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CONDITIONAL CONFIDENCE INTERVAL FOR THE SCALE PARAMETER OF A WEIBULL DISTRIBUTION

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ABSTRACT. A two-sided conditional confidence interval for the scale parameter θ of a Weibull distribution is constructed. The construction follows the rejection of a preliminary test for the null hypothesis: $\theta = \theta_0$ where θ_0 is a given value. The confidence bounds are derived according to the method set forth by Meeks and D'Agostino (1983) and subsequently used by Arabatzis et al. (1989) in Gaussian models and more recently by Chiou and Han (1994, 1995) in exponential models. The derived conditional confidence interval also suits non large samples since it is based on the modified pivot statistic advocated in Bain and Engelhardt (1981, 1991). The average length and the coverage probability of this conditional interval are compared with whose of the corresponding optimal unconditional interval through simulations. The study has shown that both intervals are similar when the population scale parameter is far enough from θ_0. However, when θ is in the vicinity of θ_0, the conditional interval outperforms the unconditional one in terms of length and also maintains a reasonably high coverage probability. Our results agree with the findings of Chiou and Han and Arabatzis et al. which contrast with whose of Meeks and D'Agostino stating that the unconditional interval is always shorter than the conditional one. Furthermore, we derived the likelihood ratio confidence interval for θ and compared numerically its performance with the two other interval estimators.

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Key words: Weibull distribution, rejection of a preliminary hypothesis, conditional and unconditional interval estimator, likelihood ratio interval, coverage probability, average length, simulation.

1. Introduction. The estimation following rejection of a preliminary hypothesis used, for instance, in Chiou and Han[5, 6], Meeks and D'Agostino [14], and, Arabatzis et al. [1] is certainly the closest inference procedure to the one initiated in Bancroft [4]. The distinction between these two procedures has been pointed out, among others, in Mahdi [9]. For a detailed account on the use of preliminary test procedures, see, for instance, Jugde and Bock [8], Mahdi [10, 12], Rai and Srivastava [15], Giles et al. [7] and reference therein. In Chiou and Han [5, 6], the effect of the use of conditional interval estimation for the shape and scale parameters, following rejection of a preliminary test, in a two-parameter exponential population has been investigated. The inference is based on a type II censored single sample. The authors have shown that the conditional confidence interval is more accurate than the usual unconditional interval for some values of the parameter space. On the other hand, the conditional interval estimation based on two-sample data from exponential populations has been recently investigated in Mahdi [9], and, the case of normal populations has been treated in Mahdi and Gupta [11].

In this paper, we consider the problem of interval estimation for the scale parameter θ of a Weibull distribution when it is suspected that $\theta = \theta_0$ for some given θ_0. This interval is compared in terms of length and coverage probability to the optimal corresponding unconditional interval with same targeted confidence level. Furthermore, we compare these confidence intervals to the likelihood ratio based confidence interval recommended in Meeker and Escobar [13] for the shape parameter of the Weibull distribution. It is worth noting that the Weibull distribution was introduced in 1939 by a Swedish scientist on empirical ground in the statistical theory of the strength of materials. This law has proven, since then, to be a successful analytical model for many phenomena in reliability engineering, infant mortality and extreme value problems.

We organize this paper as follows. In Section 2, we state the considered problem and in Section 3, we derive the bounds for the optimal unconditional interval for θ. The bounds for the conditional confidence interval are derived in Section 4. In Section 5, we compute the actual coverage probability of the unconditional interval and in Section 6 we present the likelihood ratio based confidence interval for θ. Simulations results are discussed in Section 7. Table 1 and Figure 1, illustrating the main simulation results, are displayed in Appendix. We finally conclude in Section 8.

2. Statement of the problem. Suppose that $X_1, \cdots, X_n$ constitute a random sample from a Weibull distribution with shape parameter β and scale parameter θ. It is suspected that $\theta \geq \theta_0$ where θ_0 is a prefixed value. The null hypothesis $H_0 : \theta = \theta_0$ versus the alternative $H_1 : \theta > \theta_0$ is then tested and in the case of rejection a two sided confidence interval for θ is thus constructed. Otherwise, θ_0 is substituted for θ. To test H_0 and construct the confidence interval bounds for θ we use the pivotal quantity derived in Bain and Engelhardt [2, 3], that is,

$$\sqrt{n-1}\frac{\hat{\beta}\ln\dfrac{\hat{\theta}}{\theta}}{c} \sim t_{n-1} \tag{1}$$

where $c = 1.053$, $\hat{\beta}$ is the maximum likelihood estimator of β and $\hat{\theta}$ is the maximum likelihood estimator of θ. The critical region associated with the preliminary test of H_0 versus H_1 performed at the significance level α is given by

$$\mathcal{R} = \left\{\hat{\theta} : \ \hat{\theta} > \theta_0 \exp\left[\frac{ct_{n-1}(\alpha)}{\sqrt{n-1}\hat{\beta}}\right]\right\}$$

where $t_{n-1}(\alpha)$ denote the quantile of order $(1-\alpha)100\%$ of the Student variable with $n-1$ degrees of freedom.

Remark 1. The value of the statistic $\hat{\beta}$ is independent of θ and its estimator $\hat{\theta}$.

P r o o f. The two-parameter Weibull probability density function is given by

$$g(x;\theta,\beta) = \beta\theta^{-\beta}x^{\beta-1}\exp\left[-\left(\frac{x}{\theta}\right)^{\beta}\right] \tag{2}$$

for $x > 0$, $\beta > 0$ and $\theta > 0$. The corresponding log-likelihood function based on the observed random sample $x_1, \cdots, x_n$ is

$$LnL(\theta,\beta) = n(\ln(\beta) - \beta\ln(\theta)) + (\beta-1)\sum_{i=1}^{n}\ln(x_i) - \sum\left(\frac{x_i}{\theta}\right)^{\beta}. \tag{3}$$

After some algebraic simplifications, we find that the maximum likelihood estimator of β is given by the solution $\beta = \hat{\beta}$ of the non linear equation

$$\beta = \left[\left(\sum_{i=1}^{n} x_i^{\beta} \ln(x_i) \right) \left(\sum_{i=1}^{n} x_i^{\beta} \right)^{-1} - \frac{1}{n} \sum_{i=1}^{n} \ln(x_i) \right]^{-1}. \tag{4}$$

The maximum likelihood estimator for θ is then obtained as

$$\hat{\theta} = \left[n^{-1} \sum_{i=1}^{n} x_i^{\hat{\beta}} \right]^{\frac{1}{\hat{\beta}}}. \tag{5}$$

We derive below the bounds of the optimal length unconditional confidence interval for θ. For the benefit of using optimal confidence intervals, see, for instance, Wardell [16].

3. Optimal unconditional confidence interval. The upper bound θ_U and lower bound θ_L of a $100(1-p)\%$ unconditional confidence interval, based on the pivot statistic (1) for θ are given by

$$\theta_L = \hat{\theta} \exp \left[\frac{c t_{n-1}(1-p_1)}{\hat{\beta}\sqrt{n-1}} \right] \tag{6}$$

and

$$\theta_U = \hat{\theta} \exp \left[\frac{c t_{n-1}(p_2)}{\hat{\beta}\sqrt{n-1}} \right] \tag{7}$$

for all p_i, $i = 1, 2$ satisfying $0 < p = p_1 + p_2 \leq 1$. However, we can state the following result.

Theorem 1. *For any positive p_1 and p_2 such that $0 < p = p_1 + p_2 < 1$, the $(1-p)100\%$ unconditional confidence interval for θ has optimal length when $p_1 = p_2 = \frac{p}{2}$. In such a case, the lower and upper confidence bounds are, respectively, given by*

$$\theta_L = \hat{\theta} \exp \left[-\frac{c t_{n-1}(p/2)}{\hat{\beta}\sqrt{n-1}} \right] \leq \hat{\theta} \tag{8}$$

and

$$\theta_U = \hat{\theta} \exp \left[\frac{c t_{n-1}(p/2)}{\hat{\beta}\sqrt{n-1}} \right] \geq \hat{\theta}. \tag{9}$$

P r o o f. Consider the intervals

$$I = \hat{\theta} \exp \left[\frac{ct_{n-1}(1-p/2)}{\hat{\beta}\sqrt{n-1}} \right] < \theta < \hat{\theta} \exp \left[\frac{ct_{n-1}(p/2)}{\hat{\beta}\sqrt{n-1}} \right] \tag{10}$$

and

$$I^{'} = \hat{\theta} \exp \left[\frac{ct_{n-1}(1-((k-1)p)/k)}{\hat{\beta}\sqrt{n-1}} \right] < \theta < \hat{\theta} \exp \left[\frac{ct_{n-1}(p/k)}{\hat{\beta}\sqrt{n-1}} \right] \tag{11}$$

where $k > 2$. Both I and I' are $(1-p)100\%$ confidence intervals for θ. Let LI and LI' denote the length of I and I', respectively. To prove that $LI \leq LI'$, it suffices to prove that

$$t_{n-1}(p/2) - t_{n-1}(1-p/2) \leq t_{n-1}(p/k) - t_{n-1}(1-((k-1)p)/k) \tag{12}$$

which is equivalent to prove the inequality,

$$t_{n-1}(1-((k-1)p)/k)) - t_{n-1}(1-p/2) \leq t_{n-1}(p/k) - t_{n-1}(p/2). \tag{13}$$

Inequality (13) is true since the intervals $(t_{n-1}(1-p/2), t_{n-1}(1-((k-1)p)/k))$ and $(t_{n-1}(p/2), t_{n-1}(p/k))$ intercept the same area $\dfrac{(k-2)p}{2k}$ under the even probability density function of T_{n-1} and that the curve of this probability density function is higher above the interval $(t_{n-1}(1-p/2), t_{n-1}(1-((k-1)p)/k))$. This proves then inequality (13) and therefore Theorem 1 in the case $k > 2$. The proof in the case $1 < k < 2$ can be done in a similar way. □

Corollary 1. *The two-sided optimal length unconditional* $(1-p)100\%$ *confidence interval for* θ *has a smaller length than the usual one-sided unconditional interval of the form*

$$\left(0, \hat{\theta} \exp \left[\frac{ct_{n-1}(p)}{\hat{\beta}\sqrt{n-1}} \right] \right).$$

P r o o f. The intervals $(0, t_{n-1}(1-p/2))$ and $(t_{n-1}(p), t_{n-1}(p/2))$ have the same probability value but the length of the former is larger according to the position of the points 0, $t_{n-1}(1-p/2), t_{n-1}(p)$ and $t_{n-1}(p/2)$ under the curve of the probability density function of the Student variable with $n-1$ degrees of freedom. □

4. Conditional confidence interval. The bounds θ_L^C and θ_U^C of the $(1-p)100\%$ conditional confidence interval for θ are computed using the conditional sampling distribution of the statistic $T = \dfrac{\sqrt{n-1}\,\hat{\beta}\ln\dfrac{\hat{\theta}}{\theta}}{c}$ given rejection of H_0, that is, $T > t_{n-1}(\alpha) + \dfrac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}$ where $\psi = \dfrac{\theta_0}{\theta}$. The conditional cumulative function of T, given rejection of H_0, is $F_C(t) = P[T \leq t | T > t_1]$ where $t_1 = t_{n-1}(\alpha) + \dfrac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}$. Thus,

$$F_C(t) = \begin{cases} 0 & \text{if } t < t_1 \\ \dfrac{F(t)-F(t_1)}{1-F(t_1)} & \text{if } t \geq t_1 \end{cases} \tag{14}$$

where F is the cumulative function of a Student variable with $n-1$ degrees of freedom. The corresponding probability density function is

$$f_C(t) = \begin{cases} 0 & \text{if } t < t_1 \\ \dfrac{f(t)}{1-F(t_1)} & \text{if } t \geq t_1 \end{cases} \tag{15}$$

where f denote the probability density function of a Student variable with $n-1$ degrees of freedom. The bounds of the conditional confidence set are solutions of the system of inequations

$$F_C\left(\frac{\sqrt{n-1}\hat{\beta}\ln\dfrac{\hat{\theta}}{\theta}}{c}\right) = \frac{F\left(\dfrac{\sqrt{n-1}\hat{\beta}\ln\dfrac{\hat{\theta}}{\theta}}{c}\right) - F\left(t_{n-1}(\alpha) + \dfrac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}\right)}{1 - F\left(t_{n-1}(\alpha) + \dfrac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}\right)} \geq p_1 \tag{16}$$

$$F_C\left(\frac{\sqrt{n-1}\hat{\beta}\ln\dfrac{\hat{\theta}}{\theta}}{c}\right) = \frac{F\left(\dfrac{\sqrt{n-1}\hat{\beta}\ln\dfrac{\hat{\theta}}{\theta}}{c}\right) - F\left(t_{n-1}(\alpha) + \dfrac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}\right)}{1 - F\left(t_{n-1}(\alpha) + \dfrac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}\right)} \tag{17}$$

$$\leq 1 - p_2$$

such that $p_1 + p_2 = p$. Now from the monotony property of the function $\xi(\theta) = F_C(\frac{\sqrt{n-1}\hat{\beta}\ln\frac{\hat{\theta}}{\theta}}{c})$, the above system reduces to simplified following system of equations

$$\frac{1 - F\left(\frac{\sqrt{n-1}\hat{\beta}\ln\frac{\hat{\theta}}{\theta_U^c}}{c}\right)}{1 - F\left(t_{n-1}(\alpha) + \frac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}\right)} = p_1 \tag{18}$$

and

$$\frac{1 - F\left(\frac{\sqrt{n-1}\hat{\beta}\ln\frac{\hat{\theta}}{\theta_L^c}}{c}\right)}{1 - F\left(t_{n-1}(\alpha) + \frac{\sqrt{n-1}\hat{\beta}\ln(\psi)}{c}\right)} = 1 - p_2 \tag{19}$$

which gives the upper and lower conditional confidence bounds. In the case $\alpha = 1$, that is, we always reject H_0, the above system reduces to the following system

$$F\left(\frac{\sqrt{n-1}\hat{\beta}\ln\left(\frac{\hat{\theta}}{\theta_U^c}\right)}{c}\right) = p_2 \tag{20}$$

and

$$F\left(\frac{\sqrt{n-1}\hat{\beta}\ln(\frac{\hat{\theta}}{\theta_L^c})}{c}\right) = 1 - p_2 \tag{21}$$

which also guarantees that the conditional confidence set to be an interval since

$$\frac{dF\left(\frac{\sqrt{n-1}\hat{\beta}\ln\left(\frac{\hat{\theta}}{\theta}\right)}{c}\right)}{d\theta} = -\frac{-\hat{\beta}\sqrt{n-1}}{c\theta} f\left(\frac{\sqrt{n-1}\hat{\beta}\ln\left(\frac{\hat{\theta}}{\theta}\right)}{c}\right) \leq 0 \tag{22}$$

for any value $\theta > 0$. Note that the conditional confidence bounds are the same as the unconditional confidence bounds in this case. Setting again $p_1 = p_2 = p/2$ in the above system of equations yields the bounds of the minimum length unconditional confidence interval.

5. Coverage probability of the unconditional interval. The coverage probability of the conditional interval is $1-p$ since it is derived under this nominal level. However, the actual coverage probability of the unconditional confidence interval has to be computed under the conditional probability function of the pivot statistic (1), given rejection of H_0. This yields the following result.

Theorem 2. *The coverage probability CP of the unconditional interval (θ_L, θ_U) is given by $CP = 0$ if $t_{n-1}(p/2) < t_1$; $CP = \frac{1-p/2-F(t_1)}{1-F(t_1)}$ if $-t_{n-1}(p/2) < t_1 < t_{n-1}(p/2)$ and by $CP = \frac{1-p}{1-F(t_1)}$ if $t_1 < -t_{n-1}(p/2) < t_{n-1}(p/2)$ where t_1 is the previously defined quantity.*

Proof. The coverage probability is given by

$$CP = \int_A f_C(t)dt \tag{23}$$

where f_C is defined in formula (15) and $A = \{t : \{-t_{n-1}(p/2) < t < t_{n-1}(p/2)\}$ and $\{t > t_1\}\}$. Now, if $t_{n-1}(p/2) < t_1$, then $A = \emptyset$ and therefore $CP = 0$. On the other hand, if $-t_{n-1}(p/2) < t_1 < t_{n-1}(p/2)$, then $A = (t_1, t_{n-1}(p/2))$ and the formula (23) gives $CP = \frac{F(t_{n-1}(p/2)) - F(t_1)}{1-F(t_1)} = \frac{1-p/2-F(t_1)}{1-F(t_1)}$. Finally, when $t_1 < -t_{n-1}(p/2) < t_{n-1}(p/2)$, then $A = (-t_{n-1}(p/2), t_{n-1}(p/2))$ and thus $CP = \frac{F(t_{n-1}(p/2)) - F(-t_{n-1}(p/2))}{1-F(t_1)} = \frac{1-p}{1-F(t_1)} \geq 1-p$. $\square$

Remark 2. The coverage probability CP may exceed the confidence level $1-p$ when $1-F(t_1)<1$ and this occurs over a significant region of the parameter space.

We derive below the likelihood ratio confidence interval.

6. Likelihood ratio based confidence interval. The likelihood function based on the two parameter Weibull distribution and the random sample $x_1,\ldots,x_n$ is given by

$$Lw(\theta,\beta)=\beta^n\theta^{-n\beta}\prod_{i=1}^{n}x_i^{\beta-1}\exp-\left[\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta}\right]. \tag{24}$$

For a fixed value θ, the maximum likelihood estimator of β is given by the solution $\tilde{\beta}$ of the profile likelihood gradient equation

$$\frac{\partial\ln(Lw(\theta,\beta))}{\partial\beta}=\beta\left[\sum_{i=1}^{n}\ln(\frac{x_i}{\theta})\left[(\frac{x_i}{\theta})^{\beta}-1\right]\right]-n=0. \tag{25}$$

which has the following property.

Theorem 3. *The profile likelihood gradient equation*

$$\beta\left[\sum_{i=1}^{n}\ln\left(\frac{x_i}{\theta}\right)\left[\left(\frac{x_i}{\theta}\right)^{\beta}-1\right]\right]-n=0. \tag{26}$$

admits a unique solution.

P r o o f. Consider the function

$$g(\beta)=\sum_{i=1}^{n}\ln(y_i^{\beta})[y_i^{\beta}-1]-n \tag{27}$$

where $y_i=\dfrac{x_i}{\theta}$ for $i=1,\cdots,n$. The derivative of the function g is given by

$$g'(\beta)=\sum_{i=1}^{n}\ln(y_i)[y_i^{\beta}[\ln(y_i^{\beta})+1]-1] \tag{28}$$

which is greater or equal to zero as sum of non negative terms. Indeed, in the case, $0<y_i<1$, $\ln(y_i)<0$ and $y_i^{\beta}\ln(y_i^{\beta})+y_i^{\beta}-1\le 0$ since $y_i^{\beta}\ln(y_i^{\beta})<0$ and

$y_i^\beta - 1 \leq 0$. Thus $\ln(y_i)[y_i^\beta[\ln(y_i^\beta)+1]-1] \geq 0$ for $0 < y_i < 1$. Now in the case $y_i > 1$, we have that $\ln(y_i) > 0$, $y_i^\beta \geq 1$ and $\ln(y_i^\beta) > 0$. Thus

$$\ln(y_i)[y_i^\beta[\ln(y_i^\beta)+1]-1] \geq 0.$$

Finally in the case $y_i = 1$, it is easy to see that

$$\ln(y_i)[y_i^\beta[\ln(y_i^\beta)+1]-1] = 0.$$

We conclude then that the continuous function g is non decreasing for $\beta \geq 0$ provided that not all y_i worth 1; which occurs almost surely. On the other hand, we have that $\lim_{\beta\to\infty} g(\beta) = \infty$ since not all $y_i = 1$ almost surely. Thus $\lim_{\beta\to\infty} g(\beta) = \infty$ as sum of almost sure infinite positive quantities. Futhermore, for $\beta = 0$, $g(\beta) = -n < 0$. Consequently, the equation $g(\beta) = 0$ admits a unique positive root $\tilde{\beta}$ by virtue of the intermediate property of continuous functions.

The profile likelihood ratio for θ is then given by

$$PLR(\theta) = \frac{Lw(\theta, \tilde{\beta})}{Lw(\hat{\theta}, \hat{\beta})}, \tag{29}$$

where $(\hat{\theta}, \hat{\beta})$ denote the unrestricted maximum likelihood estimate of (θ, β). Thus $(1-p)100\%$ likelihood ratio confidence interval for θ is then given by the solution set of the inequation

$$PLR(\theta) > \exp - \left[\frac{\chi^2_{(1-p,1)}}{2}\right] \tag{30}$$

where $\chi^2_{(1-p,1)}$ is the $(1-p)100\%$ percentile of the chi-squared distribution whith one degree of freedom. □

7. Simulation results. To illustrate the performance of the conditional interval over the corresponding unconditional one and the likelihood ratio interval, we present simulation results obtained in the cases of $n = 50$, $\alpha = .05, 0.10, 0.20, 0.40, 0.80, 1.0$, $\Psi = 0.90(0.1)1$, $\beta = 5$ and $p = .10$. Note that the scale parameter θ is often set to unity in real life applications. We therefore took $\theta_0 = 1$ without loss of generality. Probability plots, as illustrated in Figure1, have confirmed the fit accuracy of the pivot (1) statistic by the Student variable with $n-1$ degrees of freedom for moderate and large sample sizes and for sufficiently large β in the case of small samples. We have therefore chosen $\beta = 5$. The chosen values for α and $\psi = 0.01(0.01)1$ account for small to large possible values. Now, under each possible value of the parameter space, we randomly selected 1000 times random samples of size n and for each sample we carried out the testing of H_0 and the computation of the conditional, unconditional and likelihood ratio confidence bounds in the case of rejection of H_0. The Newton-Raphson procedure using the starting point for the scale parameter advocated in Zanakis [18] has been successfully used to solve equations (4) and (5). Furthermore, the bisection method has been used for solving equations (18) and (19) in order to find the bounds of the conditional interval. The Newton-Raphson method has also been used to compute the bounds of the likelihood ratio interval. The results have shown that for $\alpha < 1$, the respective ratios of the coverage probability and length of the conditional interval and the likelihood ratio interval to the unconditional interval are just slightly smaller than unity, with equality, when $\alpha = 1$. Therefore, these ratios are not significantly affected by the parameter α. Furthermore, ψ values below .89 yielded similar lengths and coverage probabilities for the conditional and unconditional confidence intervals, see, Table1 in Appendix. The likelihood ratio confidence interval has a slighly smaller average length but also a smaller coverage probability with respect to the unconditional interval which always maintain a coverage probability close to the nominal level. However, when $.90 \leq \psi \leq 1.0$, the length of the conditional interval i s significantly smaller than the length of the unconditional one. Furthermore, the coverage probability of the conditional confidence interval is reasonably close to the nominal level when $\Psi < .95$ as illustrated in Table 1 displayed in Appendix. Thus the unconditional confidence interval does not always outperform the corresponding conditional interval. Note that the critical region $\mathcal{R}$ can also be expressed using the estimator $\hat{\Psi} = \dfrac{\theta_0}{\hat{\theta}}$ as

$$\mathcal{R} = \left\{ \hat{\Psi} : \hat{\Psi} < \exp\left[-\frac{ct_{n-1}(\alpha)}{\sqrt{n-1}\hat{\beta}} \right] \leq 1 \right\}$$

where the bound $\tilde{\Psi} = \exp\left[-\dfrac{ct_{n-1}(\alpha)}{\sqrt{n-1}\hat{\beta}} \right]$ increases rapidly towards 1 as n becomes large. We have noticed that the region over which the conditional interval performs very well corresponds to the vicinity of the point $\Psi = \tilde{\Psi}$ in the critical region $\mathcal{R}$. This agrees with the results of Chiou and Han [5, 6]. There is then indeed a gain in using the conditional interval based on the technique advocated in Meeks and D'Agostino [14]. Contrarily to the finding of Meeks and D'Agostino [14], the length of the obtained conditional confidence interval never exceeds the length of the unconditional confidence interval. On the other hand, the conditional interval also performs better than the likelihood ratio interval over the region $.90 \leq \Psi < .95$. However, as ψ increases from $\Psi = .95$, the coverage probability of the conditional interval decreases significantly and the unconditional interval and the likelihood ratio interval become very similar in terms of length and coverage probability. We recommand then to just use the unconditional interval in such a case since it is less computational. Consequently, we recommand to always use the unconditional confidence interval based on Bain and Engelhardt statistic expect when Ψ or its estimate $\hat{\Psi}$ is close to the critical point $\tilde{\Psi}$.

8. Conclusion. We have investigated here the conditional estimation by confidence interval of the scale parameter θ in Weibull model. The case of shape parameter will be treated in a separate paper. The confidence interval is constructed only after rejection of a one-sided preliminary test of significance for the null hypothesis $H_0 : \theta = \theta_0$. Conditional confidence bounds are obtained following the procedure set forth by Meeks and D'Agostino [14]. This interval is compared in terms of coverage probability and average length to the optimal corresponding unconditional interval and to the likelihood ratio confidence interval. The simulation study has also shown that the likelihood ratio interval, recommended in Meeker and Escobar for the estimation of the shape parameter, is not very appropriate for the estimation of the scale parameter. The coverage probability of the unconditional confidence interval, evaluated under the conditional distribution of the pivot statistic (1) is also given. It has been noticed that as the values of ψ move away from the vicinity of $\Psi = \tilde{\Psi}$ both intervals become similar in terms of average length and coverage probability. However, in the neighborhood of $\psi = \tilde{\Psi}$, the study has shown that the length of the conditional confidence interval is significantly smaller than the length of the unconditional one. Moreover, it has a reasonably high coverage probability. It is then worth using the conditional interval in such situations. The study has also shown that none of these two intervals outperforms completely the other over the whole parameter space. Therefore, the always use of unconditional confidence intervals independently of the preliminary test outcome may lead to inaccurate estimates.

The numerical study has been carried out with Gauss, SPSS and Mathematica [17].

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Appendix A – Table

Table 1. Empirical length and coverage probability ratios of 90% confidence intervals based on 1000 simulations. $\overline{LCCI}/\overline{LUCI}$ and $\overline{LLRCCI}/\overline{LUCI}$ represent, respectively, the average length ratios of the conditional confidence interval and the likelihood ratio confidence interval to the unconditional confidence interval. $\overline{CPCI}/\overline{CPUCI}$ and $\overline{CPLLRCI}/\overline{CPUCI}$ represent, respectively, the average coverage probability ratios of the conditional confidence interval and the likelihood ratio confidence interval to the unconditional confidence interval.

Ψ	$\overline{LCCI}/\overline{LUCI}$	$\overline{CPCI}/\overline{CPUCI}$	$\overline{LLRCCI}/\overline{LUCI}$	$\overline{CPLLRCI}/\overline{CPUCI}$
$\leq$ .89	.999	1.000	.910	.940
.90	.984	.999	.911	.940
.91	.960	.999	.911	.940
.92	.917	.997	.911	.940
.93	.860	.991	.920	.940
.94	.781	.954	.931	.940
.95	.711	.883	.959	.947
.96	.657	.611	.990	.974
.97	.617	.210	.911	1.001
.98	.602	.210	.999	.999
.99	.574	.210	.999	1.000
1.00	.560	.209	1.001	1.000

Appendix B – Figure

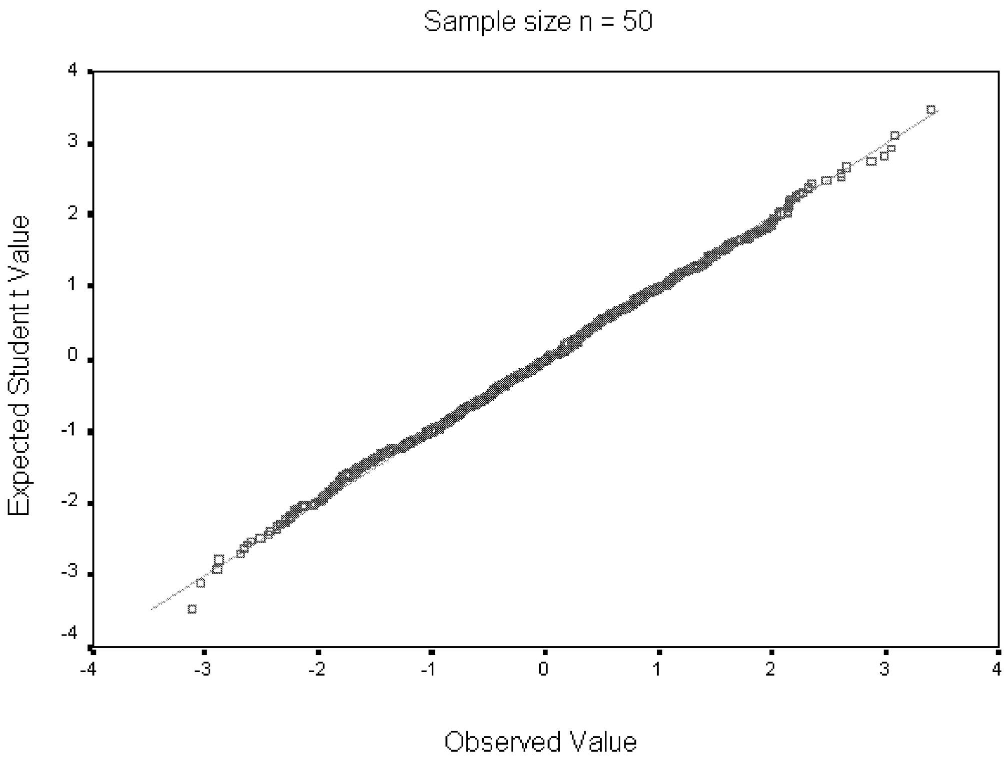

Fig. 1. Student t Q-Q Plot Pivot

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