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ZERO-DIMENSIONALITY AND SERRE RINGS

D. Karim

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Abstract. This paper deals with zero-dimensionality. We investigate the problem of whether a Serre ring $R < X >$ is expressible as a directed union of Artinian subrings. In particular, we show that $\prod_{\alpha\in A} (R_\alpha < X_\alpha >)$ is not a directed union of Artinian subrings, where $\{R_\alpha\}_{\alpha\in A}$ is an infinite family of zero–dimensional rings and each X_α is an indeterminate over R_α.

1. Introduction. All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of a ring S, we assume that the unity element of S belongs to R, and hence is the unity of R. We let Spec(R) and $Z(R)$, respectively, denote the spectrum of R (the set of prime ideals of R) and the set of zero–divisors of R. We use the term dimension of R, denoted dim$R$, to refer to the Krull dimension of R. Thus dim$R$ is the non–negative integer n if there exists a chain $P_o \subset P_1 \subset \ldots \subset P_n$ of proper prime ideals of R, but no longer such chain; if there is no upper bound on the lengths of such chains, we write dim$R = \infty$.

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This paper is concerned with zero–dimensionality in a Serre rings with coefficients in a commutative ring R, say $R < X >$. We investigate the problem of whether a Serre ring $R < X >$ is expressible as a directed union of Artinian subrings. We also show that the infinite product of $R_\alpha < X_\alpha >$, where $\{R_\alpha\}_{\alpha \in A}$ is a family of zero-dimensional rings and $\{X_\alpha\}_{\alpha \in A}$ is a family of indeterminates, is not a directed union of Artinian subrings. It is worth reminding that if R is a ring and X, Y are two indeterminates over R then:

(i) $R < X, Y >= R < Y, X >$ if and only if $dimR = 0$ [8, Theorem 17.12].

(ii) $R(X) = R < X >$ if and only if $dimR = 0$ [8, Theorem 17.11].

2. Serre rings as a directed unions of Artinian subrings.

Let R be a commutative ring and X an indeterminate over R. Let $R[X]$ be the ring of polynomials in one indeterminate over R. The set $N = \{f \in R[X] : f \text{ is monic}\}$ and $S = R[X] \setminus M[X]$, M ranges over all maximal ideals of R are two multiplicatively closed subsets of $R[X]$.

Let $R < X >= R[X]_N$ (resp., $R(X) = S^{-1}R[X]$) denote the Serre (resp., Nagata) ring with coefficients in R. We have $R[X] \subset R < X > \subset R(X))$ $R(X)$. The Serre ring in k indeterminates with coefficients in R is the ring $R < X_1, \ldots, X_k >= R < X_1, \ldots, X_{k-1} >< X_k >$. If R is zero–dimensional, then by [11, Theorem 2.1], dim$R < X >$=dim$R(X)$=dim$R[X] - 1$, and hence by [12, Theorem 2], $R < X >$ (resp., $R(X)$) is a zero–dimensional ring.

Let (R_j, f_{jk}) be a directed system of rings, indexed by a directed set $(I, \leq)$. Let $R = \bigcup_{j \in I} R_j$, together with the canonical maps $f_j : R_j \longrightarrow R$. The ring R is said to be a directed union of the R_j's if the f_{jk}'s are inclusion maps. Thus, directed unions can be treated by assuming all f_{jk} to be monomorphisms. If R_j is a ring for each $j \in I$, then R is also a ring. However, R need not be Noetherian even if each R_j is. If $R = \bigcup_{j \in I} R_j$ is a directed union of Artinian subrings, then we regard each R_i as a subring of R, i.e., it contains the same unity element.

Proposition 2.1. *Let* $\prod_{i=1}^{n} K_i$ *be a finite product of fields* K_i, *then* $(\prod_{i=1}^{n} K_i) < X > \simeq \prod_{i=1}^{n} (K_i < X >)$ *is a finite product of fields.*

Proof. Let K and L be two fields and X an indeterminate over both K and L. We show that $(K \times L) < X > \simeq K < X > \times L < X >= K(X) \times L(X)$.

It is well–known that $(K \times L)[X]$ is isomorphic to $K[X] \times L[X]$, then every polynomial of $(K \times L)[X]$ is seen as a pair of polynomials $(f_1, f_2) \in K[X] \times L[X]$. Let $\frac{g}{h} \in (K \times L) < X >$ then g and h are elements of $(K \times L)[X]$ with h a monic polynomial. We have $g = (g_1, g_2)$, where $g_1 \in K[X]$ and $g_2 \in L[X]$, and $h = (h_1, h_2)$, such that $h_1 \in K[X] \diagdown (0)$ and $h_2 \in L[X] \diagdown (0)$. Hence $\frac{g}{h} = \left(\frac{g_1}{h_1}, \frac{g_2}{h_2}\right) \in K < X > \times L < X >$. Therefore, we have $K < X > \times L < X > \simeq (K \times L) < X >$. □

We now present a result which will be useful later.

Lemma 2.2. *Let R be a zero–dimensional ring with a finite spectrum, then R is expressible as a finite product of zero–dimensional quasi–local subrings.*

P r o o f. Let $Spec(R) = \{M_1, \dots, M_1\}$, $\varphi_i : R \to R_{M_i}$, $\varphi_i(r) = \frac{r}{1}$ be the canonical homomorphism, $i = 1, \dots, n$ and $S_{M_i}(0) = Ker\varphi_i$ for each $i = 1, \dots, n$. Since $Rad(S_{M_i}(0)) = M_i$, $S_{M_i}(0)$ is a primary ideal. Note that $\bigcap_{i=1}^{n} S_{M_i}(0) = (0)$ and $S_{M_i}(0) + S_{M_j}(0) = R$, for each $i \neq j$ in { 1,. . . , n}. Therefore, $R \simeq \frac{R}{\bigcap\limits_{i=1}^{n} S_{M_i}(0)}$. By the Chinese remainder Theorem, $R \simeq \prod\limits_{i=1}^{n} \frac{R}{S_{M_i}(0)}$, where $\frac{R}{S_{M_i}(0)}$ is quasi–local and zero–dimensional, for $i = 1, \dots, n$. □

Notice that if R is a Von Neumann regular ring[1], then R is an Artinian ring if and only if R is a finite product of fields; if and only if R is Noetherian. Indeed, if R is a Von Neumann regular ring and Artinian then by [2, Corollary 8.2], Spec(R) is finite and hence $R = R_1 \oplus \ldots \oplus R_n$, where each R_i is a quasi–local and zero–dimensional ring, for $i = 1, \dots, n$. By [3, Result 3.2], each R_i is a Von Neumann regular ring. As R_i is a quasi–local ring, by [3, Theorem 3.1], R_i is a field for $i = 1, \dots, n$. It follows that R is a finite product of fields.

Proposition 2.3. *Let R be a ring and X an indeterminate over R. Then*

(1) *If R is a directed union of Artinian subrings then so is $R < X >$.*

(2) *If R is a reduced ring, then $R < X >$ is a directed union of Artinian subrings implies that R has the same property.*

P r o o f. (1) If $R = \bigcup_{i \in I} R_i$ is a directed union of Artinian subrings, then $R < X > = \bigcup_{i \in I}(R_i < X >)$. Since each R_i is Noetherian, by [10, (6.17)],

[1] R is reduced ((0) is the only nilpotent element of R) and R is zero–dimensional.

$R_i < X >$ is also Noetherian and each $R_i < X >$ is zero–dimensional as each R_i is zero–dimensional (cf.[1, Proposition 1.21]). By [2, Theorem 8.5], $R_i < X >$ is an Artinian ring for each $i \in I$. Since the family $\{R_i\}_{i\in I}$ is directed then so is $\{R_i < X >\}_{i\in I}$. Hence $R < X >$ is a directed union of Artinian subrings.

(2) If $R < X >= \bigcup_{j\in I} S_j$ is a directed union of Artinian subrings, by [5, Theorem 2.4 (a)], each $R_j = S_j \bigcap R$ is zero–dimensional. Since $R_j \subseteq S_j$ and $\mathrm{Spec}(S_j)$ is finite (cf.[2, Theorem 8.3]), this yields that each $\mathrm{Spec}(R_j)$ is finite. As R is reduced and by [3, Teorem 3.1], each R_j is a Von Neumann regular ring with finite spectrum. It follows that R_j is Artinian and hence $R = \bigcup_{j\in I} R_j$ is a directed union of Artinian subrings. □

Remark 2.4. (1) Let R be a hereditarily zero-dimensional ring, that is all subrings of R are zero-dimensional. Then R is a directed union of Artinian subrings. Therefore $R < X >$ is a directed union of Artinian subrings that is not hereditarily zero-dimensional, indeed, $R[X] \subset R < X >$ and $\dim(R[X]) = 1$ (cf. [12, Theorem 2]).

(2) Let R be a Von Neumann regular ring and $X_1, \ldots, X_n$ indeterminates over R. We denote $R < X_1, \ldots, X_n >= R < n >$, for each $n \in \mathbb{Z}^+$. Then by Proposition 2.3 and [8, Lemma 15.3], R is a directed union of Artinian subrings if and only if $R < n >$ is a directed union of Artinian subrings, for each $n \in \mathbb{Z}^+$.

(3) Let R be a ring and X a family of indeterminates over R. Then R is a directed union of zero-dimensional semi-quasilocal subrings if and only if $R < X >$ has the same property. This follows from the fact that $Spec(R < X >) = \{\mathfrak{m} < X >: \mathfrak{m} \in Spec(R)\}$ and hence $|Spec(R)| = |Spec(R < X >)|$.

If $x \in N(R)$[2], we denote by $\eta(x)$ the index of nilpotency of x, that is, $\eta(x) = k$ if $x^k = 0$ but $x^{k-1} \neq 0$. We define $\eta(R)$ to be $\mathrm{Sup}\{\eta(x) : x \in N(R)\}$; if the set $\{\eta(x) : x \in N(R)\}$ is unbounded, then we write $\eta(R) = \infty$. From [6, Theorem 3.4], we have $\dim \prod_{\alpha\in A} T_\alpha = 0$ if and only if $\{\alpha \in A : \eta(T_\alpha) > k\}$ is finite for some $k \in \mathbb{Z}^+$, where $\{T_\alpha\}_{\alpha\in A}$ is a family of zero–dimensional rings.

Proposition 2.5. *Let R be a ring and X an indeterminate over R, then $\eta(R[X]) = \eta(R)$.*

P r o o f. Assume that $\eta(R) < k$. The inequality $\eta(R) \leq \eta(R[X])$ is clear. Let $f = a_nX^n + \ldots + a_1X + a_o \in N(R[X]) = N(R)[X]$. We denote by $I_f = (a_o, \ldots, a_n)$ the ideal of R generated by the coefficients of f. We claim that $f^l = 0$ for some $l \in \mathbb{Z}^+$. Let $y \in I_f$, then $y = \sum_{i=0}^{n} r_ia_i$, where $r_i \in R$, $i = 0, \ldots, n$. Since $a_i \in N(R)$ for each i, we have $y \in N(R)$ and hence there exists $e \in \mathbb{Z}^+$

[2] $N(R) = \{x \in R :$ there exists $k \in \mathbb{Z}^+$ such that $x^k = 0\}$.

such that $e < k$ and $y^e = 0$. Therefore, there exists a positive integer $l < k$ such that $I_f^l = 0$. If S is a ring, it is well-known that $I[X]J[X] \subseteq (IJ)[X]$, where I and J are two ideals of S. It follows that $(I_f[X])^l \subseteq I_f^l[X]$. In other words, $(I_f[X])^l = 0$. We conclude that $f^l = 0$. Thus, $\eta(R[X]) = \eta(R)$.

Theorem 2.6. *Let $\{R_\alpha\}_{\alpha\in A}$ be an infinite family of Von Neumann regular rings and X_α an indeterminate over R_α, for each $\alpha \in A$. Then $\prod_{\alpha\in A}(R_\alpha < X_\alpha >)$ is not a directed union of Artinian subrings.*

Before proving this Theorem, we establish the following Lemmas.

Lemma 2.7. *Let R be a ring and U a multiplicatively closed subset of R. If R is reduced then $U^{-1}R$ is also reduced.*

Proof. Let $r/s \in N(U^{-1}R)$, where $N(U^{-1}R)$ is the nilradical of $U^{-1}R$. Then there exists $n_o \in \mathbb{N}^*$ such that $(r/s)^{n_o} = 0$; there exists $u \in U$ such that $r^{n_o}u = 0$, i.e., $(ru)^{n_o} = 0$, since R is reduced, we have $ru = 0$, and hence $r/s = 0$. In other words, $N(U^{-1}R) = (0)$ and $U^{-1}R$ is reduced. □

Lemma 2.8. *Let R be a ring and X an indeterminate over R. The following statements are equivalent:*

(i) R is reduced;

(ii) $R < X >$ is reduced.

Proof. (i) ⇒ (ii). Let $f = a_nX^n + \ldots + a_1X + a_o \in N(R[X])$. So there exists $l \in \mathbb{Z}^+$ such that $f^l = 0$. Therefore, $a_n^l = a_{n-1}^l = \ldots = a_o^l = 0$. Since R is reduced, we have $a_n = a_{n-1} = \cdots = a_o = 0$, and hence $f = 0$. It follows that $N(R[X]) = (0)$. By Lemma 2.7, $R < X >$ is also reduced because $R < X > = R[X]_N$ is a localization of $R[X]$.
(ii) ⇒ (i). It follows from the fact that every subring of a reduced ring is reduced. □

We can replace (ii) by $R < n >$ is reduced, since $R < n > = R < n-1 >< X >$, for each $n \in \mathbb{Z}^+$ (cf. [8, Lemma 15.3]).

Proof of Theorem 2.6. By [6, Theorem 6.7], $\prod_{\alpha\in A}(R_\alpha < X_\alpha >)$ is a directed union of Artinian subrings if and only if there exists $k \in \mathbb{Z}^+$ such that $\{\alpha \in A :$ there exists $M \in Spec(R_\alpha < X_\alpha >)$ with $|R_\alpha/M| > k\}$ is finite. It was shown, [10, (6.17)], that $Spec(R_\alpha < X_\alpha >) = \{M < X_\alpha >: M \in Spec(R_\alpha)\}$ for each $\alpha \in A$. Furthermore, $R_\alpha < X > /M_\alpha < X_\alpha > \simeq (R_\alpha/M_\alpha) < X_\alpha >$ and $(R_\alpha/M_\alpha) < X_\alpha >$ is at least a countably denumerable field. Therefore, for each

$\alpha \in A$ and for each $M_\alpha \in Spec(R_\alpha)$, we have for each $k \in \mathbb{Z}^+$ $|(R_\alpha/M_\alpha) < X_\alpha > | > k$. Then $\{\alpha \in A :$ there exists $M_\alpha \in Spec(R_\alpha)$ with $|(R_\alpha/M_\alpha) < X_\alpha > | > k\}$ is infinite for each $k \in \mathbb{Z}^+$. Thus, $\prod_{\alpha\in A}(R_\alpha < X_\alpha >)$ is not a directed union of Artinian subrings. $\square$

Remark 2.9. (i) Let $\{R_\alpha\}_{\alpha\in A}$ be an infinite family of Von Neumann regular rings and X an indeterminate. Then $\prod_{\alpha\in A}(R_\alpha < X >)$ is not a directed union of Artinian subrings.
(ii) Let $\{R_\alpha\}_{\alpha\in A}$ be a family of zero–dimensional rings (not Von Neumann regular rings) such that $\dim(\prod_{\alpha\in A} R_\alpha) = 0$, and X an indeterminate over $\prod_{\alpha\in A} R_\alpha$. Assume that each R_α is a directed union of Artinian subrings. By Proposition 2.3, $R_\alpha < X >$ is also a directed union of Artinian subrings for each $\alpha \in A$. Now, suppose that $\prod_{\alpha\in A} R_\alpha$ is a directed union of Artinian subrings, by [6, Theorem 6.7], there exists $k \in \mathbb{Z}^+$ such that $\{\alpha \in A : \eta(R_\alpha) > k$ or there exists $M \in Spec(R_\alpha) : |R_\alpha/M| > k\}$ is finite. By Proposition 2.5, $\eta(R_\alpha) = \eta(R_\alpha < X >$ since $R_\alpha < X >= R[X]_U$, where $U = \{f \in R[X] : f \text{ is monic}\}$. It follows that $\{\alpha \in A : \eta(R_\alpha < X >) > k\}$ is finite, from [6, Theorem 3.4], $dim \prod_{\alpha\in A}(R_\alpha < X >) = 0$. Suppose that $\left|\frac{R_\alpha}{M}\right| < k$ for each $M \in Spec(R_\alpha)$ and for each $\alpha \in A$. Therefore, $\frac{R_\alpha}{M} < X >$ is an infinite field for each $M \in Spec(R_\alpha)$ and each $\alpha \in A$. According to [6, Theorem 6.7], $\prod_{\alpha\in A}(R_\alpha < X >)$ is not a directed union of Artinian subrings.
(iii) If R is a zero–dimensional ring, then the Serre ring $R < X >$ is equal to the Nagata ring $R(X)$. To show this result, it is enough to prove that $S \subseteq N$, where S and N are sets defined in the introduction. Let $f \in S$ with $f = a_0 + a_1X + \ldots + a_nX^n$ and P be a prime ideal of R such that $a_{j+1}, \ldots, a_n \in P$ and $a_j \notin P$ for some $j \in \{1, \ldots, n\}$. Then $\overline{a}_j \in \frac{R}{P}\setminus\{\overline{0}\}$ and hence $\overline{a}_j$ is a unity element of $\frac{R}{P}$. By [7, Theorem 7], $f \in N$. Thus, $S \subseteq N$.

Example 2.10. Let p be a prime integer and X an indeterminate over $GF(p)$, where $GF(p)$ is the Galois field with p elements. Let $R = (GF(p) < X >)^{\omega_o}$ be a countable direct product of copies of $GF(p) < X >$. We note that R is a Von Neumann regular ring as a direct product of fields. By Theorem 2.6, the ring R is not a directed union of Artinian subrings. Let $\mathcal{S} = \{\{x_i\}_{i=1}^{\infty} \in R : \{x_i\}_{i=1}^{\infty}$ has only finitely many components$\}$. By [6, Proposition 5.1], the ring $\mathcal{S}$ is a directed union of Artinian subrings.

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Department of Mathematics
Faculty of Sciences Semlalia
P. O. Box 2390
Marrakech-Morocco
e-mail: ikarim@ucam.ac.ma

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