WEIERSTRASS POINTS WITH FIRST NON-GAP FOUR ON A DOUBLE COVERING OF A HYPERELLIPTIC CURVE

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ABSTRACT. Let $H$ be a 4-semigroup, i.e., a numerical semigroup whose minimum positive element is four. We denote by $4r(H) + 2$ the minimum element of $H$ which is congruent to 2 modulo 4. If the genus $g$ of $H$ is larger than $3r(H) - 1$, then there is a cyclic covering $\pi : C \to \mathbb{P}^1$ of curves with degree 4 and its ramification point $P$ such that the Weierstrass semigroup $H(P)$ of $P$ is $H$ (Komeda [1]). In this paper it is showed that we can construct a double covering of a hyperelliptic curve and its ramification point $P$ such that $H(P)$ is equal to $H$ even if $g \leq 3r(H) - 1$.

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1. Introduction. Let $\mathbb{Z}_{\geq 0}$ be the additive semigroup of non-negative integers. A subsemigroup of $\mathbb{Z}_{\geq 0}$ is called a *numerical semigroup* if the complement of $H$ in $\mathbb{Z}_{\geq 0}$ is finite. The cardinality of $\mathbb{Z}_{\geq 0}\setminus H$ is said to be the *genus* of $H$, which is denoted by $g(H)$. For a 4-semigroup $H$ let $S(H) = \{4, s_1, s_2, s_3\}$ be the standard basis for $H$; i.e., $s_i = \min\{h \in H | h \equiv i \text{ mod } 4\}$ for $i = 1, 2, 3$. We set $s_2 = 4r(H) + 2$. On the other hand, let $C$ be a complete non-singular irreducible curve over an algebraically closed field $k$ of characteristic 0, which is called a *curve* in this paper. For any point $P$ of $C$ we define the *Weierstrass semigroup* $H(P)$ of $P$ as follows:

$$H(P) = \{n \in \mathbb{Z}_{\geq 0} | \text{there exists } f \in \mathbb{K}(C) \text{ with } (f)_{\infty} = nP\}$$

where $\mathbb{K}(C)$ denotes the function field of $C$. It is known that $H(P)$ is a numerical semigroup whose genus is equal to the genus of the curve $C$. In the case where $g(H) \leq 3r(H) - 1$ it was only shown that the moduli space $\mathcal{M}_H$ of pointed curves $(C, P)$ with $H(P) = H$ is non-empty (Komeda [1] Corollary 4.13). We did not give a curve $C$ and its point $P$ with $H(P) = H$. In this paper even if $g(H) \leq 3r(H) - 1$, it will be shown that we can find a double covering $C$ of a hyperelliptic curve with its ramification point $P$ such that $H(P) = H$. We note that such a curve $C$ is not a cyclic covering of $\mathbb{P}^1$ with degree 4.

2. On double coverings of a hyperelliptic curve. In this section we construct a double covering of a hyperelliptic curve using the method of Mumford [2] and investigate the Weierstrass semigroup of a ramification point of the covering. Let $C$ be a curve. For any even number $t$ let $P_1, \ldots, P_t$ be distinct points of $C$. Let us take an invertible sheaf $\mathcal{L}$ and an isomorphism $\phi$ such that

$$\phi: \mathcal{L}^\otimes 2 \cong \mathcal{O}_C \left(- \sum_{i=1}^{t} P_i \right) \subset \mathcal{O}_C.$$ 

Let $S$ be a sheaf of $\mathcal{O}_C$–algebras of the form $S \cong \mathcal{O}_C \oplus \mathcal{L}$ where multiplication is given by

$$(a, l) \cdot (b, m) = (a \cdot b + \phi(l \otimes m), a \cdot m + b \cdot l).$$

Then the canonical morphism $\pi: \tilde{C} = \text{Spec } S \rightarrow C$ is a double covering of curves whose branch locus is $\sum_{i=1}^{t} P_i$ (Mumford [2]). Hence, if $r$ is the genus of $C$, 

$$H(P) = \{n \in \mathbb{Z}_{\geq 0} | \text{there exists } f \in \mathbb{K}(C) \text{ with } (f)_{\infty} = nP\}$$
then by Riemann-Hurwitz formula the genus of $\tilde{C}$ is $2r - 1 + \frac{t}{2}$. For any $i$ let $\tilde{P}_i \in \tilde{C}$ such that $\pi(\tilde{P}_i) = P_i$.

**Proposition 2.1.** For any $i$ and any positive integer $n$ we have

$$h^0(C, \pi_* O_{\tilde{C}}(2n\tilde{P}_i)) = h^0(C, \mathcal{O}_C(nP_i)) + h^0(C, \mathcal{L} \otimes \mathcal{O}_C(nP_i)).$$

**Proof.** First we note that $\pi_* O_{\tilde{C}} \cong \mathcal{O}_C \oplus \mathcal{L}$. Hence for any point $P$ of $C$ we get

$$\pi_* O_{\tilde{C}}(n\pi^* P) \cong \pi_* (O_{\tilde{C}} \otimes O_{\tilde{C}}(n\pi^* P)) \cong \pi_* (O_{\tilde{C}} \otimes \pi^* O_C(nP))$$

$$\cong \pi_* O_{\tilde{C}} \otimes O_C(nP) \cong (O_C \oplus \mathcal{L}) \otimes O_C(nP) \cong O_C(nP) \oplus (\mathcal{L} \otimes O_C(nP)).$$

Since we have $\pi_* O_{\tilde{C}}(n\pi^* P_i) = \pi_* O_{\tilde{C}}(2n\tilde{P}_i)$, we get the desired equality. $\square$

From now on we consider the case where $C$ is a hyperelliptic curve of genus $r \geq 2$.

**Lemma 2.2.** Let $P_i$ be a Weierstrass point on $C$. Then $\tilde{C}$ is non-hyperelliptic. Moreover, $H(\tilde{P}_i)$ is a 4-semigroup.

**Proof.** Since the curve $C$ is not rational, $H(\tilde{P}_i) \not\supseteq 2$ follows from Proposition 2.1. Next we will show that $H(\tilde{P}_i) \not\supseteq 3$. Assume that $H(\tilde{P}_i) \supseteq 3$. We know that $H(\tilde{P}_i)$ also contains 4, because $P_i$ is a Weierstrass point on a hyperelliptic curve $C$. Hence, we obtain $g(H(\tilde{P}_i)) \leq 3$. But

$$3 \geq g(H(\tilde{P}_i)) = 2r - 1 + \frac{t}{2} \geq 2 \times 2 - 1 + 1 = 4,$$

which is a contradiction. Therefore, $H(\tilde{P}_i)$ is a 4-semigroup. Assume that $\tilde{C}$ were hyperelliptic. Since $H(\tilde{P}_i)$ is a 4-semigroup, $\tilde{P}_i$ is not a Weierstrass point. Therefore, we get

$$\mathbb{Z}_{\geq 0} \setminus H(\tilde{P}_i) = \{1, \ldots, n\}$$

for some $n$. But $H(\tilde{P}_i)$ is a 4-semigroup, which implies that $n + 1 = 4$. Hence, we get $\mathbb{Z}_{\geq 0} \setminus H(\tilde{P}_i) = \{1, 2, 3\}$. Thus, we see that $H(\tilde{P}_i)$ is generated by 4, 5, 6 and 7, which implies that there is $f \in \mathbb{K}(\tilde{C})$ such that $(f)_\infty = 6\tilde{P}_i$. Let us take a local
parameter $t$ at $\tilde{P}_i$ such that $\sigma^*t = -t$ where $\sigma$ is the involution on $\tilde{C}$ such that $\tilde{C}/<\sigma> \cong C$. Then $f$ is written by

$$f = \frac{a}{t^6} + \text{(higher order)}$$

locally at $\tilde{P}_i$ where $a$ is a non-zero element of $k$. Moreover, we have

$$\sigma^*f = \frac{a}{t^6} + \text{(higher order)}$$

locally at $\tilde{P}_i$. Hence $f + \sigma^*f$ is a non-zero function on $\tilde{C}$ such that $(f + \sigma^*f) \infty = 6\tilde{P}_i$ on $\tilde{C}$, which implies that $(f + \sigma^*f) \infty = 3P_i$ on $C$. Hence we get $H(P_i) \ni 3$. Since $P_i$ is a Weierstrass point on a hyperelliptic curve of genus $r \geq 2$, we get $H(P_i) \ni 2$. Thus, $2 \leq r = g(H(P_i)) \leq 1$, which is a contradiction. Hence $\tilde{C}$ is non-hyperelliptic. □

**Lemma 2.3.** Let the notation be as in Lemma 2.2. If $r \geq 3$, then $\tilde{C}$ is not bielliptic.

**Proof.** We note that

$$g(\tilde{C}) = 2r - 1 + \frac{t}{2} \geq 2 \times 3 - 1 + \frac{2}{2} = 6.$$  

If $H(\tilde{P}_i) \ni 6$, then $H(P_i) \ni 3$, which is a contradiction. Thus, $H(\tilde{P}_i) \not\ni 6$. Since by Lemma 2.2 $H(\tilde{P}_i)$ is a 4-semigroup, $\tilde{C}$ is not bielliptic (Komeda [1] Lemma 2.8.) □

**Proposition 2.4.** Let $P_i$ be a Weierstrass point on a hyperelliptic curve $C$ of genus $r \geq 5$. There exists an odd number $s$ with $1 \leq s \leq t - 1$ such that

$$S(H(\tilde{P}_i)) = \{4, 2r + s, 2r + 2t - s, 4r + 2\}.$$  

**Proof.** In view of $r \geq 5$ the genus of $\tilde{C}$ is at least 10. By Lemmas 2.2 and 2.3 we must have

$$S(H(\tilde{P}_i)) = \{4, 4r + 2, 4m + 1, 4n + 3\}$$

(Komeda [1] Proposition 3.1). We get

$$\min\{h \in H(\tilde{P}_i) | h \text{ is odd}\} \geq 2r + 1,$$
because $4r + 2 \in S(H(\tilde{P}_i))$. We set

$$\text{Min} \{h \in H(\tilde{P}_i) | h \text{ is odd} \} = 2r + s$$

with odd $s \geq 1$. If $s > t - 1$, then we obtain

$$2r + \frac{t}{2} - 1 = g(H(\tilde{P}_i)) \geq r + \left[\frac{2r + t + 1}{4}\right] + \left[\frac{2r + t + 3}{4}\right] = 2r + \frac{t}{2}$$

where for any real number $x$ the symbol $[x]$ denotes the largest integer less than or equal to $x$. This is a contradiction. Thus, $s \leq t - 1$. Let $S(H(\tilde{P}_i)) = \{4, 2r + s, h, 4r + 2\}$. Then we must have

$$\left\lfloor \frac{h}{4} \right\rfloor = r - 1 + \frac{t}{2} - \left\lfloor \frac{2r + s}{4} \right\rfloor.$$ 

Since $h$ is an odd number such that $h \not\equiv 2r + s \mod 4$, we obtain $h = 2r + 2t - s$. □

**Example 2.5.** Let the notation be as in Proposition 2.4. If $t = 2$, then

$$S(H(\tilde{P}_i)) = \{4, 2r + 1, 2r + 3, 4r + 2\}.$$ 

In this case the semigroup $H(\tilde{P}_i)$ is generated by $4, 2r + 1$ and $2r + 3$.

Combining Proposition 2.4 with Proposition 2.1 we get the following:

**Theorem 2.6.** Let $P_i$ be a Weierstrass point on a hyperelliptic curve $C$ of genus $r \geq 5$. Let $t \leq 2r$ and $s$ an odd number with $1 \leq s \leq t - 1$. Then the following conditions are equivalent:

i) $S(H(\tilde{P}_i)) = \{4, 2r + s, 2r + 2t - s, 4r + 2\}$.

ii) $h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left((r + \frac{s + 1}{2})P_i\right)\right) = 1$ and $h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left((r + \frac{s - 1}{2})P_i\right)\right) = 0$.

**Proof.** By Proposition 2.1 we have

$$h^0\left(C, \pi_* \mathcal{O}_{\tilde{C}}\left((2r + s + 1)\tilde{P}_i\right)\right) = h^0\left(C, \pi_* \mathcal{O}_{\tilde{C}}\left(2 \left( r + \frac{s + 1}{2} \right) \tilde{P}_i\right)\right)$$

$$= h^0\left(C, \mathcal{O}_C\left(\left( r + \frac{s + 1}{2} \right) P_i\right)\right) + h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C\left(\left( r + \frac{s + 1}{2} \right) P_i\right)\right).$$
Since $P_i$ is a Weierstrass point on a hyperelliptic curve and we have $t \leq 2r$ and $s \leq t - 1$, we get

$$h^0(C, \pi_* \mathcal{O}_\tilde{C}((2r+s+1)\tilde{P}_i)) = \left[\frac{2r + s + 1}{4}\right] + 1 + h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C \left(\left(r + \frac{s + 1}{2}\right)P_i\right)\right).$$

First we show that i) implies ii). Since $2r + s \in H(\tilde{P}_i)$, we have

$$h^0(\mathcal{O}_\tilde{C}((2r+s)\tilde{P}_i)) = \left[\frac{2r + s}{4}\right] + 2.$$

Hence, we get

$$h^0\left(\mathcal{O}_\tilde{C} \left(\left(2r + s + 1\right)\tilde{P}_i\right)\right) = \left[\frac{2r + s + 1}{4}\right] + 2.$$

By the above formula we obtain

$$h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C \left(\left(r + \frac{s + 1}{2}\right)P_i\right)\right) = 1.$$

Since we have

$$h^0\left(\mathcal{O}_\tilde{C} \left(\left(2r + s - 1\right)\tilde{P}_i\right)\right) = \left[\frac{2r + s - 1}{4}\right] + 1,$$

we get

$$h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C \left(\left(r + \frac{s - 1}{2}\right)P_i\right)\right) = 0.$$

Assume that ii) holds. By Proposition 2.4 there exists an odd number $s'$ with $1 \leq s' \leq t - 1$ such that

$$S(H(\tilde{P}_i)) = \{4, 2r + s', 2r + 2t - s', 4r + 2\}.$$

If $s' \leq s - 2$, we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((2r + s' + 1)\tilde{P}_i))$$

$$= h^0\left(C, \mathcal{O}_C \left(\left(r + \frac{s' + 1}{2}\right)P_i\right)\right) + h^0\left(C, \mathcal{L} \otimes \mathcal{O}_C \left(\left(r + \frac{s' + 1}{2}\right)P_i\right)\right).$$
\[
= \left[ \frac{2r + s' + 1}{4} \right] + 1
\]

because of

\[
h^0 \left( C, \mathcal{L} \otimes \mathcal{O}_C \left( \left( r + \frac{s - 1}{2} \right) P_i \right) \right) = 0.
\]

Hence \(2r + s' \not\in H(\tilde{P}_i)\), which is a contradiction. Assume that \(s' \geq s + 2\). Since we have

\[
h^0(C, \mathcal{L} \otimes \mathcal{O}_C((r + \frac{s + 1}{2})P_i)) = 1,
\]

we know that

\[
h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((2r + s + 1)\tilde{P}_i)) = \left[ \frac{2r + s + 1}{4} \right] + 2.
\]

Therefore there exists an odd number \(h\) with \(h \leq 2r + s\) such that \(h \in H(\tilde{P}_i)\). Then \(h < 2r + s'\), which is a contradiction. Hence \(s' = s\). \(\Box\)

Since for a 4-semigroup \(H\) with \(g(H) \geq 3r(H)\) there exist a cyclic covering of the projective line \(\mathbb{P}^1\) with degree 4 and its total ramification point \(P\) such that \(H(P) = H\) (Komeda [1] \S 4), we want to investigate 4-semigroups \(H\) with \(g(H) \leq 3r(H) - 1\).

**Proposition 2.7.** Let \(H\) be a 4-semigroup with \(g(H) \leq 3r(H) - 1\). Then there exist \(2 \leq t \leq 2r\) and an odd number \(s\) with \(1 \leq s \leq t - 1\) such that

\[
S(H) = \{4, 2r + s, 2r + 2t - s, 4r + 2\}.
\]

**Proof.** If a 4-semigroup \(H\) satisfies \(g(H) \leq 3r(H) - 1\), by Komeda [1] it is one of the semigroups with the following standard basis:

i) \(\{4, 4n + 1, 4m + 3, 4 \cdot 2n + 2\}, 1 \leq n \leq m \leq 3n - 1\),

ii) \(\{4, 4n + 3, 4m + 1, 4(2n + 1) + 2\}, 2 \leq n + 1 \leq m \leq 3n + 1\),

iii) \(\{4, 4n+1, 4m+2, 4l+3\}, 1 \leq n \leq m \leq 2n-1, m \leq l \leq n+m-1, n+l \leq 2m-1\),

vi) \(\{4, 4n + 1, 4m + 3, 4l + 2\}, 2 \leq n \leq m \leq 2n - 2, m + 1 \leq l \leq 2n - 1\),

v) \(\{4, 4n + 3, 4m + 1, 4l + 2\}, 2 \leq n + 1 \leq m \leq 2n, m \leq l \leq 2n\),

vi) \(\{4, 4n+3, 4m+2, 4l+1\}, 2 \leq n + 1 \leq m \leq 2n, m + 1 \leq l \leq n+m, n+l \leq 2m-1\). In the case i) let \(r = 2n, s = 1\) and \(t = 2m - 2n + 2\). Then the set \(\{4, 2r + s, 2r + 2t - s, 4r + 2\}\) coincides with the set \(\{4, 4n+1, 4m+3, 4 \cdot 2n+2\}\). In the case ii) let \(r = 2n + 1, s = 1\) and \(t = 2m - 2n\). In the case iii) let \(r = m, s = 4n + 1 - 2m\) and \(t = 2n - 2m + 2l + 2\). In the case vi) let \(r = l, s = 4n + 1 - 2l\) and \(t = 2n + 2m - 2l + 2\).
In the case v) let \( r = l, s = 4n + 3 - 2l \) and \( t = 2n + 2m - 2l + 2 \). In the case vi) let \( r = m, s = 4n + 3 - 2m \) and \( t = 2n - 2m + 2l + 2 \). \( \square \)

3. Construction of a point on a double covering of a hyperelliptic curve with a given semigroup. In this section we construct a point \( \tilde{P}_1 \) satisfying the conditions in Theorem 2.6 ii). For that purpose we need a hyperelliptic curve \( C \) which is a covering of degree \( n \) of another hyperelliptic curve. First we build a hyperelliptic curve \( C' \) which is the base of the covering. For a homogeneous polynomial \( F \in \mathbb{C}[x, z] \) of degree \( 2b + 2 \) which has no multiple factor we set

\[
C_1(F) = \{(s, x)|s^2 = F(x, 1)\}, \quad (C_1(F))_0 = \{(s, x)|s^2 = F(x, 1), x \neq 0\},
\]

\[
C_2(F) = \{(t, z)|t^2 = F(1, z)\}, \quad (C_2(F))_0 = \{(t, z)|t^2 = F(1, z), z \neq 0\}.
\]

Through the isomorphism between \( (C_1(F))_0 \) and \( (C_2(F))_0 \) sending \((s, x)\) to \( \left( \frac{s}{x^b+1}, \frac{1}{x} \right) \) we can construct the nonsingular curve \( C' = HC(F) \) by patching \( C_1(F) \) and \( C_2(F) \). We can define a morphism \( h : C' = HC(F) \rightarrow \mathbb{P}^1 \) sending an element \((s, x)\) of \( C_1(F) \) (resp. \((t, z)\) of \( C_2(F) \)) to \((x : 1)\) (resp. \((1 : z)\)). Since the degree of \( h \) is two, \( HC(F) \) is a hyperelliptic curve of genus \( b \). On the other hand, let \( \rho : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) be the morphism defined by sending \((u : v)\) to \((x(u, v) : z(u, v))\) where \( z(u, v) = v^n \) and \( x(u, v) = u^\lambda(u - \tau_1 v)(u - \tau_2 v)\ldots(u - \tau_{n-\lambda} v) \) with distinct non-zero elements \( \tau_1, \ldots, \tau_{n-\lambda} \) of \( k \). Then \((0 : 1)\) and \((1 : 0)\) are ramification points with indices \( \lambda \) and \( n \) respectively. Let \( q_1 = (0 : 1), q_2, \ldots, q_{\alpha-1}, q_\alpha = (1 : 0) \) be the branch points of \( \rho \). We set \((\rho^*F)(u, v) = F(x(u, v), z(u, v))\). We consider the curve \( HC(\rho^*F) \) in the following cases:

i) The case where the zeros of \( F(x, y) \) in \( \mathbb{P}^1 \) are different from \( q_1, \ldots, q_\alpha \). Then \( HC(\rho^*F) \) is a non-singular curve of genus \( nb + n - 1 \).

ii) The case where one of the zeros of \( F(x, y) \) is equal to \( q_1 \) and the other zeros are different from \( q_2, \ldots, q_\alpha \). Then \( HC(\rho^*F) \) is a singular curve with only one singular point. The singular point is analytically isomorphic to the point \((0, 0)\) on the curve defined by the equation \( y^2 = u^\lambda \). Since the singularity is resolved by \( \left\lfloor \frac{\lambda}{2} \right\rfloor \) blowing-ups where \( \left\lfloor x \right\rfloor \) means the largest integer less than or equal to \( x \), the genus of \( HC(\rho^*F) \) is \( nb + n - 1 - \left\lfloor \frac{\lambda}{2} \right\rfloor \).
iii) The case where $q_1$ and $q_α$ are zeros of $F(x, y)$ and the other zeros of $F(x, y)$ are different from $q_2, \ldots, q_{α-1}$. By the similar method to the case ii) the genus of of $HC(ρ^*F)$ is $nb + n - 1 - \left\lfloor \frac{λ}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor$.

Let $η : C → HC(ρ^*F)$ be the normalization. Then we get a commutative diagram

$$
\begin{array}{ccc}
C = HC(ρ^*F) & \xrightarrow{η} & HC(ρ^*F) \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{ρ} & \mathbb{P}^1
\end{array}
$$

Thus $C$ is a hyperelliptic curve whose genus takes any value of $nb, nb+1, \ldots, nb+n-1$. Moreover, the morphism $\tilde{φ} = φ \circ η : C → HC(F)$ is of degree $n$, which implies that

$$
\tilde{φ}^*g_2^1(HC(F)) = φ^*h^*O_{\mathbb{P}^1}(1) = h_C^*ρ^*O_{\mathbb{P}^1}(1) = h_C^*O_{\mathbb{P}^1}(n) = ng_2^1(C),
$$

where $h_C$ is the composite map of $η$ and the morphism $HC(ρ^*F) → \mathbb{P}^1$ of degree 2.

**Lemma 3.1.** Let the notation be as in the above. We denote the genus of $C$ by $r$. We set $t = 2n$ with a positive integer $n ≤ r$. Let $s$ be an odd integer with $1 ≤ s ≤ t - 1$. Then there exist points $P_1, \ldots, P_t, Q_1, \ldots, Q_{s+1-t+r}$ of $C$ such that

$$
P_1 + P_2 + \cdots + P_t + (r - t + \frac{s + 1}{2})g_2^1(C) \sim 2(Q_1 + \cdots + Q_{s+1-t+r}),
$$

where $P_1, \ldots, P_n$ are Weierstrass points and $Q_1, \ldots, Q_{s+1-t+r}$ are different from $P_1$. Moreover, we get $h^0(O_C(Q_1 + \cdots + Q_{s+1-t+r})) = 1$.

**Proof.** Let $p$ be a point on $C' = HC(F)$. First we show that there are points $q, q_1, \ldots, q_{b-1}$ of $C'$ such that

$$
p + q + (b - 2)g_2^1(C') \sim 2(q_1 + \cdots + q_{b-1}).
$$

Let $2_A : \text{Pic}^b(C') → \text{Pic}^{2b-2}(C')$ be the morphism defined by $2_A(\mathcal{L}) = 2\mathcal{L}$. We set

$$
Θ = \{ O(T_1 + \cdots + T_{b-1}) | T_1, \ldots, T_{b-1} ∈ C' \},
$$

which is a theta divisor on the abelian variety $\text{Pic}^b(C')$. Hence $Θ$ is an ample divisor, which implies that the divisor $2_A(Θ) ⊂ \text{Pic}^{2b-2}(C')$ is ample. By Nakai’s
criterion for any 1-dimensional subvariety $\Sigma \subset \text{Pic}^{2b-2}(C')$ we have $(\Sigma, 2_A(\Theta)) > 0$, that is to say, $\Sigma \cap 2_A(\Theta) \neq \emptyset$. Now we set

$$\Sigma = \{p + q + (b - 2)g_2^1(C') | q \in C'\},$$

which is a 1-dimensional locus in $\text{Pic}^{2b-2}(C')$. Therefore we get $\Sigma \cap 2_A(\Theta) \neq \emptyset$, which implies that

$$p + q + (b - 2)g_2^1(C') \sim 2(q_1 + \cdots + q_{b-1})$$

for some points $q, q_1, \ldots, q_{b-1}$ of $C'$. Here let $p$ be a Weierstrass point on the hyperelliptic curve $C'$. We may assume that $q_1, \ldots, q_{b-1}$ are distinct from $p$. In fact, let $q_1 = \cdots = q_t = p$ and let $q_{t+1}, \ldots, q_{b-1}$ be distinct from $p$. Then we get

$$p + q + (b - 2 - 1)g_2^1(C') \sim 2(q_{t+1} + \cdots + q_{b-1})$$

because of $2p \sim g_2^1(C')$. Take Weierstrass points $q'_1, \ldots, q'_t$ on $C'$ which are distinct from $p$. Then we obtain

$$p + q + (b - 2)g_2^1(C') \sim 2(q'_1 + \cdots + q'_t + q_{t+1} + \cdots + q_{b-1}).$$

Let $\tilde{\phi}^*p = P_1 + \cdots + P_n$ and $\tilde{\phi}^*q = P_{n+1} + \cdots + P_{2n}$. Since $p$ is a Weierstrass point on $C'$, $P_1, \ldots, P_n$ are also Weierstrass points on $C$. We obtain

$$P_1 + \cdots + P_t + \left( r - t + \frac{s + 1}{2} \right) g_2^1(C) \sim \tilde{\phi}^*(p + q + (b - 2)g_2^1(C')) + \left( \left( r - t + \frac{s + 1}{2} \right) - (nb - 2n) \right) g_2^1(C)$$

because of $\tilde{\phi}^*g_2^1(C') = ng_2^1(C)$. Since $r - t + \frac{s + 1}{2} \geq nb - 2n$, we get

$$P_1 + \cdots + P_t + \left( r - t + \frac{s + 1}{2} \right) g_2^1(C) \sim 2 \left( Q_1 + \cdots + Q_{\frac{s+1}{2} + t} \right)$$

for some points $Q_1, \ldots, Q_{\frac{s+1}{2} + t}$ of $C$ distinct from $P_1$. Lastly we may assume that $h^0 \left( \mathcal{O}_C \left( Q_1 + \cdots + Q_{\frac{s+1}{2} + t} \right) \right) = 1$. In fact, if $h^0 \left( \mathcal{O}_C \left( Q_1 + \cdots + Q_{\frac{s+1}{2} + t} \right) \right) \geq 2$, then we must have (upon renumbering of the points $Q_t$)

$$Q_1 + \cdots + Q_{\frac{s+1}{2} + t} \sim l g_2^1(C) + Q_1 + \cdots + Q_{\frac{s+1}{2} + t - 2l}.$$
Hence we get

\[ P_1 + \cdots + P_t + \left( r - t + \frac{s + 1}{2} - 2t \right) g_2^1(C) \sim 2 \left( Q_1 + \cdots + Q_{\frac{s+1}{2}+r-2t} \right). \]

Let us take distinct Weierstrass points \( Q_{\frac{s+1}{2}+r-2t+1}, \ldots, Q_{\frac{s+1}{2}+r} \) on \( C \) which are different from \( P_1, Q_1, \ldots, Q_{\frac{s+1}{2}+r-2t} \). Then we get

\[ P_1 + \cdots + P_t + \left( r - t + \frac{s + 1}{2} - 2t \right) g_2^1(C) \sim 2 \left( Q_1 + \cdots + Q_{\frac{s+1}{2}+r} \right) \]

again where \( h^0 \left( O_C \left( Q_1 + \cdots + Q_{\frac{s+1}{2}+r} \right) \right) = 1 \) and \( Q_1, \ldots, Q_{\frac{s+1}{2}+r} \) are different from \( P_1 \). □

We set

\[ \mathcal{L} = O_C \left( Q_1 + \cdots + Q_{\frac{s+1}{2}+r} - \left( r + \frac{s + 1}{2} \right) P_1 \right). \]

Then by Lemma 3.1 we get

\[ \mathcal{L} \otimes 2 \cong O_C (P_1 + P_2 + \cdots + P_t - tg_2^1(C)) \cong O_C (-\iota(P_1) - \cdots - \iota(P_t)) \]

where \( \iota \) is the hyperelliptic involution on \( C \).

**Theorem 3.2.** Let the notation be as in the above. Let \( \pi : \tilde{C} = \text{Spec}(O_C \oplus \mathcal{L}) \longrightarrow C \) be the canonical morphism. We set \( \pi^{-1}(P_1) = \{ \tilde{P}_1 \} \). If \( r \geq 5 \), then we get

\[ S(H(\tilde{P}_1)) = \{ 4, 2r + s, 2r + 2t - s, 4r + 2 \} \]

**Proof.** By Lemma 3.1 we get

\[ h^0 \left( C, \mathcal{L} \otimes O_C \left( \left( r + \frac{s + 1}{2} \right) P_1 \right) \right) = h^0 \left( O_C \left( Q_1 + \cdots + Q_{\frac{s+1}{2}+r} \right) \right) = 1 \]

and

\[ h^0 \left( C, \mathcal{L} \otimes O_C \left( \left( r + \frac{s - 1}{2} \right) P_1 \right) \right) = h^0 \left( O_C \left( Q_1 + \cdots + Q_{\frac{s+1}{2}+r} - P_1 \right) \right) = 0 \]

By Theorem 2.6 we get our desired result. □

Combining Theorem 3.2 with Proposition 2.7 we get the following:
Main Theorem 3.3. Let $H$ be a 4-semigroup of genus $g(H) \geq 10$ with

$$S(H) = \{4, 4r_1 + 1, 4r_2 + 2, 4r_3 + 3\}.$$ 

Assume that $g(H) \leq 3r_2 - 1$. Then there exist a double covering $\pi : \tilde{C} \to C$ of a hyperelliptic curve and its ramification point $\tilde{P} \in \tilde{C}$ such that $H(\tilde{P}) = H$.

Considering the result of the case where $H$ is a 4-semigroup with $4r_2 + 2 \in S(H)$ and $g(H) \geq 3r_2$, the following statement holds:

Corollary 3.4. Let $H$ be a 4-semigroup of genus $\geq 10$. Then there exist a double covering of a hyperelliptic curve and its ramification point $\tilde{P}$ such that $H(\tilde{P}) = H$.

REFERENCES


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