq s \leq t-1$ such that $S(H) = \{4, 2r+s, 2r+2t-s, 4r+2\}$.*

Proof. If a 4-semigroup H satisfies $g(H) \leq 3r(H)-1$, by Komeda [1] it is one of the semigroups with the following standard basis:
i) $\{4, 4n+1, 4m+3, 4\cdot 2n+2\}, 1 \leq n \leq m \leq 3n-1,$
ii) $\{4, 4n+3, 4m+1, 4(2n+1)+2\}, 2 \leq n+1 \leq m \leq 3n+1,$
iii) $\{4, 4n+1, 4m+2, 4l+3\}, 1 \leq n \leq m \leq 2n-1, m \leq l \leq n+m-1, n+l \leq 2m-1,$
vi) $\{4, 4n+1, 4m+3, 4l+2\}, 2 \leq n \leq m \leq 2n-2, m+1 \leq l \leq 2n-1,$
v) $\{4, 4n+3, 4m+1, 4l+2\}, 2 \leq n+1 \leq m \leq 2n, m \leq l \leq 2n,$
vi) $\{4, 4n+3, 4m+2, 4l+1\}, 2 \leq n+1 \leq m \leq 2n, m+1 \leq l \leq n+m, n+l \leq 2m-1.$
In the case i) let $r = 2n$, $s = 1$ and $t = 2m-2n+2$. Then the set $\{4, 2r+s, 2r+2t-s, 4r+2\}$ coincides with the set $\{4, 4n+1, 4m+3, 4\cdot 2n+2\}$. In the case ii) let $r = 2n+1$, $s = 1$ and $t = 2m-2n$. In the case iii) let $r = m$, $s = 4n+1-2m$ and $t = 2n-2m+2l+2$. In the case vi) let $r = l$, $s = 4n+1-2l$ and $t = 2n+2m-2l+2$.

In the case v) let $r = l$, $s = 4n + 3 - 2l$ and $t = 2n + 2m - 2l + 2$. In the case vi) let $r = m$, $s = 4n + 3 - 2m$ and $t = 2n - 2m + 2l + 2$. $\square$

3. Construction of a point on a double covering of a hyperelliptic curve with a given semigroup. In this section we construct a point $\tilde{P}_i$ satisfying the conditions in Theorem 2.6 ii). For that purpose we need a hyperelliptic curve C which is a covering of degree n of another hyperelliptic curve. First we build a hyperelliptic curve C' which is the base of the covering. For a homogeneous polynomial $F \in \mathbb{C}[x, z]$ of degree $2b+2$ which has no multiple factor we set

$$C_1(F) = \{(s, x)|s^2 = F(x, 1)\},\ (C_1(F))_0 = \{(s, x)|s^2 = F(x, 1), x \neq 0\},$$

$$C_2(F) = \{(t, z)|t^2 = F(1, z)\},\ (C_2(F))_0 = \{(t, z)|t^2 = F(1, z), z \neq 0\}.$$

Through the isomorphism between $(C_1(F))_0$ and $(C_2(F))_0$ sending (s, x) to $\left(\frac{s}{x^{b+1}}, \frac{1}{x}\right)$ we can construct the nonsingular curve $C' = HC(F)$ by patching $C_1(F)$ and $C_2(F)$. We can define a morphism $h : C' = HC(F) \longrightarrow \mathbb{P}^1$ sending an element (s, x) of $C_1(F)$ (resp. (t, z) of $C_2(F)$) to $(x : 1)$ (resp. $(1 : z)$). Since the degree of h is two, $HC(F)$ is a hyperelliptic curve of genus b. On the other hand, let $\rho : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the morphism defined by sending $(u : v)$ to $(x(u, v) : z(u, v))$ where $z(u, v) = v^n$ and $x(u, v) = u^\lambda(u - \tau_1 v)(u - \tau_2 v) \cdots (u - \tau_{n-\lambda} v)$ with distinct non-zero elements $\tau_1, \ldots, \tau_{n-\lambda}$ of k. Then $(0 : 1)$ and $(1 : 0)$ are ramification points with indices λ and n respectively. Let $q_1 = (0 : 1), q_2, \ldots, q_{\alpha-1}, q_\alpha = (1 : 0)$ be the branch points of ρ. We set $(\rho^* F)(u, v) = F(x(u, v), z(u, v))$. We consider the curve $HC(\rho^* F)$ in the following cases:

i) The case where the zeros of $F(x, y)$ in $\mathbb{P}^1$ are different from $q_1, \ldots, q_\alpha$. Then $HC(\rho^* F)$ is a non-singular curve of genus $nb + n - 1$.

ii) The case where one of the zeros of $F(x, y)$ is equal to q_1 and the other zeros are different from $q_2, \ldots, q_\alpha$. Then $HC(\rho^* F)$ is a singular curve with only one singular point. The singular point is analytically isomorphic to the point $(0, 0)$ on the curve defined by the equation $y^2 = u^\lambda$. Since the singularity is resolved by $\left[\frac{\lambda}{2}\right]$ blowing-ups where $[x]$ means the largest integer less than or equal to x, the genus of $HC(\rho^* F)$ is $nb + n - 1 - \left[\frac{\lambda}{2}\right]$.

iii) The case where q_1 and q_α are zeros of $F(x,y)$ and the other zeros of $F(x,y)$ are different from $q_2,\dots,q_{\alpha-1}$. By the similar method to the case ii) the genus of of $HC(\rho^*F)$ is $nb+n-1-\left[\dfrac{\lambda}{2}\right]-\left[\dfrac{n}{2}\right]$.

Let $\eta : C \longrightarrow HC(\rho^*F)$ be the normalization. Then we get a commutative diagram

$$\begin{array}{ccccc} C=\widetilde{HC(\rho^*F)} & \stackrel{\eta}{\to} & HC(\rho^*F) & \stackrel{\phi}{\to} & HC(F) \\ & & \downarrow & & \downarrow \\ & & \mathbb{P}^1 & \stackrel{\rho}{\to} & \mathbb{P}^1 \end{array}$$

Thus C is a hyperelliptic curve whose genus takes any value of $nb, nb+1,\dots,nb+n-1$. Moreover, the morphism $\tilde{\phi}=\phi\circ\eta : C \longrightarrow HC(F)$ is of degree n, which implies that

$$\tilde{\phi}^*g_2^1(HC(F))=\tilde{\phi}^*h^*\mathcal{O}_{\mathbb{P}^1}(1)=h_C^*\rho^*\mathcal{O}_{\mathbb{P}^1}(1)=h_C^*\mathcal{O}_{\mathbb{P}^1}(n)=ng_2^1(C),$$

where h_C is the composite map of η and the morphism $HC(\rho^*F)\longrightarrow\mathbb{P}^1$ of degree 2.

Lemma 3.1. *Let the notation be as in the above. We denote the genus of C by r. We set $t=2n$ with a positive integer $n\le r$. Let s be an odd integer with $1\le s\le t-1$. Then there exist points $P_1,\dots,P_t,Q_1,\dots,Q_{\frac{s+1-t}{2}+r}$ of C such that*

$$P_1+P_2+\cdots+P_t+(r-t+\frac{s+1}{2})g_2^1(C)\sim 2(Q_1+\cdots+Q_{\frac{s+1-t}{2}+r})$$

where $P_1,\dots,P_n$ are Weierstrass points and $Q_1,\dots,Q_{\frac{s+1-t}{2}+r}$ are different from P_1. Moreover, we get $h^0(\mathcal{O}_C(Q_1+\cdots+Q_{\frac{s+1-t}{2}+r}))=1$.

P r o o f. Let p be a point on $C'=HC(F)$. First we show that there are points $q,q_1,\dots,q_{b-1}$ of C' such that

$$p+q+(b-2)g_2^1(C')\sim 2(q_1+\cdots+q_{b-1}).$$

Let $2_A : \mathrm{Pic}^{b-1}(C')\longrightarrow\mathrm{Pic}^{2b-2}(C')$ be the morphism defined by $2_A(\mathcal{L})=2\mathcal{L}$. We set

$$\Theta=\{\mathcal{O}(T_1+\cdots+T_{b-1})|T_1,\dots,T_{b-1}\in C'\},$$

which is a theta divisor on the abelian variety Pic $^{b-1}(C')$. Hence Θ is an ample divisor, which implies that the divisor $2_A(\Theta)\subset\mathrm{Pic}^{2b-2}(C')$ is ample. By Nakai's criterion for any 1-dimensional subvariety $\Sigma \subset \mathrm{Pic}^{2b-2}(C')$ we have $(\Sigma.2_A(\Theta)) > 0$, that is to say, $\Sigma \cap 2_A(\Theta) \neq \emptyset$. Now we set

$$\Sigma = \{p + q + (b-2)g_2^1(C') | q \in C'\},$$

which is a 1-dimensional locus in $\mathrm{Pic}^{2b-2}(C')$. Therefore we get $\Sigma \cap 2_A(\Theta) \neq \emptyset$, which implies that

$$p + q + (b-2)g_2^1(C') \sim 2(q_1 + \cdots + q_{b-1})$$

for some points $q, q_1, \ldots, q_{b-1}$ of C'. Here let p be a Weierstrass point on the hyperelliptic curve C'. We may assume that $q_1, \ldots, q_{b-1}$ are distinct from p. In fact, let $q_1 = \cdots = q_l = p$ and let $q_{l+1}, \ldots, q_{b-1}$ be distinct from p. Then we get

$$p + q + (b-2-l)g_2^1(C') \sim 2(q_{l+1} + \cdots + q_{b-1})$$

because of $2p \sim g_2^1(C')$. Take Weierstrass points $q_1', \ldots, q_l'$ on C' which are distinct from p. Then we obtain

$$p + q + (b-2)g_2^1(C') \sim 2(q_1' + \cdots + q_l' + q_{l+1} + \cdots + q_{b-1}).$$

Let $\tilde{\phi}^* p = P_1 + \cdots + P_n$ and $\tilde{\phi}^* q = P_{n+1} + \cdots + P_{2n}$. Since p is a Weierstrass point on C', $P_1, \cdots, P_n$ are also Weierstrass points on C. We obtain

$$P_1 + \cdots + P_t + \left(r - t + \frac{s+1}{2}\right) g_2^1(C) \sim$$

$$\tilde{\phi}^*(p + q + (b-2)g_2^1(C')) + \left(\left(r - t + \frac{s+1}{2}\right) - (nb - 2n)\right) g_2^1(C)$$

because of $\tilde{\phi}^* g_2^1(C') = n g_2^1(C)$. Since $r - t + \dfrac{s+1}{2} \geq nb - 2n$, we get

$$P_1 + \cdots + P_t + \left(r - t + \frac{s+1}{2}\right) g_2^1(C) \sim 2\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}\right)$$

for some points $Q_1, \ldots, Q_{\frac{s+1-t}{2}+r}$ of C distinct from P_1. Lastly we may assume that $h^0\left(\mathcal{O}_C\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}\right)\right) = 1$. In fact, if $h^0\left(\mathcal{O}_C\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}\right)\right) \geq 2$, then we must have (upon renumbering of the points Q_i)

$$Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r} \sim l g_2^1(C) + Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r-2l}.$$

Hence we get

$$P_1 + \cdots + P_t + \left(r - t + \frac{s+1}{2} - 2l\right) g_2^1(C) \sim 2\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r-2l}\right).$$

Let us take distinct Weierstrass points $Q_{\frac{s+1-t}{2}+r-2l+1}, \ldots, Q_{\frac{s+1-t}{2}+r}$ on C which are different from $P_1, Q_1, \ldots, Q_{\frac{s+1-t}{2}+r-2l}$. Then we get

$$P_1 + \cdots + P_t + \left(r - t + \frac{s+1}{2}\right) g_2^1(C) \sim 2\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}\right)$$

again where $h^0\left(\mathcal{O}_C\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}\right)\right) = 1$ and $Q_1, \ldots, Q_{\frac{s+1-r}{2}+r}$ are different from P_1. □

We set

$$\mathcal{L} = \mathcal{O}_C\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r} - \left(r + \frac{s+1}{2}\right) P_1\right).$$

Then by Lemma 3.1 we get

$$\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(P_1 + P_2 + \cdots + P_t - t g_2^1(C)) \cong \mathcal{O}_C(-\iota(P_1) - \cdots - \iota(P_t))$$

where ι is the hyperelliptic involution on C.

Theorem 3.2. *Let the notation be as in the above. Let* $\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$ *be the canonical morphism. We set* $\pi^{-1}(P_1) = \{\tilde{P}_1\}$. *If* $r \geq 5$, *then we get*

$$S(H(\tilde{P}_1)) = \{4, 2r+s, 2r+2t-s, 4r+2\}$$

Proof. By Lemma 3.1 we get

$$h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s+1}{2}\right) P_1\right)\right) = h^0\left(\mathcal{O}_C\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}\right)\right) = 1$$

and

$$h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left(r + \frac{s-1}{2}\right) P_1\right)\right) = h^0\left(\mathcal{O}_C\left(Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}\right) - P_1\right) = 0$$

By Theorem 2.6 we get our desired result. □

Combining Theorem 3.2 with Proposition 2.7 we get the following:

Main Theorem 3.3. *Let H be a 4-semigroup of genus $g(H) \geq 10$ with*

$$S(H) = \{4, 4r_1 + 1, 4r_2 + 2, 4r_3 + 3\}.$$

Assume that $g(H) \leq 3r_2 - 1$. Then there exist a double covering $\pi : \tilde{C} \longrightarrow C$ of a hyperelliptic curve and its ramification point $\tilde{P} \in \tilde{C}$ such that $H(\tilde{P}) = H$.

Considering the result of the case where H is a 4-semigroup with $4r_2+2 \in S(H)$ and $g(H) \geq 3r_2$, the following statement holds:

Corollary 3.4. *Let H be a 4-semigroup of genus ≥ 10. Then there exist a double covering of a hyperelliptic curve and its ramification point $\tilde{P}$ such that $H(\tilde{P}) = H$.*

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