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INVOLUTION MATRIX ALGEBRAS – IDENTITIES AND GROWTH

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Abstract. The paper is a survey on involutions (anti-automorphisms of order two) of different kinds. Starting with the first systematic investigations on involutions of central simple algebras due to Albert the author emphasizes on their basic properties, the conditions on their existence and their correspondence with structural characteristics of the algebras.

Focusing on matrix algebras a complete description of involutions of the first kind on $M_n(F)$ is given. The full correspondence between an involution of any kind for an arbitrary central simple algebra A over a field F of characteristic 0 and an involution on $M_n(A)$ specially defined is studied.

The research mainly in the last 40 years concerning the basic properties of involutions applied to identities for matrix algebras is reviewed starting with the works of Amitsur, Rowen and including the newest results on the topic. The cocharactes, codimensions and growth of algebras with involutions are considered as well.

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I. History and background. An involution is an anti-automorphism of order two of an algebra. The most elementary example is the transpose for matrix algebras. A more complicated example of an algebra over $\mathbf{Q}$ admitting an involution is the multiplication algebra of a Riemann surface. The motivation of Albert for the first systematic investigations is connected with the genus g of such a surface and a complex $g \times 2g$ matrix of periods P for which there exists a nonsingular alternating matrix $C \in M_{2g}(\mathbf{Q})$ such that $PCP^t = 0$ and $iPC\overline{P}^t$ is positive definite hermitian. We will not go into details of algebraic geometry. We only mention that the genus of a surface is a numerical birational invariant of an algebraic variety of dimension two over an algebraically closed field. The study of correspondences on the Riemann surface leads to considering the matrices $M \in M_{2g}(\mathbf{Q})$ for which there exists a matrix $K \in M_g(\mathbf{Q})$ such that $KP = PM$. They form a subalgebra of $M_{2g}(\mathbf{Q})$ known as the *multiplication algebra*. As observed by [52] and [57] this algebra admits an involution $X \to C^{-1}X^tC$. Albert completely determined the structure of the multiplication algebra in *Annals of Mathematics* in 1934–1935, an improved version of it is [1].

The central problem completely solved by Albert is to give necessary and sufficient conditions on a division algebra over $\mathbf{Q}$ to be a multiplication algebra. To achieve this, Albert developed a theory of central simple algebras with involution, based on the theory of simple algebras initiated a few years earlier by Brauer, Noether, and also Albert and Hasse, and gave a complete classification over $\mathbf{Q}$.

We expose the main contributions of the theory of central simple algebras with involution following [35], Chapter I.

1. Bilinear forms. A bilinear form $b : V \times V \to F$ on a finite dimensional vector space V over an arbitrary field F is called *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in V$, *skew-symmetric* if $b(x, y) = -b(y, x)$ for all $x, y \in V$ and *alternating* if $b(x, x) = 0$ for all $x \in V$. Thus, the notion of skew-symmetric and alternating (resp. symmetric) forms coincide if char $F \neq 2$ (resp. char $F = 2$). Alternating forms are skew-symmetric in every characteristic.

If b is a symmetric or alternating bilinear form on a (finite dimensional) vector space V, the induced map

$$\hat{b} : V \to V^* = \mathrm{Hom}_F(V, F)$$

is defined by $\hat{b}(x)(y) = b(x, y)$ for $x, y \in V$. The bilinear form b is *nonsingular* (*or regular, or nondegenerate*) if $\hat{b}$ is bijective. Alternately, b is nonsingular if and only if the determinant of its Gram matrix with respect to an arbitrary basis of

V is nonzero:

$$\det\left(b(e_i, e_j)\right)_{1\le i,j\le n} \neq 0.$$

The square class of this determinant is called the *determinant* of b:

$$\det\, b = \det\left(b(e_i, e_j)\right)_{1\le i,j\le n}.F^{\times 2} \in F^{\times}/F^{\times 2},$$

where (and up to the end) the notation $F^{\times}$ stands for $F \setminus \{0\}$.

The *discriminant* of b is the signed determinant:

$$\operatorname{disc} b = (-1)^{n(n-1)/2} \det\, b \in F^{\times}/F^{\times 2},$$

where $n = \dim V$. If $b : V \times V \to F$ is a symmetric bilinear form, we denote by $q_b : V \to F$ the associated quadratic map, defined by

$$q_b(x) = b(x, x) \quad \text{for } \ x, y \in V.$$

For any $f \in \operatorname{End}_F(V)$ we may define $\sigma_b(f) \in \operatorname{End}_F(V)$ by

$$\sigma_b(f) = \hat{b}^{-1}\ f^t\ \hat{b},$$

where $f^t \in \operatorname{End}_F(V^*)$ is the *transpose* of f, defined by mapping $\varphi \in V^*$ to $\varphi\ f$. Alternately, $\sigma_b(f)$ may be defined by the following property:

$$b(x, f(y)) = b(\sigma_b(f)(x), y) \text{ for } x, y \in V.$$

The map $\sigma_b : \operatorname{End}_F(V) \to \operatorname{End}_F(V)$ is then an anti-automorphism of $\operatorname{End}_F(V)$ which is known as the *adjoint anti-automorphism* with respect to the nonsingular bilinear form b. The map σ_b is F-linear.

The basic result in [35] is the following.

Theorem 1.1. *The map which associates to each nonsingular bilinear form b on V its adjoint anti-automorphism σ_b induces a one-to-one correspondence between equivalence classes of nonsingular bilinear forms on V modulo multiplication by a factor in $F^{\times}$ and linear anti-automorphisms of* $\operatorname{End}_F(V)$*. Under this correspondence, F-linear involutions on* $\operatorname{End}_F(V)$ *correspond to nonsingular bilinear forms which are either symmetric or skew-symmetric.*

2. Central simple algebras. For any finite-dimensional algebra A with 1 over a field F and any field extension K/F, we write A_K for the K-algebra obtained from A by extending scalars to K:

$$A_K = A \otimes_F K.$$

We also define the *opposite algebra* A^{op} by

$$A^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in A\},$$

with the operations defined as follows:

$$a^{\mathrm{op}} + b^{\mathrm{op}} = (a+b)^{\mathrm{op}},\ a^{\mathrm{op}}b^{\mathrm{op}} = (ba)^{\mathrm{op}},\ \alpha \cdot a^{\mathrm{op}} = (\alpha \cdot a)^{\mathrm{op}}$$

for $a, b \in A$ and $\alpha \in F$.

A *central simple algebra* over a field F is a (finite dimensional) algebra $A \neq \{0\}$ with centre $F(= F \cdot 1)$ which has no two-sided ideals except $\{0\}$ and A. An algebra $A \neq \{0\}$ is a *division algebra* (or a *skew field*) if every non-zero element in A is invertible.

The structure of these algebras is determined by the following

Theorem 1.2 (Wedderburn)**.** *For an algebra A over a field F, the following conditions are equivalent:*
(1) *A is central simple.*
(2) *The canonical map $A \otimes_F A^{\mathrm{op}} \to \mathrm{End}_F(A)$ which associates to $a \otimes b^{\mathrm{op}}$ the linear map $x \to axb$ is an isomorphism.*
(3) *There is a field K containing F such that A_K is isomorphic to a matrix algebra over K, i.e. $A_K \simeq M_n(K)$ for some n.*
(4) *If Ω is an algebraically closed field containing F,*

$$A_\Omega \simeq M_n(\Omega) \ \textit{for some } n.$$

(5) *There is a finite dimensional central division algebra D over F and an integer r such that $A \simeq M_r(D)$.*
Moreover, if these conditions hold, all the simple left (or right) A-modules are isomorphic, and the division algebra D is uniquely determined up to an algebra isomorphism as $D = \mathrm{End}_A(M)$ for any simple left A-module M.

The fields K for which condition (3) holds are called *splitting fields of A*. Accordingly, the algebra A is called *split* if it is isomorphic to a matrix algebra $M_n(F)$ (or to $\mathrm{End}_F(V)$ for some vector space V over F).

Since the dimension of an algebra does not change under an extension of scalars, it follows from the above theorem that the dimension of every central simple algebra is a square: $\dim_F A = n^2$ if $A_K \simeq M_n(K)$ for some extension K/F. The integer n is called the *degree* of A and is denoted by $\deg A$. The degree of the division algebra D in condition (5) is called the *index* of A and is denoted by $\mathrm{ind}\, A$. Alternately, the index of A can be defined by the relation

$$\deg A \cdot \mathrm{ind}\, A = \dim_F M,$$

where M is any simple left module over A. This relation readily follows from the fact that if $A \simeq M_r(D)$, then D^r is a simple left module over A (via matrix multiplication, writing the elements of D^r as column vectors).

Let Ω denote an algebraic closure of F. Under scalar extension to Ω, every central simple F-algebra A of degree n becomes isomorphic to $M_n(\Omega)$. We may therefore fix an F-algebra embedding $A \hookrightarrow M_n(\Omega)$ and view every element $a \in A$ as a matrix in $M_n(\Omega)$. Its characteristic polynomial has coefficients in F and is independent of the embedding of A in $M_n(\Omega)$. It is called the *reduced characteristic polynomial* of A and is denoted

$$\mathrm{Prd}_{A,a}(X) = X^n - s_1(a)X^{n-1} + s_2(a)X^{n-2} - \cdots + (-1)^n s_n(a).$$

The *reduced trace* and *reduced norm* of a are denoted $\mathrm{Trd}_A(a)$ and $\mathrm{Nrd}_A(a)$:

$$\mathrm{Trd}_A(a) = s_1(a), \;\; \mathrm{Nrd}_A(a) = s_n(a).$$

Proposition 1.3. *The bilinear form $T_A : A \times A \to F$ defined by*

$$T_A(x, y) = \mathrm{Trd}_A(xy) \;\; \textit{for } x, y \in A$$

is nonsingular.

3. Involutions of the first kind. An *involution* on a central simple algebra A over a field F is a map $\sigma : A \to A$ subject to the following conditions:
(a) $\sigma(x + y) = \sigma(x) + \sigma(y)$ for $x, y \in A$,
(b) $\sigma(xy) = \sigma(y)\sigma(x)$ for $x, y \in A$,
(c) $\sigma^2(x) = x$ for $x \in A$.

Though the map σ is not required to be F-linear it is easily checked that the centre $F(= F \cdot 1)$ is preserved under σ. The restriction of σ to F is therefore an automorphism which is either the identity or of order 2. Involutions which leave the centre elementwise invariant are called *involutions of the first kind.* Involutions whose restriction to the centre is an automorphism of order 2 are called *involutions of the second kind.*

Proposition 1.4. *Let (A, σ) be a central simple F-algebra of degree n with involution of the first kind and let L be a splitting field of A. Let V be an L-vector space of dimension n. There is a nonsingular symmetric or skew-symmetric bilinear form b on V and an invertible matrix $g \in GL_n(L)$ such that $g^t = g$ if b is symmetric and $g^t = -g$ if b is skew-symmetric, and*

$$(A_L, \sigma_L) \simeq (\mathrm{End}_L(V), \sigma_b) \simeq (M_n(L), \sigma_g).$$

Corollary 1.5. *For all $a \in A$, the elements a and $\sigma(a)$ have the same reduced characteristic polynomial. In particular,* $\mathrm{Trd}_A(\sigma(a)) = \mathrm{Trd}_A(a)$ *and* $\mathrm{Nrd}_A(\sigma(a)) = \mathrm{Nrd}_a(a)$.

Two types of involutions of the first kind can be distinguished which correspond to symmetric and to alternating forms. This distinction is made on the basis of properties of symmetric elements.

In a central simple F-algebra A with involution of the first kind σ, the sets of symmetric, skew-symmetric, symmetrized and alternating elements in A are defined as follows:

$$\begin{aligned} S(A,\sigma) = \mathrm{Sym}(A,\sigma) &= \{a \in A \mid \sigma(a) = a\}, \\ K(A,\sigma) = \mathrm{Skew}(A,\sigma) &= \{a \in A \mid \sigma(a) = -a\}, \\ \mathrm{Symd}(A,\sigma) &= \{a + \sigma(a) \mid a \in A\}, \\ \mathrm{Alt}(A,\sigma) &= \{a - \sigma(a) \mid a \in A\}. \end{aligned}$$

If $\mathrm{char}\, F \neq 2$, then $\mathrm{Symd}(A,\sigma) = \mathrm{Sym}(A,\sigma)$, $\mathrm{Alt}(A,\sigma) = \mathrm{Skew}(A,\sigma)$ and $A = \mathrm{Sym}(A,\sigma) \oplus \mathrm{Skew}(A,\sigma)$ since every element $a \in A$ decomposes as $a = \frac{1}{2}(a + \sigma(a)) + \frac{1}{2}(a - \sigma(a))$. If $\mathrm{char}\, F = 2$, then $\mathrm{Symd}(A,\sigma) = \mathrm{Alt}(A,\sigma) \subset \mathrm{Skew}(A,\sigma) = \mathrm{Sym}(A,\sigma)$.

Lemma 1.6. *Let $n = \deg A$, then* $\dim \mathrm{Sym}(A,\sigma) + \dim \mathrm{Alt}(A,\sigma) = n^2$. *Moreover,* $\mathrm{Alt}(A,\sigma)$ *is the orthogonal space of* $\mathrm{Sym}(A,\sigma)$ *for the bilinear form T_A on A induced by the reduced trace:*

$$\mathrm{Alt}(A,\sigma) = \{a \in A \mid \mathrm{Trd}_A(as) = 0 \textit{ for } s \in \mathrm{Sym}(A,\sigma)\}.$$

Similarly, $\dim \mathrm{Skew}(A,\sigma) + \dim \mathrm{Symd}(A,\sigma) = n^2$, *and* $\mathrm{Symd}(A,\sigma)$ *is the orthogonal space of* $\mathrm{Skew}(A,\sigma)$ *for the bilinear form T_A.*

In arbitrary characteristic, the property of b being symmetric or skew-symmetric or alternating depends only on the involution and not on the choice of L nor of b. We may set the following definition:

Definition 1.7. *An involution σ of the first kind is said to be of symplectic type (or simply* symplectic*) if for any splitting field L and any isomorphism $(A_L, \sigma_L) \simeq (\mathrm{End}_L(V), \sigma_b)$, the bilinear form b is alternating; otherwise it is called* of orthogonal type *(or simply* orthogonal*). In the latter case, for any splitting field L and any isomorphism $(A_L, \sigma_L) \simeq (End_L(V), \sigma_b)$, the bilinear form b is symmetric (and notalternating).*

Proposition 1.8. *Let* (A,σ) *be a central simple* F*-algebra of degree* n *with involution of the first kind.*
(1) *Suppose that* $\operatorname{char} F \neq 2$, *hence* $\operatorname{Symd}(A,\sigma) = \operatorname{Sym}(A,\sigma)$ *and* $\operatorname{Alt}(A,\sigma) = \operatorname{Skew}(A,\sigma)$. *If* σ *is of orthogonal type, then*

$$\dim_F \operatorname{Sym}(A,\sigma) = \frac{n(n+1)}{2} \quad \textit{and} \quad \dim_F \operatorname{Skew}(A,\sigma) = \frac{n(n-1)}{2}.$$

If σ *is of symplectic type, then*

$$\dim_F \operatorname{Sym}(A,\sigma) = \frac{n(n-1)}{2} \quad \textit{and} \quad \dim_F \operatorname{Skew}(A,\sigma) = \frac{n(n+1)}{2}.$$

Moreover, in this case n *is necessarily even.*
(2) *Suppose that* $\operatorname{char} F = 2$, *hence* $\operatorname{Sym}(A,\sigma) = \operatorname{Skew}(A,\sigma)$ *and* $\operatorname{Symd}(A,\sigma) = \operatorname{Alt}(A,\sigma)$. *Then*

$$\dim_F \operatorname{Sym}(A,\sigma) = \frac{n(n+1)}{2} \quad \textit{and} \quad \dim_F \operatorname{Alt}(A,\sigma) = \frac{n(n-1)}{2}.$$

The following proposition highlights a special feature of symplectic involutions.

Proposition 1.9. *Let* A *be a central simple* F*-algebra with involution* σ *of symplectic type. The reduced characteristic polynomial of every element in* $\operatorname{Symd}(A,\sigma)$ *is a square. In particular,* $\operatorname{Nrd}_A(s)$ *is a square in* F *for all* $s \in \operatorname{Symd}(A,\sigma)$.

4. Involutions of the second kind. In the case of involutions of the second kind on a simple algebra B, the base field F is usually not the centre of the algebra, but the subfield of central invariant elements which is of codimension 2 in the centre. Under scalar extension to an algebraic closure of F, the algebra B decomposes into a direct product of two simple factors. We consider a finite dimensional F-algebra with a centre K which is either simple (if K is a field) or a direct product of two simple algebras (if $K \simeq F \times F$). We denote by ι the nontrivial automorphism of K/F and by τ an involution of the second kind on B, whose restriction to K is ι. A homomorphism $f : (B,\tau) \to (B',\tau')$ is an F-algebra homomorphism $f : B \to B'$ such that $\tau' \, f = f \, \tau$.

Proposition 1.10. *If* $K \simeq F \times F$, *there is a central simple* F*-algebra* E *such that*

$$(B,\tau) \simeq (E \times E^{\mathrm{op}}, \varepsilon),$$

where the involution ε is defined by $\varepsilon(x, y^{\mathrm{op}}) = (y, x^{\mathrm{op}})$. This involution is called the exchange involution.

We may define the *degree* of the central simple F-algebra (B, τ) with involution of the second kind by

$$\begin{aligned} \deg(B,\tau) &= \deg B \text{ if } K \text{ is a field,} \\ \deg(B,\tau) &= \deg E \text{ if } K \simeq F \times F \text{ and } (B,\tau) \simeq (E \times E^{\mathrm{op}}, \varepsilon). \end{aligned}$$

Proposition 1.11. *Suppose that the centre K of B is a field. There is a canonical isomorphism of K-algebras with involution*

$$\varphi : (B_K, \tau_K) \to (B \times B^{\mathrm{op}}, \varepsilon),$$

which maps $b \otimes \alpha$ to $(b\alpha, (\tau(b)\alpha)^{\mathrm{op}})$ for $b \in B$ and $\alpha \in K$.

Corollary 1.12. *For every $b \in B$, the reduced characteristic polynomials of b and $\tau(b)$ are related by*

$$\mathrm{Prd}_{B,\tau(b)} = \iota(\mathrm{Prd}_{B,b}) \text{ in } K[X].$$

In particular, $\mathrm{Trd}_B(\tau(b)) = \iota(\mathrm{Trd}_B(b))$ *and* $\mathrm{Nrd}_B(\tau(b)) = \iota(\mathrm{Nrd}_B(b))$.

As for involutions of the first kind, we may define the sets of symmetric, skew-symmetric, symmetrized and alternating elements in (B, τ). They are vector spaces over F. In contrast with the case of involutions of the first kind, there is a straightforward relation between symmetric, skew-symmetric and alternating elements.

Proposition 1.13. $\mathrm{Symd}(B,\tau) = \mathrm{Sym}(B,\tau)$ *and* $\mathrm{Alt}(B,\tau) = \mathrm{Skew}(B,\tau)$ *for any $\alpha \in K^{\times}$ such that $\tau(\alpha) = -\alpha$,*

$$\mathrm{Skew}(B,\tau) = \alpha \cdot \mathrm{Sym}(B,\tau).$$

If $\deg(B,\tau) = n$, *then*

$$\dim_F \mathrm{Sym}(B,\tau) = \dim_F \mathrm{Skew}(B,\tau) = \dim_F \mathrm{Symd}(B,\tau) = \dim_F \mathrm{Alt}(B,\tau) = n^2.$$

As for the involutions of the first kind, all the involutions of the second kind on B which have the same restriction to K as τ are obtained by composing τ with an inner automorphism.

Proposition 1.14. *Let (B, τ) be a central simple F-algebra with involution of the second kind, and let K denote the centre of B.*
(1) *For every unit $u \in B^\times$ such that $\tau(u) = \lambda u$ with $\lambda \in K^\times$, the map* $\mathrm{Int}(u)\ \tau$ *is an involution of the second kind on B.*
(2) *Conversely, for every involution τ' on B whose restriction to K is ι, there exists some $u \in B^\times$, uniquely determined up to a factor in $F^\times$, such that*

$$\tau' = \mathrm{Int}(u)\ \tau \ \textit{ and } \ \tau(u) = u.$$

In this case,

$$\mathrm{Sym}(B, \tau') = u \cdot \mathrm{Sym}(B, \tau) = \mathrm{Sym}(B, \tau) \cdot u^{-1}.$$

5. Matrix algebras. The discussion before Theorem 1.1 and the theorem itself could give much information since the choice of a basis in an n-dimensional vector space V over F yields an isomorphism $\mathrm{End}_F(V) \simeq M_n(F)$. However, matrix algebras are endowed with a canonical involution of the first kind, namely the transpose involution t. Therefore a complete description of involutions of the first kind on $M_n(F)$ can also be easily derived from Proposition 1.9.

Proposition 1.15. *Every involution of the first kind σ on $M_n(F)$ is of the form*

$$\sigma = \mathrm{Int}(u)\ t$$

for some $u \in GL_n(F)$, uniquely determined up to a factor in $F^\times$, such that $u^t = \pm u$. Moreover, the involution σ is orthogonal if $u^t = u$ and $u \notin \mathrm{Alt}(M_n(F), t)$, and it is symplectic if $u \in \mathrm{Alt}(M_n(F), t)$.

If $M_n(F)$ is identified with $\mathrm{End}_F(F^n)$, the involution $\sigma = \mathrm{Int}(u)\ t$ is the adjoint involution with respect to the nonsingular form b on F^n defined by

$$b(x, y) = x^t \cdot u^{-1} \cdot y \ \text{ for } x, y \in F^n.$$

Suppose now that A is an arbitrary central simple algebra over a field F and $^-$ is an involution (of any kind) on A. We define an involution $*$ on $M_n(A)$ by

$$(a_{ij})^*_{1 \le i,j \le n} = (\overline{a_{ij}})^t_{1 \le i,j \le n}.$$

Proposition 1.16. *The involution $*$ is of the same type as $^-$. Moreover, the involutions σ on $M_n(A)$ such that $\sigma(\alpha) = \overline{\alpha}$ for all $\alpha \in F$ can be described as*

follows:
(1) *If* $^-$ *is of the first kind, then every involution of the first kind on* $M_n(A)$ *is of the form* $\sigma = \mathrm{Int}(u)^*$ *for some* $u \in GL_n(A)$*, uniquely determined up to a factor in* $F^\times$*, such that* $u^* = \pm u$*. If* char $F \neq 2$*, the involution* $\mathrm{Int}(u)^*$ *is of the same type as* $^-$ *if and only if* $u^* = u$*. If* char $F = 2$*, the involution* $\mathrm{Int}(u)^*$ *is symplectic if and only if* $u \in Alt(M_n(A),^*)$*.*
(2) *If* $^-$ *is of the second kind, then every involution of the second kind* σ *on* $M_n(A)$ *such that* $\sigma(\alpha) = \overline{\alpha}$ *for all* $\alpha \in F$ *is of the form* $\sigma = \mathrm{Int}(u)^*$ *for some* $u \in GL_n(A)$*, uniquely determined up to a factor in* $F^\times$ *invariant under* $^-$*, such that* $u^* = u$*.*

6. Lie and Jordan structures. Every associative algebra A over an arbitrary field F is endowed with a Lie algebra structure for the bracket $[x, y] = xy - yx$. We denote this Lie algebra by $L(A)$. Similarly, if char $F \neq 2$, a Jordan algebra can be defined on A by $x \circ y = \frac{1}{2}(xy + yx)$. If A is viewed as a Jordan algebra for the product $\circ$, we denote it by A^+.

The relevance of the Lie and Jordan structures for algebras with involution stems from the observation that for every algebra with involution (A, σ) (of any kind), the spaces Skew (A, σ) and Alt (A, σ) are Lie subalgebras of $L(A)$, and the space Sym (A, σ) is a Jordan subalgebra of A^+ if char $F \neq 2$. For $x, y \in \mathrm{Skew}(A, \sigma)$ we have

$$[x, y] = xy - \sigma(xy) \in \mathrm{Alt}(A, \sigma) \subset \mathrm{Skew}(A, \sigma),$$

hence $\mathrm{Alt}(A, \sigma)$ and $\mathrm{Skew}(A, \sigma)$ are Lie subalgebras of $L(A)$. On the other hand, for $x, y \in \mathrm{Sym}(A, \sigma)$

$$x \circ y = \frac{1}{2}(xy + \sigma(xy)) \in \mathrm{Sym}(A, \sigma),$$

hence $\mathrm{Sym}(A, \sigma)$ is a Jordan subalgebra of A^+. This Jordan subalgebra is usually denoted by $H(A, \sigma)$.

The algebra $\mathrm{Skew}(A, \sigma)$ is the kernel of the Lie algebra homomorphism

$$\mu : g(A, \sigma) \to F$$

defined by $\mu(a) = a + \sigma(a)$, for $a \in g(A, \sigma)$. The map μ is surjective, except when char $F = 2$ and σ is orthogonal, since the condition $1 \in \mathrm{Symd}(A, \sigma)$ characterizes symplectic involutions among involutions of the first kind in characteristic 2, and $\mathrm{Symd}(A, \sigma) = \mathrm{Sym}(A, \sigma)$ if σ is of the second kind. Thus, $g(A, \sigma) = \mathrm{Skew}(A, \sigma)$

if σ is orthogonal and char $F = 2$, and dim $g(A,\sigma) =$ dim Skew$(A,\sigma) + 1$ in the other cases.

Proposition 1.17. (1) *Let (A,σ) and (A',σ') be central simple F-algebras with involution of the first kind and let L/F be a field extension. Suppose that* deg $A > 2$ *and let*

$$f:\ \mathrm{Alt}(A,\sigma) \to \mathrm{Alt}(A',\sigma')$$

be a Lie isomorphism which has the following property: there is an isomorphism of L-algebras with involution $(A_L,\sigma_L) \to (A'_L,\sigma'_L)$ whose restriction to Alt(A,σ) *is f. Then f extends uniquely to an isomorphism of F-algebras with involution $(A,\sigma) \to (A',\sigma')$.*
(2) *Let (B,τ) and (B',τ') be central simple F-algebras with involution of the second kind and let L/F be a field extension. Suppose that* deg$(B,\tau) > 2$ *and let*

$$f: \mathrm{Skew}(B,\tau)^0 \to \mathrm{Skew}(B',\tau')^0$$

be a Lie isomorphism which has the following property: there is an isomorphism of L-algebras with involution $(B_L,\tau_L) \to (B'_L,\tau'_L)$ whose restriction to Skew$(B,\tau)^0$ *is f. Then f extends uniquely to an isomorphism of F-algebras with involution $(B,\tau) \to (B',\tau')$.*

Before giving necessary and sufficient conditions for existence of involutions we define the *norm $N_{K/F}(A)$ of the K-algebra A* as the F-subalgebra of ${}^{\iota}A \otimes_K A$ elementwise invariant under the map $s({}^{\iota}a \otimes b) = b^{\iota} \otimes a$. We remind that K/F is a quadratic extension and ι is the nontrivial automorphism of K/F.

Theorem 1.18 (Albert)**.** (1) *Let A be a central simple algebra over a field F. There is an involution of the first kind on A if and only if $A \otimes_F A$ splits.*
(2) (Albert-Riehm-Scharlau) *Let K/F be a separable quadratic extension of fields and let B be a central algebra over K. There is an involution of the second kind on B which leaves F elementwise invariant if and only if the norm $N_{K/F}(B)$ splits.*

Another sources investigating matrix algebras with involutions are [16, 19].

In [16] involution algebras (algebras with involution) are introduced and C^*-algebras as a special type of them. In [19] the authors considered noncommutative rings with an axiomatically introduced operation involution ($*$-operation). Using positive functionals the representations of these rings are investigated vice operators in a Hilbert space. The definition of a normed ring R with involution $*$ is given. Hermitian elements are introduced ($x^* = x$) and the presentation $x = x_1 + x_2$ for $x \in R$ and x_1, x_2 hermitian elements is given ($x_1 = \frac{x+x^*}{2}$, $x_2 = \frac{x-x^*}{2i}$). A typical example of a ring with involution is the ring of all limited linear operators in a Hilbert space.

Obviously the source of all investigations is functional analysis – the theory of normed rings and its applications, specially the theory of C^*-algebras. It is a typical aparatus in quantum mechanics, field theory and statistical physics.

II. Identities for involition matrix algebres. Let F be a field of characteristic zero and R an associative F-algebra with involution $*$. Consider $X = \{x_1, x_2, \ldots\}$, a countable set, and let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ be the free associative algebra with involution $*$ on X. An element $f(x_1, x_1^*, \ldots x_m, x_m^*)$ of $F\langle X, *\rangle$ is a $*$-polynomial identity for R if $f(r_1, r_1^*, \ldots, r_m, r_m^*) = 0$ for all substitutions $r_1, \ldots, r_m \in R$. The set $T(R, *)$ of all $*$-polynomial identities of R is a $*$-T-ideal of $F\langle X, *\rangle$, i.e. an ideal invariant under all endomorphisms of $F\langle X, *\rangle$ commuting with $*$.

For $i = 1, 2, \ldots$, we define $y_i = \dfrac{x_i + x_i^*}{2}$ and $z_i = \dfrac{x_i - x_i^*}{2}$. Thus we have that $F\langle X, *\rangle = F\langle Y \cup Z\rangle$.

We start our survey with the works of S.A. Amitsur. In [2] the notion of a ring R with involution is considered as well as the set $S = \mathrm{Sym}(R, *)$ for its symmetric with respect to the involution variables and the set $K = \mathrm{Skew}(R, *)$ for its anti-(or skew-)symmetric variables. Using structure theorems concerning the lower radical and the union of all nilpotent ideals of a ring R Amitsur proves a general result, namely:

Theorem 2.1 [2, Theorem 6]**.** *If R is a ring with involution such that the set S of symmetric elements satisfies a polynomial identity of degree d, then R satisfies an identity $S_{2d}(x)^m = 0$ for some m (the same for the anti-symmetric elements of R). Here the notation S_n stands for the standard polynomial in n variables.*

The paper [3] generalizes the above theorem:

Theorem 2.2 [3, Theorem 1]**.** *If R satisfies a polynomial identity of the form $p[x_1, \ldots, x_r; x_1^*, \ldots, x_r^*] = 0$ of degree d, then R satisfies an identity $S_{2d}(x)^m = 0$. If R is semi-prime, then $m = 1$.*

We mention the classical result of S. A. Amitsur and J. Levitzki stated in [4] that S_{2n} is an identity for the matrix algebra $M_n(F)$.

We give the following definition.

Definition 2.3. *Let g be a polynomial in m variables. We say that $g(k - l, l, *) = 0$ is an identity for $M_n(F)$ with respect to the involution $*$ if*

$g(k_1,\dots,k_{m-l},s_{m-l+1},\dots,s_m)=0$ *for all* $k_1,\dots,k_{m-l}\in K=K(M_n(F),*)$ *and for all* $s_{m-l+1},\dots,s_m\in S=S(M_n(F),*)$.

Using Lie group theory B. Kostant has shown in [36] that for n even $S_{2n-2}(n-2,0,*)$ is an identity for $M_n(F)$ and $*$ being the transpose involution. In [55] Rowen obtained the following strengthening of Kostant's theorem.

Theorem 2.4 [48, Theorem 1]**.** *Over a field* F *of characteristic zero the following identities hold for* $M_n(F)$*, where* (t) *is the transpose:*
(i) $S_{2n-1}(2n-1,0,t)=0$,
(ii) $S_{2n-1}(2n-2,1,t)=0$,
(iii) $S_{2n-2}(2n-2,0,t)=0$ *for all* n,
(iv) $S_{2n-2}(2n-3,1,t)=0$ *for* n *odd.*

The method of proof is largely graph-theoretic – exploiting certain properties of the trace of a matrix in connection with an undirected graph whose edges correspond to elementary symmetric and anti-symmetric matrices.

The object of the paper [55] is threefold – to give an easy proof of Kostant's theorem, to give an analogue of it for the symplectic involution and to show that these results follow from the same trace identity (arising from the generic minimal polynomial for a symmetric element with respect to the symplectic involution).

Working in $M_n(\mathbf{C})$, up to isomorphism, there are two types of involutions – the transpose, denoted (t), and the canonical symplectic involution (s), defined (only when n is even) by the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix},$$

where the original matrix is partitioned into $(n/2)\times(n/2)$ blocks.

According to [51, Theorem 2] and [47] all trace identities for M_n follow formally from the Cayley-Hamilton equation. Considering the set T_l of the generalized multilinear polynomials of $x_1,\dots,x_l$ of type $\underbrace{(1,\dots,1)}_{l\text{ times}}$, the algebra T_l due to a special operation could be identified with the group algebra $F[\mathrm{Sym}_{l+1}]$ of the symmatric group Sym_{l+1}. Thus the problem of finding the multilinear trace identities of degree l is reduced to the problem of classifying the two-sided ideals of the group algebra $F[\mathrm{Sym}_{l+1}]$.

The same point of view is presented in [55] relying on the following well-known analogue to the Cayley-Hamilton theorem and Newton's formulae.

Suppose $n=2m$ and $x\in S$. Then x satisfies a polynomial of the form

$p(\lambda)=\sum\limits_{k=0}^{m}(-1)^k\mu_k\lambda^{m-k}$, where $\mu_0=1$ and, inductively, $2k\mu_k=\sum\limits_{i=1}^{k}(-1)^{i-1}\mu_{k-i}\operatorname{tr}(x^i)$ for $1\le k\le m$. This is obtained by taking the pfaffian instead of the determinant in order to halve the degree of the "characteristic polynomial", and then obtaining the above formulae in a manner analogous to the proof of Newton's formulae. This polynomial is called the "generic minimal equation of x".

The following three remarks are essential in proving Theorem 2.8.

Remark 2.5. Given an involution $*$, for any $k\in K$ and $s\in S$, we have $\operatorname{tr}(ks)=0$ because $\operatorname{tr}(ks)=\operatorname{tr}(ks)^*=\operatorname{tr}(s^*k^*)=\operatorname{tr}(k^*s^*)=-\operatorname{tr}(ks)$. Consequently, the trace is a nondegenerate bilinear form both on S and K (i.e. if $k_1\in K$ and $\operatorname{tr}(kk_1)=0$ for all $k\in K$ then $k_1=0$, the same for S).

Remark 2.6. Suppose $*$ is a given involution with $k_1,\dots,k_{l-t}\in K$ and $s_{l-t+1},\dots,s_k\in S$, then

$$S_k(k_1,\dots,k_{l-t},s_{l-t+1},\dots,s_k)^*=(-1)^{[k/2]+k-l+t}S_k(k_1,\dots,k_{l-t},s_{l-t+1},\dots,s_k).$$

Remark 2.7. For all n and $x_1,\dots,x_{2k}\in M_n(\mathbf{C})$ one gets

$$\operatorname{tr}(S_{2k}(x_1,\dots,x_{2k}))=0\quad\text{and}$$
$$\operatorname{tr}(S_{2k-1}(x_1,\dots,x_{2k-1}))=(2k-1)\operatorname{tr}(S_{2k-2}(x_1,\dots,x_{2k-2})x_{2k-1}).$$

Multilinearizing the identity $p(s)=0$ for $s\in S$, substituting $s_1=[k_1,s_1]$ for $k_1\in K$ and making special substitutions for s_i $i\ge 2$, taking into account the above remarks, Rowen gets

Theorem 2.8 [55, Theorem 3]**.** *Considering the symplectic involution* (s)*, the following identities hold for* $M_n(\mathbf{C})$ *and* n *even, namely:*

(i) $S_{2n-2}(0,2n-2,s)=0$,
(ii) $S_{2n-2}(1,2n-3,s)=0$,
(iii) $S_{2n-1}(0,2n-1,s)=0$,
(iv) $S_{2n-1}(1,2n-2,s)=0$.

These are identities of minimal degree.

A basic paper on algebras with involution $*$ and their relations with forms is [47]. Given a finite-dimensional vector space V over F and a nondegenerate ε-symmetric form

$$\langle(v,\xi),(w,\zeta)\rangle=\frac{1}{2}[\langle v,\xi\rangle+\varepsilon\langle w,\zeta\rangle]$$

the algebra $\operatorname{End}(V)$ is equipped with a canonical involution $*$ having the property $(a^*v,w)=(v,aw)$. If $\varepsilon=1$ we refer to the involution as transposition, for $\varepsilon=-1$ – as symplectic involution, respectively.

When the field F is $\mathbf{C}$ and V is equipped with a nondegenerate Hermitian form, $\mathrm{End}(V)$ is endowed with an involution called adjoint.

Given a $*$-algebra R, a $*$-representation of R in the vector space V will be $*$-map $\varphi : R \to \mathrm{End}(V)$. It gives rise to an R-module structure on V and $(r^*v, w) = (v, rw)$ for all $v, w \in V$, $r \in R$. According to the nature of the form we speak of orthogonal, symplectic, or unitary representation, respectively. Two representations $\varphi : R \to \mathrm{End}(V)$, $\psi : R \to \mathrm{End}(W)$ will be called *equivalent*, if there exists an isomorphism $u : V \to W$ for which $u(\varphi(r)v) = \psi(r)u(v)$.

Let R be a free $*$-algebra. Given a set I, we construct on the category of $*$-algebras the three set valued functors: $R \hookrightarrow R^I$, $R \hookrightarrow R^{+I}$, $R \hookrightarrow R^{-I}$ ($S = R^+$ and $K = R^-$ always denoting the sets of the symmetric, respectively, anti-symmetric elements of R). Each of the three given functors is representable and the representing algebras are constructed in this way:
(i) The free algebra $F\langle x_i, y_i | i \in I\rangle$ with the involution assigned by the rule $x_i^* = y_i$ – the free $*$-algebra $F\langle x_i, x_i^* | i \in I\rangle$.
(ii) The free algebra $F\langle x_i | i \in I\rangle$ with the involution defined by $x_i^* = x_i$ – the free $*$-algebra in the symmetric variables s_i.
(iii) The free algebra $F\langle x_i | i \in I\rangle$ with the involution defined by $x_i^* = -x_i$ – the free $*$-algebra in the anti-symmetric variables k_i.

The canonical decomposition $R = R^+ \oplus R^-$ gives rise to the canonical isomorphism $F\langle x_i, x^* | i \in I\rangle \cong F\langle y_i | i \in I\rangle \bigsqcup F\langle z_i | i \in I\rangle$, where $\bigsqcup$ denotes the free product and $y_i = \dfrac{x_i + x_i^*}{2}$, $z_i = \dfrac{x_i - x_i^*}{2}$.

Let us give a vector space V, without any form, and a map $\varphi : R \to \mathrm{End}(V)$, with R a $*$-algebra. We can deduce, from this map, an orthogonal and a symplectic representation as follows:
(a) Construct the dual representation $\varphi^* : R \to \mathrm{End}(V^*)$ by the formula $\varphi^*(r) = \varphi(r^*)^t$.
(b) Construct the space $W = V \oplus V^*$ and the direct sum representation, $R \to \mathrm{End}(W)$, of φ and φ^*.
(c) Equip W with the canonical ε-symmetric form.
In each case φ is a $*$-map.

Let R be a semisimple Artinian $*$-algebra $R = \oplus_{i=1}^m R_i$, R_i a simple algebra. The involution $*$ induces a map of order 2 on the set of simple factors R_i. Therefore, we can subdivide this set in the factors R_i that are fixed under $*$, and the remaining ones exchanged:

$$R = \oplus_{i=1}^h R_i \oplus_{i=1}^t (S_j \oplus S_j^*),\ \ (R_i = R_i^*).$$

S_j^* is isomorphic, via $*$, to S_j^{op} (the opposite algebra).

The $*$-algebra $S_j \oplus S_j^*$ is thus isomorphic to the $*$-algebra $S_j \oplus S_j^{\mathrm{op}}$ with the exchange involution $(a,b)^* = (b,a)$.

Analyzing modules over R we came to the following

Theorem 2.9 [47, Theorem 14.1]. *Let R be a simple $*$-algebra, V – an irreducible R-module and $\Delta = \mathrm{End}_R(V)$, the centralizer of R.*
(a) *There exists an involution $*$ on Δ and a nonzero biadditive map $B : V \times V \to \Delta$ such that:*

(i) *$B(r^*v, w) = B(v, rw)$ for all $v, w \in V$, and $r \in R$.*

(ii) *$B(dv, w) = dB(v, w)$; $B(v, dw) = B(v, w)d^*$ for $v, w \in V$, $d \in \Delta$.*

(iii) *$B(v, w) = \varepsilon B(w, v)^*$, ε fixed, and $\varepsilon = \pm 1$.*

(b) *Condition* (i) *implies that B is nondegenerate, i.e. $B(v, w) = 0$ for all $v \in V$ implies $w = 0$ and symmetrically for all $w \in V$ implies $v = 0$.*
(c) *The involution on Δ and the form B are unique up to the following changes. If $*, \sharp$ are two involutions on Δ and B_1, B_2 the corresponding forms on V, there is an element $a \in \Delta, a \neq 0$ and $a^* = \varepsilon a$, $(\varepsilon = \pm 1)$ such that:*

(iv) *$b^\sharp = ab^*a^{-1}$, $B_2(v, w)a = B_1(v, w)$ for all $v, w \in V$.*

(d) *If Δ is finite dimensional over its centre F, every involution on Δ, coinciding on F with the automorphism induced by the involution on R, is obtained in the way described before. Provided that, given $\varepsilon \in F$, if $\varepsilon\varepsilon^* = 1$, then $\varepsilon = \pm(\alpha/\alpha^*), \alpha \in F$.*

It is important to endow directly the vector space V with a form for which φ is $*$-representation. Such a form will be called a compatible form. In some cases, such a form may not exist at all, in other cases, many inequivalent compatible forms may be constructed. An important special case for which one has existence and uniqueness is the following:

Corollary 2.10 [47, Corollary 14.2]. *If R is a simple $*$-algebra over an algebraically closed field F ($*$ being the identity on F), R is isomorphic to one of the two algebras:*
(i) *$n \times n$ matrices with transposition,*
(ii) *$2n \times 2n$ matrices with symplectic involution.*
Any irreducible module has a unique compatible form up to a scalar multiple.
In case (i)*, every irreducible $*$-representation of R is orthogonal, and any two such representations are equivalent.*
In case (ii)*, every irreducible $*$-representation of R is symplectic and any two such representations are equivalent.*

This corollary can be read in the language of representations as well. This extends the theory of representation of algebras relating to the problem of equivalence of representations to the invarinat theory of matrices.

The conclusive theorem on semisimple modules gives:

Theorem 2.11 [47, Theorem 14.7]**.** *Let*

$$R = (\oplus_{i=1}^{s} R_i) \oplus (\oplus_{j=1}^{t} S_j) \oplus (\oplus_{k=1}^{u} (T_k + T_k^{\mathrm{op}}))$$

be a semisimple $*$-algebra. The terms R_i are the ones with transpose involution, the S_j the ones with symplectic involution, and T_k is exchanged with T_k^{op}. Let $V_i\,(i = 1, \ldots, s)$; $W_j\,(j = 1, \ldots, t)$; $Z_k, Z_k^0\,(k = 1, \ldots, u)$ be irreducible modules over R_i, S_j, T_k and T_k^0, respectively. Considering an R-module M, isomorphic to $\sum n_i V_i + \sum m_j W_j + \sum p_k Z_k + \sum q_k Z_k^0$ we have:
(i) *M has a compatible symmetric form if and only if m_j is even $(j = 1, \ldots, t)$, and $p_k = q_k$ for $k = 1, \ldots, u$. Any two such forms are isomorphic over R.*
(ii) *M has a compatible anti-symmetric form if and only if n_i is even $(i = 1, \ldots, s)$, and $p_k = q_k$ for $k = 1, \ldots, u$. Any such forms are isomorphic over R.*

Giambruno determines in [21] the structure of the ring with involution by imposing algebraic conditions on the symmetric elements of the ring. He proved the following

Theorem 2.12 [20, Theorem 5]**.** *Let R be a primitive ring with involution and a centre Z such that for all $s_1, \ldots, s_n \in S$, there exists an integer $m = m(s_1, \ldots, s_n) \geq 1$ with*

$$[s_1, \ldots, s_n]^{m(s_1,\ldots,s_n)} \in Z.$$

Then R is a simple algebra of dimension at most 16 over its centre.

In [17] Drensky and Giambruno use a generic construction considering the 2×2 matrix algebra $M_2(F[\xi], *)$ with entries from the polynomial algebra $F[\xi] = F[\xi_{ij}^{(h)} | i, j = 1, 2,\ h = 1, \ldots, m]$ and equipped with either the transpose or the symplectic involution. The generic matrix algebra $G_m(*)$ is $*$-generated by the generic matrices $\overline{X_h} = \begin{pmatrix} \xi_{11}^{(h)} & \xi_{12}^{(h)} \\ \xi_{21}^{(h)} & \xi_{22}^{(h)} \end{pmatrix}, h = 1, \ldots, m$ and is isomorphic to the relatively free algebra $F_m(M_2(F), *)$. Via special good working algebras isomorphic to the generic algebra $G_m(*)$ with transpose involution or to the $G_m(*)$ with symplectic involution Drensky and Giambruno prove the following

Theorem 2.13 [17, Theorem 3.3]**.** *Let $\lambda=(\lambda_1, \ldots, \lambda_p)$ and $\mu=(\mu_1, \ldots, \mu_q)$ be partitions such that $\lambda_p \neq 0$, $\mu_q \neq 0$ and let*

$$f_{\lambda,\mu}(s_1, \ldots, s_p, k_1, \ldots, k_q) = f_\lambda(s_1, \ldots, s_p) f_\mu(k_1, \ldots, k_q) \sum_{\gamma \in \mathrm{Sym}_n} \alpha_\sigma \sigma,\ \alpha_\sigma \in F$$

for symmetric s_i $(i = 1, \ldots, p)$ *and skew-symmetric* k_j $(j = 1, \ldots, q)$ *be a proper polynomial generating a* $GL_m \times GL_m$*-submodule* $N_{\lambda,\mu}$ *of the space* $B_m(*)$ *of the proper polynomials of* $F_m\langle s_1, \ldots, s_m, k_1, \ldots, k_m\rangle$.

(i) *Let* $*$ *be the transpose involution of* $M_2(F)$. *If* $p > 2$ *or* $q > 1$, *then* $f_{\lambda,\mu}(s_1, s_2, k_1)$ *is a polynomail identity for* $(M_2(F), *)$ *if and only if* $f_{\lambda,\mu}(a_1, a_2, a_3)$ $= 0$ *for the matrices* $a_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $a_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(ii) *For the symplectic involution and* $p > 0$ *or* $q > 3$, $f_{\lambda,\mu}$ *is a polynomial identity for* $(M_2(F), *)$. *If* $p = 0$ *and* $q \leq 3$, $f_{\lambda,\mu} = f_\mu(k_1, k_2, k_3)$ *is a polynomial identity for* $(M_2(F), *)$ *if and only if* $f_\mu(a_1, a_2, a_3) = 0$.

Here the approach is based on the proper (or commutator) polynomial identities. For $Z_2 = \{1, *\}$ being the cyclic group of order 2 the wreath product $Z_2 \wr \mathrm{Sym}_n$ is defined. For every $*$-T-ideal $T(R, *)$ the vector spaces $P_n(*) \cap T(R, *)$ and $F_m(*) \cap T(R, *)$ are invariant under the actions of $Z_2 \wr \mathrm{Sym}_n$ and $GL_m \times GL_m$, respectively. Drensky and Giambruno in [17] view the space of multilinear $*$-polynomials $P_n(R, *)$ and the relatively free algebra $F_m(R, *)$ of rank m respectively as $Z_2 \wr \mathrm{Sym}_n$- and $GL_m \times GL_m$-modules and their module structure is related very closely.

Considering $(M_{2n}(F), *)$ with symplectic involution $* = s$ over a field of characteristic zero an important problem is to find the minimal degree of a $*$-polynomial identity for the algebra. If x_1, x_2 are symmetric variables, $[x_1, x_2] = 0$ is a $*$-identity of minimal degree for $(M_2(F), *)$ and $[[x_1, x_2]^2, x_3] = 0$ is a $*$-identity of minimal degree for $(M_4(F), *)$.

In the general case in [22] it was shown that if $f = 0$ is a $*$-polynomial identity for $(M_{2n}(F), * = s)$ and $n > 1$, then $\deg(f) \geq 2n + 1$. This result is improved in [18] and if $n > 2$, $(M_{2n}(F), *)$ does not satisfy identities of degree $2n+1$ in symmetric variables only. Therefore the minimal degree of the identities in symmetric variables for $M_{2n}(F, *)$ is greater than $2n+1$. The authors' approach in [22] is based on the following idea. Every $*$-polynomial identity in symmetric variables for $(M_{2n}(F), *)$ is an ordinary polynomial identity for the $n \times n$ matrix algebra $M_n(F)$. The ordinary polynomial identities of degree $2n + 1$ for $M_n(F)$ for $n > 2$ have been described by [37]. It turns out that all they follow from the standard identity $S_{2n}(x_1, \ldots, x_{2n}) = 0$. Hence it is sufficient to show that no multilinear consequence of degree $2n + 1$ of $S_{2n}(x_1, \ldots, x_n) = 0$ vanishes on the symmetric elements from $(M_{2n}(F), *)$. The authors apply the representation theory of the symmetric group Sym_p and of the general linear group GL_m.

In 1996 Rashkova [48, Theorem 3.1] made one more step in the direction of determining the minimal degree, showing that $(M_6(F), * = s)$ has no identities of degree 8 in symmetric variables. In 1995, it was proved by Giambruno and Valenti [30] that if f is a $*$-polynomial identity in skew-symmetric variables for $(M_{2n}(F), s)$ then $\deg f > 3n$. It is easy to show that S_{2n} is a $*$-polynomial identity for $(M_n(F), t)$ of minimal degree among $*$-polynomial identities in symmetric variables. But it was pointed in [9] that in general $*$-polynomial identities of minimal degree for matrices with involution need not resemble standard identities. In [49, Theorem 1] it was established that if a polynomial $f(x, y_1, \ldots, y_n)$ which is linear in each $y_i, i = 1, \ldots, n$, is a $*$-identity for $K(M_{2n}(F), s)$ then it is a $*$-identity for $S(M_{2n}(F), s)$ as well. A description of these polynomials being identities of minimal degree for $n = 2, 3$ is given in [49] as well.

In [56] Ma and Racine exploit the idea of weak identities. These are polynomials which evaluate to zero on some fixed subspace of an algebra. For a fixed algebra A and a subspace V of A, we denote by $T(V)$ the ideal of the weak identifies of V. The authors of [56] determine the weak identities of minimal degree for the subspace $S(M_n(F), t)$ of symmetric matrices of the full matrix algebra $M_n(F)$.

Let

$$T_k^i(x_1, \ldots, x_k) = \sum_{\substack{\sigma\in \mathrm{Sym}_k \\ 1\le i\le k, \\ \sigma^{-1}(i)\equiv 1,2 \,(\mathrm{mod}\, 4)}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}.$$

The notation $T_k^1(y, x_, \ldots, x_{k-1}) = T_k(x_1, \ldots, x_{k-1}; y)$ will be used as well. For $[a, b] = ab - ba$ and $\{abc\} = abc + cba$ we introduce $Q(x_1, \ldots, x_6) = \sum_{(123),(456)} \{[x_1, x_2][x_3, x_4][x_5, x_6]\}$, where the commutators are the arguments of the triple product and the sum is taken over cyclic permutations of (123) and (456), so that Q is the sum of nine triple products.

For A a central simple associative algebra over the field F we consider an involution $*$ of A of the first kind. It has the property that $(A\otimes\overline{F}, *) \cong (M_n(\overline{F}), t)$, where $\overline{F}$ is the algebraic closure of F and (t) – the transpose, giving it the name of orthogonal one.

Ma and Racine prove the following:

Corollary 2.14 [56, Corollary 3]**.** *If R is commutative ring with unit element* 1*, then* $T_{2n}(x_1, \ldots, x_{2n-1}; y) \in T(S(M_n(R), t))$.

If A is a central simple associative algebra of degree n over its centre and $*$ an orthogonal involution of A, then $T_{2n}(x_1, \ldots, x_{2n-1}; y) \in T(S(A, *))$. *If $n = 3$, then* $Q(x_1, \ldots, x_6) \in T(S(A, *))$.

The corollary has a graph-theoretic interpretation and is equivalent to a result on Eulerian paths on indirected graphs.

Let (V, E) be a finite graph with vertices $\{v_1, \ldots, v_n\}$ and edges $\{w_1, \ldots, w_r\}$ in an arbitrary but fixed ordering. Eulerian paths (if they exist) correspond to permutations $\sigma \in \mathrm{Sym}_r$, where the path is $w_{\sigma(1)}, \ldots, w_{\sigma(r)}$.

The following is a generalization of the above corollary.

Theorem 2.15 [56, Theorem 2]. *For $l(n) = 2[(n+1)/2]$, char F not dividing $[l(n)]!$ and $|F| > 2n$ all identities for $S(M_n(F), t)$ for $n \neq 3$ of degree $2n$ are consequences of T^1_{2n}. If $n = 3$, then all identities of degree 6 for $S(M_3(F), t)$ are consequences of T^1_6 and Q.*

For proving the above results Ma and Racine consider a symplectic involution (s) of a central simple associative algebra A and use the fact that the symmetric elements of A with respect to (s) satisfy a polynomial of degree m, where $n = 2m$ and $(A \otimes \overline{F}, *) \cong (M_n(\overline{F}), s)$. This polynomial is analogous to the characteristic polynomial but obtained using the Pfaffian instead of the determinant. Linearizing the polynomial and substituting the variables with proper commutators of symmetric elements using the properties of the trace, Ma and Racine get the validity of the stated results.

Considering $*$-identities for matrix algebras an important problem is to find their bases. Levchenko [38, 39] finds an explicit finite basis of the identities with involution for the second order matrix algebra over a field of characteristic zero.

Theorem 2.16 [38, Theorem 1]. *Let $(M_2(F), *)$ be the second order matrix algebra with symplectic involution $*$. All identities with involution for it are consequences of the identity $[x + x^*, y] = 0$.*

The next theorem concerns an involution $*$ of transpose type (being a first kind involution), namely $X^* = \begin{pmatrix} a & \alpha^{-1}c \\ \alpha b & d \end{pmatrix}$ for $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $0 \neq \alpha \in F$.

Theorem 2.17 [38, Theorem 2]. *For $(M_2(F), *)$ with an involution $*$ of transpose type all identities with involution follow from the identities:*

$$
\begin{aligned}
& [(x - x^*)(y - y^*), z] = 0, \\
& [x - x^*, y - y^*] = 0, \\
& [x_1 + x_1^*, x_2 + x_2^*][x_3 + x_3^*, x_4 + x_4^*] \\
+ \quad & [x_2 + x_2^*, x_3 + x_3^*][x_1 + x_1^*, x_4 + x_4^*] \\
+ \quad & [x_3 + x_3^*, x_1 + x_1^*][x_2 + x_2^*, x_4 + x_4^*] = 0,
\end{aligned}
$$

$$[x - x^*, y + y^*, z - z^*, t + t^*] = 4(x - x^*)(z - z^*)[t + t^*, y + y^*].$$

The proof is done by induction on the degree of the polynomial identities with involution and using the full description of the types of monomials in the linear combination expressing the considered multilinear identities for $(M_2(F), *)$.

The purpose of [14] is to determine the $*$-identities of minimal degree for $(M_n(F), t)$ when $n < 5$. D'Amour and Racine start with the identities mentioned by Rowen in [54], namely

$$\begin{aligned}
&x_1 - x_1^* \in T(M_1(R), t),\\
&[x_1 - x_1^*, x_2 - x_2^*] \in T(M_2(R), t),\\
&[S_3(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*), x_4] \in T(M_3(R), t),\\
&S_6(x_1 - x_1^*, \dots, x_6 - x_6^*) \in T(M_4(R), t),\\
&[x_1 + x_1^*, x_2] \in T(M_2(R), s),\\
&[[x_1 + x_1^*, x_2 - x_2^*]^2, x_3] \in T(M_4(R), s),
\end{aligned}$$

where R is a unital commutative ring.

It is seen in [14] that more polynomials are required to obtain all the identities of minimal degree, at least in the case of transpose involution. Defining the derivation $xD_{y,z} := \{xyz\} - \{xzy\} = xyz + zyx - xzy - yzx$ D'Amour and Racine prove

Proposition 2.18 [14, Proposition 2.7]**.** *The polynomials*

$$\begin{aligned}
p(x_1, x_2, x_3, x_4) &= \sum_{(123)} \{x_1[x_2, x_4]x_3\},\\
q(x_1, x_2, x_3, x_4) &= \sum_{(123)} \{x_1[x_2, x_3]x_4\} + \sum_{(124)} \{x_1[x_2, x_4]x_3\}\\
&+ 2([x_1, x_3]D_{x_2,x_4} + [x_1, x_4]D_{x_2,x_3} - [x_2, x_3]D_{x_1,x_4} - [x_2, x_4]D_{x_1,x_3}),\\
r(x_1, x_2, x_3, x_4) &= [S_3(x_1, x_2, x_3), x_4]
\end{aligned}$$

are identities for $\mathrm{Alt}(M_n(F), t)$.

Proposition 2.19 [14, Proposition 2.16]**.** *The polynomials*

$$\begin{aligned}
g(x_1, \dots, x_4) &= \sum_{\sigma \in \mathrm{Sym}_2} (-1)^\sigma (\{x_{\sigma(1)} x_{\sigma(2)} (x_3 \circ x_4)\} - \{x_{\sigma(1)} (x_{\sigma(2)} \circ x_4) x_3\}),\\
r(x_1, \dots, x_4) &= [S_3(x_1, x_2, x_3), x_4]
\end{aligned}$$

are $(3,1,t)$*-identities for* $M_3(F)$.

Let $f(x_1,\dots,x_r,y_1,\dots,y_s,\dots,z_1,\dots,z_t)$ be a homogeneous polynomial identity of type $[m^r,n^s,\dots,u^t]$ on some subspace V on $M_n(F)$ ($V=\mathrm{Sym}(M_n(F),t)$ or $V=\mathrm{Alt}(M_n(F),t)$) and set $m_0=\max\{m,n,\dots,u,r,s,\dots,t\}$. The following theorem provides a relation of symmetry between variables of equal degree in f, depending on how many they are.

Theorem 2.20 [14, Theorem 3.1]**.** *If* char F *does not divide* $m_0!$ *and* $|F|\geq 2m_0-1$*, then* $f=f_0+f_1$*, where the* f_i*,* $i=0,1$*, are identities of the same type as* f *and for each* $k: 0<k\leq m_0$*,* f_0 *is symmetric or skew-symmetric in all variables of degree* k*, depending on whether* k *is even or odd, while* f_1 *comes from the identities of lower type.*

The proof of this result starts with the case of a pair of variables x,y of some given degree m in f. Then, by acting on all r variables of the same degree m with the symmetric group Sym_r, one obtains the desired symmetry property among those particular variables. Repeating the procedure for each degree separately yields the general result.

The theorem can be extended to $(k-l,l,t)$ identities by fixing the degree, keeping the l symmetric and $k-l$ skew-symmetric variables apart, and acting on them via $\mathrm{Sym}_l\times\mathrm{Sym}_{k-l}$. Thus, when considering a typical homogeneous $(k-l,l,t)$ identity, we may assume the symmetry properties of Theorem 2.20, and with this, reduce the number of arbitrary coefficients involved in the calculations.

Theorem 2.21 [14, Theorem 3.2]**.** *If* $|F|>2$*, then any polynomial identity of* $\mathrm{Alt}(M_2(F),t)$ *of minimal degree is a scalar multiple of* $S_2(x_1-x_1^*,x_2-x_2^*)$.

Lemma 2.22 [14, Lemma 3.4]**.** *For* $n\geq 2$ *and any* $(n+1-s,s,t)$*-identity for* $M_n(F)$ *we have* $s\leq 1$.

Theorem 2.23 [14, Theorem 3.5]**.** *Under the hypotheses of Theorem* 2.20 *on the field* F*, any polynomial identity for* $\mathrm{Alt}(M_3(F),t)$ *of minimal degree is a consequence of the identities* $p(x_1,\dots,x_4)=0$*,* $q(x_1,\dots,x_4)=0$ *and* $r(x_1,\dots,x_4)=0$ *from Proposition* 2.18.

Theorem 2.24 [14, Theorem 3.7]**.** *Under the hypotheses of Theorem* 2.20 *on the field* F*, any polynomial identity for* $\mathrm{Alt}(M_4(F),t)$ *of minimal degree is a consequence of*

$$k(x_1,x_2,x_3,y):=S_4(x_1,x_2,x_3,y^2)-y\circ S_4(x_1,x_2,x_3,y)=0,$$

and any $*$-polynomial identity for $(M_4(F), t)$ of minimal degree is a consequence of $k(x_1 - x_1^*, x_2 - x_2^*, x_3 - x_3^*, y - y^*) = 0$.

The proof of the above four statements uses concrete special symmetric or skew-symmetric matrices, calculating proper coefficients of the considered polynomials.

Analogous investigations in the symplectic case are done in [15]. The authors D'Amour and Racine established the minimal degree of $*$-identities for $(M_n(F), s)$ when $n < 5$ and provide generators for the identities of minimal degree. The paper starts with the remark that when $n = 2$, the 2×2 symmetric matrices $S(M_2(F), s) = F \cdot E$, E the identity matrix, and so $[x, y] := xy - yx$ vanishes whenever x is replaced by any element of $S(M_2(F), s)$ and y by any element of $(M_2(F), s)$. One easily checks that $(M_2(F), s)$ has no $*$-identity of degree 1 and that all $*$-identities of degree 2 comes from the above. The approach for $n = 4$ is based on the well-known fact than an identity either for $S(M_4(F), s)$ or $K(M_4(F), s)$ must be an ordinary polynomial identity for $M_2(F)$ and the initial source for such identities is the vector space of dimension 29 over F of the multilinear identities of $M_2(F)$ of degree 5 cut down considering the subspaces of symmetric and skew-symmetric elements with respect to the reversal involution, i.e. the unique involution of the free associative algebra fixing the generators of dimensions 15 and 14, respectively. As Theorem 2.20 is valid for the symplectic case as well reducing greatly the number of arbitrary coefficients involved in the calculations, the authors prove that $K(M_4(F), s)$ has no multilinear identity of degree 5 and $(M_4(F), s)$ has no multilinear identity $f(x, y_1, \dots, y_4)$ of degree 5 with $x \in S(M_4(F), s)$ and $y_i \in K(M_4(F), s), i = 1, \dots, 4$.

The following two theorems are under the hypotheses of Theorem 2.20.

Theorem 2.25 [15, Theorem 3.4]**.** *Any polynomial identity for $S(M_4(F), s)$ of minimal degree is a consequence of*

$$p_4(x_1, \dots, x_5) = [[x_1, x_2] \circ [x_3, x_4] + [x_1, x_4] \circ [x_3, x_2], x_5]$$

and the linearization in $y|_{x_4, x_5}$ of the polynomial

$$r_5(x_1, x_2, x_3; y) = S_4(x_1, x_2, x_3, y^2) - y \circ S_4(x_1, x_2, x_3, y).$$

Theorem 2.26 [15, Theorem 3.5]**.** *Any multilinear identity $f(x_1, x_2, y_1, y_2, y_3)$ of degree 5 with $x_i \in S(M_4(F), s)$, $i = 1, 2$ and $y_j \in K(M_4(F), s)$, $j = 1, 2, 3$, is a consequence of* $p_2(x_1, x_2, y_1, y_2, y_3) = [[x_1, y_1] \circ [x_2, y_2], y_3]$.

The description of the multilinear identities of degree 5 of the form $f(x_1, x_2, x_3, y_1, y_2)$ and $f(x_1, x_2, x_3, x_4, y_1)$ with $x_i \in S(M_4(F), s)$ and $y_j \in K(M_4(F), s)$, $i = 1, 2, 3, 4$, $j = 1, 2$, is given as well.

Thus [14, 15] give a complete picture of the minimal degree $*$-identities of $(M_n(F), *)$ for $n < 5$.

For special polynomials Rashkova [50] continues further, considering $(M_{2n}(F), *)$ for $n \equiv 2, 3 \pmod 4$.

Those special polynomials are inspired by the approach of Formanek and Bergman for investigating identities for matrix algebras via commutative algebra.

To a homogeneous polynomial in commuting variables

$$g(t_1, \ldots, t_{n+1}) = \sum \alpha_p t_1^{p_1} \ldots t_{n+1}^{p_{n+1}} \in F[t_1, \ldots, t_{n+1}]$$

we relate a polynomial $v(g)$ from the free associative algebra $F\langle x, y_1, \ldots, y_n \rangle$

$$v(g) = v(g)(x, y_1, \ldots, y_n) = \sum \alpha_p x^{p_1} y_1 \ldots x^{p_n} y_n x^{p_{n+1}}.$$

Any homogeneous and multilinear in $y_1, \ldots, y_n$ polynomial $f(x, y_1, \ldots, y_n)$ (we call it a Bergman type polynomial) can be written as

$$f(x, y_1, \ldots, y_n) = \sum_{i=(i_1,\ldots,i_n) \in \mathrm{Sym}_n} v(g_i)(x, y_{i_1}, \ldots, y_{i_n}),$$

where $g_i \in F[t_1, \ldots, t_{n+1}]$.

In [50] the Bergman type identity of minimal degree 14 for $K(M_6(F), *)$ is found. Necessary and sufficient conditions are given for the existence of Bergman type identities for $K(M_4(F), *)$ and of degree 15 for $K(M_6(F), *)$. A class of Bergman type identities of degree $16 + 2k$ is given as well. Some of the obtained results are generalized for $n \equiv 2, 3 \pmod 4$ and the next theorem gives the generalization.

We define the commutative polynomial

$$g_{2n,0} = \prod_{\substack{1 \le p < q \le n+1 \\ (p,q) \neq (1,n+1)}} (t_p^2 - t_q^2)(t_1 - t_{n+1}).$$

Theorem 2.27 [50, Theorem 3]**.** *For $n \equiv$ 2,3 (mod 4) every Bergman type polynomial of degree k of the form*

$$f = \alpha \sum_i v(g_i)(x, y_{i_1}, \ldots, y_{i_n}) + \beta \sum_j v(g_j)(x, y_{j_1}, \ldots, y_{j_n}),$$

where

(i) $g_i = g_{2n,0} \prod_{l=1}^{k-n^2-2n+1} \sum_{m=1}^{n} a_{i,m}^{(l)} t_m,$

$$g_{i+\frac{n!}{2}} = g_{2n,0} \prod_{l=1}^{k-n^2-2n+1} \left(- \sum_{m=1}^{n} a_{i,n+1-m}^{(l)} t_m \right),\ i = 1, \ldots, \frac{n!}{2},$$

and $t_1 + t_{n+1}$ *is not a factor of these polynomials*;

(ii) *The polynomial* $(t_1 + t_{n+1})g_{2n,0}$ *divides* g_j *and*

(iii) *The identity* $\sum v(g_i)(x, y_{i_1}, \ldots, y_{i_n}) = 0$ *follows from the identity*

$v(g_{2n,0})(x, y_{i_1}, y_{i_2}, \ldots, y_{i_n}) + v(g_{2n,0})(x, y_{i_n}, y_{i_{n-1}} \ldots, y_{i_1}),\ (i_1, i_2, \ldots, i_n) \in \mathrm{Sym}_n,$ *is a* $*$*-identity in skew-symmetric variables for* $M_{2n}(F, *)$.

Let F be a field of characteristic not 2 and let A be a central simple F-algebra, σ – an involution. The pair (A, σ) is said to be hyperbolic if A contains a right ideal I such that $I = I^{\perp}$, where $I^{\perp} = \{x \in A : \sigma(x)I = 0\}$.

Let $\pi(t) \in F[t]$ be monic and separable of even degree $2n$ and

$$F(\pi) = F[t]/\pi(t)F[t].$$

Since $\pi(t)$ is separable, the ring $F(\pi)$ is a direct product of separable field extensions of F, $F(\pi) = F(\pi_1) \times \cdots \times \pi_r(t)$. If (A, σ) is a central simple F-algebra with involution we say that (A, σ) becomes hyperbolic over $F(\pi)$ if each of the algebras $(A \otimes_F F(\pi_i), \sigma \otimes 1)$ is hyperbolic. By [32] if (Q, γ) is an F-central quaternion algebra with its unique symplectic involution, then (Q, γ) becomes hyperbolic over a field extension L of F if and only if L splits Q. In Chapter 2 of [31] there are two main results characterizing when a K-central quaternion algebra (Q, σ) with $F = K^{\sigma}$ becomes hyperbolic over a field extension L of F, where σ is either an orthogonal involution or an involution of the second kind. In Chapter 3 the case of biquaternion algebras is considered, i.e. central simple algebras A of degree 4 with an involution σ of the first kind. A criterion is given such an algebra with orthogonal involution of nontrivial discriminant to become hyperbolic over $F(\pi)$. The proof uses essentially the invariants of the involution – the index and the discriminant. The importance of the exposition lies in connecting the criteria of hyperbolicity with the properties of a special Clifford algebra of a quadratic form over a polynomial ring over F.

Another trend of investigation is the study of functional identities on prime rings with involution [7]. They play a crucial role in the solution of number of problems on Lie isomorphisms in prime rings.

Berhuy [11] considers the trace form of special central algebras with involution over a field F of characteristic different from 2. Following strictly the definitions of part **I.** of the survey, a nondegenerate symmetric bilinear form of dimension n^2 over F is introduced being the function $(x,y) \in A\times A \to \mathrm{Trd}_A(\sigma(x)y)$ with T_σ the corresponding quadratic form. If σ is of the second kind, T_σ is a nondegenerate quadratic form over the field F_0 (fixed by the trivial involution σ/F) of dimension $2n^2$. Berhuy in [11] computes explicitly the isomorphism class of the trace form T_σ of the considered F-algebras with involution of any kind for some special base fields, especially the Euclidean fields and the field of rational numbers. The proof uses essentially the properties of the invariants of the considered quadratic forms.

III. Cocharacters, Hilbert series, codimensions and growth. We recall that $F\langle X,*\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ denotes the free associative algebra with involution $*$ generated by X a field F of characteristic 0 and $F_m(*) = F\langle x_1, x_1^*, \ldots, x_m, x_m^*\rangle$ denotes the free subalgebra of rank m. As in the case of ordinary polynomial identities, in characteristic zero, the $*$-polynomial identities of an algebra are determined by the multilinear ones. If we denote by $V_k(*)$ the space of all multilinear polynomials of degree k in $x_1, x_1^*, \ldots, x_k, x_k^*$, the study of the $*$-T-ideal $T(R,*)$ of all $*$-polynomial identities of an algebra R is equivalent to the study of $V_k(*) \cap T(R,*)$ for any $k \geq 1$. We define the relatively free algebra $F(R,*) = F\langle X,*\rangle/T(R,*)$ and the relatively free algebra $F_m(R,*) = F_m(*)/F_m(*) \cap T(R,*)$ of rank m.

Another way of writing the space $V_k(*)$ is

$$\begin{aligned} V_k(*) &= \mathrm{Span}_F\{w_{\sigma(1)}\cdots w_{\sigma(k)} | \sigma \in \mathrm{Sym}_k, \\ w_i &= y_i \in Y \,\mathrm{or}\, w_i = z_i \in Z, i = 1, \ldots, k\} \end{aligned}$$

(Y denotes the symmetric while Z stands for the skew-symmetric with respect to the involution $*$ variables). We denote

$$\begin{aligned} V_k^{(r)}(*) &= \mathrm{Span}_F\{w_{\sigma(1)}\cdots w_{\sigma(k)} | \sigma \in \mathrm{Sym}_k, \\ w_i &= y_i \in Y,\, i = 1, \ldots, r \,\mathrm{and}\, w_j = z_j \in Z, j = r+1, \ldots, k\}. \end{aligned}$$

Let $V_k(R,*) = V_k(*)/V_k(*) \cap T(R,*)$ be the set of multilinear elements of degree k in $F(R,*)$. The *n-th codimension of* R is $c_n(R,*) = \dim V_n(R,*), n = 0,1,2,\ldots$. We denote $c_n^{(r)}(R,*) = \dim V_n^{(r)}(R,*)$. We have that $F_m(*) = F_m\langle y_1, \ldots, y_m, z_1, \ldots, z_m\rangle$ and assume that the symmetric variables $s_1, \ldots, s_m$ and the skew-symmetric variables $k_1, \ldots, k_m$ generate the relatively free algebra $F_m(R,*)$. The vector space $F_m(*)$ has a natural multigrading obtained by counting the degree in the symmetric and skew-symmetric variables. Since the ideal $F_m(*) \cap T(R,*)$ is multihomogeneous, $F_m(R,*)$ inherits the multigrading.

Let $F_m^{(a,b)}(R,*)$, $(a,b) = (a_1,\dots,a_m,b_1,\dots,b_m)$, be the multihomogeneous component of degree a_i in s_i and of degree b_i in $k_i, i = 1,\dots,m$. *The $*$-Hilbert series of $F_m(R,*)$* is defined as the formal power series

$$H(R,*,y_1,\dots,y_m,z_1,\dots,z_m) = \sum_{(a,b)} \dim F_m^{(a,b)}(R,*) y_1^{a_1}\cdots y_m^{a_m} z_1^{b_1}\cdots z_m^{b_m}.$$

If $Z_2 = \{1,*\}$ is the cyclic group of order 2, then B_k is the wreath product $Z_2 \wr \mathrm{Sym}_k = \{(a_1,\dots,a_k;\sigma) | a_i \in Z_2, \sigma \in \mathrm{Sym}_k\}$ with multiplication given by

$$(a_1,\dots,a_k;\sigma)(b_1,\dots,b_k;\tau) = (a_1 b_{\sigma^{-1}(1)},\dots,a_k b_{\sigma^{-1}(k)};\sigma\tau).$$

There is one-to-one correspondence between irreducible B_k-characters and pairs of partitions (λ,μ), where $\lambda \vdash r$, $\mu \vdash k-r$, for all $r = 0,1,\dots,k$. We denote by $M_{\lambda,\mu}$ and $\chi_{\lambda,\mu}$ the irreducible B_k-module corresponding to (λ,μ) and its character, respectively. The space $V_k(*)$ has a structure of a left B_k-module induced by the action of $h = (a_1,\dots,a_k;\sigma) \in B_k$ defined by $hs_i = s_{\sigma(i)}$, $hk_i = k_{\sigma(i)}^{a_{\sigma(i)}} = \pm k_{\sigma(i)}$. For every F-algebra R with involution $*$ the vector space $V_k(*) \cap T(R,*)$ is invariant under the above action of B_k, hence the space $V_k(R,*) = V_k(*)/(V_k(*) \cap T(R,*))$ has a structure of a left B_k-module. We write

$$\begin{aligned} V_k(*) \cap T(R,*) &\simeq \sum_{\lambda,\mu} m_{\lambda,\mu}(R,*) M_{\lambda,\mu}, \\ \chi_{B_k}(R,*) &= \sum_{r=0}^{k} \sum_{\substack{\lambda\vdash r,\\ \mu\vdash k-r}} m_{\lambda,\mu}(R,*)\chi_{\lambda,\mu}, \end{aligned}$$

where $m_{\lambda,\mu}(R,*) \geq 0$ are the corresponding multiplicities.

The character $\chi_k(R,*)$ of $V_k(R,*)$ is called *the k-th $*$-cocharacter of R* and has the following decomposition

$$\chi_k(R,*) = \sum_{r=0}^{k} \sum_{\substack{\lambda\vdash r,\\ \mu\vdash k-r}} \overline{m}_{\lambda,\mu}(R,*)\chi_{\lambda,\mu}. \tag{1}$$

Again, as in the case of ordinary polynomial identities, one of the main problems for the $*$-polynomial identities of R is to describe the multiplicities $m_{\lambda,\mu}(R,*)$ or, equivalently, $\overline{m}_{\lambda,\mu}(R,*)$.

Drensky and Giambruno determined in [17] the multiplicities $\overline{m}_{\lambda,\mu}(M_2(F),*)$ for any pair (λ,μ).

Theorem 3.1 [17, Theorem 4.1]. *The B_n-cocharacter of $(M_2(F), *)$ is $\chi_n(M_2(F), *) = \sum a_{\lambda,\mu}\chi_{\lambda,\mu}$, where*
(i) *For the transpose involution $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1)$ and*

$$\begin{aligned} &a_{\lambda,\mu} = 1, \ \textit{when } \lambda_2 = \mu_1 = 0;\\ &a_{\lambda,\mu} = (\lambda_1 - \lambda_2 + 1)\lambda_2, \ \textit{when } \lambda_2 \neq 0,\ \lambda_3 = \mu_1 = 0;\\ &a_{\lambda,\mu} = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1), \ \textit{for other pairs of partitions.}\end{aligned}$$

(ii) *For the symplectic involution $\lambda = (\lambda_1)$, $\mu = (\mu_1, \mu_2, \mu_3)$ and $a_{\lambda,\mu} = 1$.*

For r fixed let $V_{r,k-r}(*) = V_k^{(r)}(*)$ be the space of multilinear polynomials in $s_1, \ldots, s_r, k_{r+1}, \ldots, k_k$. If Sym_r acts on the symmetric variables $s_1, \ldots, s_r$ and Sym_{k-r} on the skew-symmetric variables $k_{r+1}, \ldots, k_k$, then we obtain an action of $\mathrm{Sym}_r \times \mathrm{Sym}_{k-r}$ on $V_{r,k-r}(R, *)$. Since $*$-T-ideals are invariant under permutations of symmetric and skew-symmetric variables, we obtain that $V_{r,k-r}(R, *) = V_{r,k-r}(*)/V_{r,k-r}(*) \cap T(R, *)$ has the induced structure of a left $\mathrm{Sym}_r \times \mathrm{Sym}_{k-r}$-module. We denote by $\chi_{r,k-r}(R, *)$ its character. There is a one-to-one correspondence between irreducible $\mathrm{Sym}_r \times \mathrm{Sym}_{k-r}$-characters and pairs of partitions (λ, μ) such that $\lambda \vdash r$, and $\mu \vdash k - r$. Hence, by the complete reducibility, we have the decomposition

$$\chi_{r,k-r}(R, *) = \sum_{\substack{\lambda \vdash r,\\ \mu \vdash k-r}} \widetilde{m}_{\lambda,\mu}(R, *)(\chi_\lambda \otimes \chi_\mu), \tag{2}$$

where χ_λ (resp. χ_μ) denotes the usual Sym_r-character (resp. Sym_{k-r}-character), $\chi_\lambda \otimes \chi_\mu$ is the irreducible $\mathrm{Sym}_r \times \mathrm{Sym}_{k-r}$-character associated with the pair (λ, μ) and $\widetilde{\mu}_{\lambda,\mu}(R, *) \geq 0$ is the corresponding multiplicity.

The relation between the B_k-character and the $\mathrm{Sym}_r \times \mathrm{Sym}_{k-r}$-character is expressed in

Theorem 3.2 [17, Theorem 1.3].

(i) *If the k-th $*$-character of R has the decomposition given in* (1) *and the* $\mathrm{Sym}_r \times \mathrm{Sym}_{k-r}$*-character of $V_{r,k-r}(R, *)$ has the decomposition* (2)*, then $\overline{m}_{\lambda,\mu}(R, *) = \widetilde{m}_{\lambda,\mu}(R, *)$, for all λ and μ.*

(ii) *The codimensions satisfy the relation*

$$c_k(R, *) = \sum_{r=0}^{k} \binom{n}{r} c_k^{(r)}(R, *).$$

Let $U = \mathrm{Span}_F\{s_1, \ldots, s_m\}$ and $V = \mathrm{Span}_F\{k_1, \ldots, k_m\}$. The group $GL(U) \times GL(V) \simeq GL_m \times GL_m$ acts from the left on the space $U \oplus V$ and we can extend this action diagonally to an action on $F_m(*)$. For every $*$-T-ideal $T(R,*)$ the space $F_m(R,*) = F_m(*)/F_m(*) \cap T(R,*)$ is a $GL_m \times GL_m$-module. Let $F_m^{(k)}(R,*)$ be its homogeneous component of degree k; it is a $GL_m \times GL_m$-submodule of $F_m(R,*)$ and we denote its character by $\psi_k(R,*)$. The irreducible polynomial $GL_m \times GL_m$-characters are described by pairs of partitions (λ, μ), where $\lambda \vdash r$ and $\mu \vdash k - r$ for all $r = 0, \ldots, k$.

A complete description of the representation theory of the group B_k on $V_k(*)$ and of $GL_m \times GL_m$ on $F_m(*)$ is given in [29, 21].

In [29] the representation theory of the wreath product $G \wr \mathrm{Sym}_n$ is applied to study algebras satisfying polynomial identities that involve a group G of (anti)-automorphisms, in the same way the representation theory of Sym_n has been applied to study ordinary P.I. algebras. The basic idea of identifying the space V_n of the multilinear polynomials in $x_1, \ldots, x_n$, with the group algebra $F[\mathrm{Sym}_n]$ is exploited for identifying the group algebra $F[G\wr\mathrm{Sym}_n]$ with $V_n(x|G) = \mathrm{Span}_F\{x_{\sigma(1)}^{g_1} \ldots x_{\sigma(n)}^{g_n} | \sigma \in \mathrm{Sym}_n, g_i \in G\}$, the multilinear G-polynomials of degree n of the associative ring $F\langle X|G\rangle$ of non-commutative F-polynomials in the variables $\langle X|G\rangle = \{x^g = g(x) | x \in X, g \in G\}$.

Let R be an F-algebra and $\mathrm{Aut}^*(R)$ be the group of automorhisms and anti-automorphisms of the algebra R. For $G \subseteq \mathrm{Aut}^*(R)$ and the G-identities $P \subseteq F\langle X|G\rangle$ of R one defines $\chi_n(R|G)$ to be the $G\wr\mathrm{Sym}_n$-character of the module $V_n(x|G)/P_n$, where $P_n = V_n(x|G| \cap P)$, and call $\chi_n(R|G)$ "the G-cocharacters of R". The "G-codimensions" of R are defined as $c_n(R|G) = \dim(V_n(X|G)/P_n)$ being the degrees of the G-cocharacters.

In the paper [29] it was proved that $c_n(R|G) \le |G|^n c_n(R)$ and as a corollary the following

Theorem 3.3 [29, Theorem 4.8]**.** *Let $G \subseteq \mathrm{Aut}^*(R)$ be a finite subgroup, and let R be a G-P.I. algebra. Then R satisfies an ordinary identity if and only if $c_n(R|G)$ is exponentially bounded (i.e. there exists $0 < a$ such that for all n, $c_n(R|G) \le a^n$).*

Thus Amitsur's theorem (Theorem 2.2) is translated to the language of codimensions, namely

Theorem 2.2′ [3, Theorem 1]**.** *A ring R with involution $*$ that is $*$-P.I. is also (ordinary) P.I. ring.*

Equivalently, such R is $*$-P.I. if and only if $c_n(R,*)$ is exponentially bounded.

The following theorem was proved as well.

Theorem 3.4 [29, Theorem 6.2]. *Let $\chi_n(M_k(F),t)$ be the $*$-cocharacter of the matrix algebra $M_k(F)$, $u=\frac{1}{2}k(k+1)$ and $v=\frac{1}{2}k(k-1)$. Then*

$$\chi_n(F_k|*)=\sum_{\substack{|\lambda|+|\mu|=n\\ \lambda_1'\le u,\mu_1'\le v}} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $\lambda_1'=h(\lambda)$ is the height of λ, etc. Moreover, there exist $n=n(k)$ and partitions λ, μ, $|\lambda|+|\mu|=n$, satisfying $\lambda_1'=u$ and $\mu_1'=v$, for which the corresponding multiplicity $m_{\lambda,\mu}=m_{\lambda,\mu}(F_k,*)$ is non-zero.

If we denote by $\psi_{\lambda,\mu}$ the irreducible $GL_m\times GL_m$-character associated to the pair (λ,μ), then we have the following decomposition

$$\psi_k(R,*)=\sum_{r=0}^{k}\sum_{\substack{\lambda\vdash r,\\ \mu\vdash k-r}}\widehat{m}_{\lambda,\mu}(R,*)\psi_{\lambda,\mu}. \tag{3}$$

The B_k-module structure of $V_k(R,*)$ and the $GL_m\times GL_m$-module structure of $F_m^{(k)}(R,*)$ are closely related.

Theorem 3.5 [21, Theorem 3]. *If the k-th $*$-cocharacter of R has the decomposition given in* (1) *and the $GL_m\times GL_m$-character of $F_m^{(k)}(R,*)$ has the decomposition* (3) *then $m_{\lambda,\mu}(R,*)=\widehat{m}_{\lambda,\mu}(R,*)$, for all λ,μ.*

For $(M_2(F),*)$ the Hilbert and codimension series, as well as the codimension sequences are computed in the next two theorems.

Theorem 3.6 [17, Theorem 4.2]. (i) *For the transpose involution $*$*

$$\begin{aligned}
&H(M_2(F),*,y_1,\dots,y_m,z_1,\dots,z_m)\\
&\quad=\prod_{i=1}^{m}\frac{1}{1-y_i}\prod_{i=1}^{m}\frac{1}{1-z_i}\sum_{(\lambda_1,\lambda_2)}S_{(\lambda_1,\lambda_2)}(y_1,\dots,y_m)\\
&\quad-\prod_{i=1}^{m}\frac{1}{(1-y_i)^2}+\prod_{i=1}^{m}\frac{1}{1-y_i}\\
&\quad=\prod_{i=1}^{m}\frac{1}{1-y_i}\prod_{i=1}^{m}\frac{1}{1-z_i}\sum_{k\ge 0}(h_k^2(y_1,\dots,y_m)\\
&\quad+h_k(y_1,\dots,y_m)h_{k+1}(y_1,\dots,y_m))-\prod_{i=1}^{m}\frac{1}{(1-y_i)^2}+\prod_{i=1}^{m}\frac{1}{1-y_i},
\end{aligned}$$

where $S_{(\lambda_1,\lambda_2)}(y_1,\ldots,y_m)$ *is the Schur function corresponding to the partition* $\lambda = (\lambda_1,\lambda_2)$ *and* $h_k(y_1,\ldots,y_m) = S_{(k)}(y_1,\ldots,y_m)$ *denotes the* k*-th complete symmetric function;*

$$\begin{aligned}
&H(M_2(F),*,y_1,y_2,y_3,z_1,z_2,z_3)\\
&\qquad =(1-y_1y_2y_3)\prod_{i=1}^{3}\frac{1}{(1-y_i)^2}\prod_{i<j}\frac{1}{1-y_iy_j}\prod_{i=1}^{3}\frac{1}{1-z_i}\\
&\qquad -\prod_{i=1}^{3}\frac{1}{(1-y_i)^2}+\prod_{i=1}^{3}\frac{1}{1-y_i}.
\end{aligned}$$

(ii) *For the symplectic involution*

$$\begin{aligned}
H(M_2(F),*,y_1,\ldots,y_m,z_1,\ldots,z_m) &= \prod_{i=1}^{m}\frac{1}{1-y_i}\sum_{\mu}S_{(\mu_1,\mu_2,\mu_3)}(z_1,\ldots,z_m)\\
&= \prod_{i=1}^{m}\frac{1}{1-y_i}\prod_{i=1}^{m}\frac{1}{1-z_i}\sum_{\mu_1\geq 0}S_{(\mu_1,\mu_1)}(z_1,\ldots,z_m),\\
H(M_2(F),*,y_1,y_2,y_3,z_1,z_2,z_3) &= \prod_{i=1}^{3}\frac{1}{1-y_i}\prod_{i=1}^{3}\frac{1}{1-z_i}\prod_{i<j}\frac{1}{1-z_iz_j}.
\end{aligned}$$

Theorem 3.7 [17, Theorem 4.3]**.** (i) *For the transpose involution the codimension series and the codimension sequence are equal respectively to*

$$\begin{aligned}
c(M_2(F),*,t_0) &= \frac{1}{2t_0}\left(-1+\sqrt{\frac{1}{1-4t_0}}\right)-\frac{1}{1-2t_0}+\frac{1}{1-t_0},\\
c_n(M_2(F),*) &= \frac{1}{2}\binom{2n+2}{n+1}-2^n+1.
\end{aligned}$$

(ii) *For the symplectic involution involution*

$$\begin{aligned}
c(M_2(F),*,t_0) &= \frac{1}{t_0^2}\left(1-2t_0-\sqrt{1-4t_0}\right),\\
c_n(M_2(F),*) &= \frac{1}{n+2}\binom{2n+2}{n+1}.
\end{aligned}$$

Let $M_3(F)$ be the algebra of 3×3 matrices with involution $*$. In [8] Benanti and Campanela study the $*$-polynomial identities of $(M_3(F),t)$ through the representation theory of the hyperoctahedral group B_n. After decomposing the space of multilinear $*$-polynomial identities of degree n under the B_n-action, they determine which irreducible B_n-modules appear with non-zero multiplicity. The main result in [8] is

Theorem 3.8 [8, Theorem 7]. *The n-th $*$-cocharacter of $M_3(F)$ is*

$$\chi_n(M_3(F),t)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r,h(\lambda)\leq 6\\ \mu\vdash n-r,h(\mu)\leq 3}} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $m_{\lambda,\mu}\neq 0$ if and only if $(\lambda,\mu)\neq((1)^6,\emptyset)$.

The proof uses essentially the technique of "gluing" Young tableaux. The symmetric group Sym_n lies naturally inside the group B_n. Hence, for any associative algebra R, the B_n-module $V_n(R,*)$ may be regarded as an Sym_n-module and for the induced Sym_n-character $\chi_{\mathrm{Sym}_n}(R)$ we have the decomposition $\chi_{\mathrm{Sym}_n}(R)=\sum\limits_{\nu\vdash n} m_\nu\chi_\nu$, where χ_ν is the irreducible Sym_n-character associated to the partition ν and m_ν is the corresponding multiplicity.

An easy corollary of the above theorem using [12] is

Corollary 3.9 [8, Corollary 9]. *The multiplicities m_ν in $\chi_{\mathrm{Sym}_n}(M_3(F),t)$ are always non-zero for all partitions ν of n.*

The natural definition of a free algebra with trace and involution and of $*$-trace polynomial identity for an algebra R given in [40] extends the above results including the trace map. Using the notations $\overline{V_n}(*)$ for the space of all multilinear mixed $*$-trace polynomials of degree n, $\overline{T}(R,*)$ for the corresponding $*$-T-ideal and $\overline{\chi_n}(R,*)$ for the $*$-trace cocharacter of R, we write

$$\overline{\chi_n}(R,*)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m'_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $m'_{\lambda,\mu}\geq 0$ is the multiplicity of the irreducible B_n-character $\chi_{\lambda,\mu}$.

Another consequence of Theorem 3.8 is

Corollary 3.10 [8, Corollary 11]. *The multiplicites $m'_{\lambda,\mu}$ in $\overline{\chi_n}(M_3(F),t)$ are always non-zero for any $\lambda\vdash r$ and $\mu\vdash n-r$ such that $h(\lambda)\leq 6$ and $h(\mu)\leq 3$.*

The idea of [29] is used in [12] as well, where the main goal is to calculate the asymptotic behaviour of the multilinear $*$-codimensions of $p \times p$ matrices with or without trace, over a field of characteristic zero. The theory of $*$-trace identities is used to calculate the asymptotic growth of the $*$-codimensions with trace $t_n(M_p(F), *)$ and to show that $t_n(M_p(F), *) \approx c_n(M_p(F), *)$. The investigations parallel the case of the ordinary cocharacter of $p \times p$ matrices. In it the identification of the space of multilinear polynomials with $F[\mathrm{Sym}_n]$ is an Sym_n-isomorphism, taking the Sym_n-action on $F[\mathrm{Sym}_n]$ to be left multiplication. In the case of multilinear $*$-polynomials there is a similar B_n-isomorphism with the regular representation $F[B_n]$. Specially $\sigma \in B_n$ may be identified with the $*$-monomial $\sigma(x_1)\sigma(x_2)\dots\sigma(x_n)$.

For trace polynomials without involution $*$, there is an identification of pure trace polynomials with $F[\mathrm{Sym}_n]$. To make the identification an Sym_n-isomorphism, one uses the conjugation action of Sym_n on $F[\mathrm{Sym}_n]$. In the case of polynomials with involution $*$ and trace according to [40] there is a B_n-isomorphism between $F[\mathrm{Sym}_{2n}/B_n]$ and the space of pure trace $*$-polynomials, modulo the relation $\mathrm{tr}(a^*) = \mathrm{tr}(a)$, for all a of the considered algebra. One gets

Theorem 3.11 [12, Theorem]**.** *Let A be a $p \times p$ matrix with involution. Then the trace $*$-cocharacters $t_n(A)$ and the $*$-cocharacters $c_n(A)$ are asymptotically equal.*

Loday and Procesi consider in [40] the infinite Lie algebra of orthogonal and symplectic matrices over an associative ring with involution over a characteristic zero field. They give a comprehensive and detailed review of invariant theory for these types of matrices. The authors in [40] make explicit the relations between the hyperoctahedral group, the trace identities for matrices and the invariant space of the tensor algebra of matrices. Introducing the universal space for trace formulas Loday and Procesi study the module structure of this space over the hyperoctahedral group and compare it to some spaces of invariants, the stability range and the existence of a stabilization homomorphism.

For a finite dimensional algebra with involution $*$ over F Giambruno and Zaicev study in [28] the asymptotic behaviour of the sequence of $*$-codimensions $c_n(A, *)$ of A and show that $\mathrm{Exp}(A, *) = \lim_{n\to\infty} \sqrt[n]{c_n(A, *)}$ exists and is an integer. They give an explicit way of computing the $*$-P.I. exponent $\mathrm{Exp}(A, *)$ and as a consequence obtain the following characterization of $*$-simple algebras: A is $*$-simple if and only if $\mathrm{Exp}(A, *) = \dim_F A$. The authors investigate in [28] algebras A for which $\mathrm{Exp}(A, *) \leq 1$.

In [46] Pipitone considers the case $\mathrm{Exp}(A, *) = 2$. The author gives a list of four finite dimensional algebras A_1, A_2, A_3 and A_4 with involution satisfying the following property: A finite dimensional algebra A with involution has $*$-P.I. exponent greater than 2 if and only if $T(A,*) \subseteq T(A_i,*)$ for some $i = 1,\dots,4$. Two of the algebras A_i are the 2×2 matrix algebras with the transpose and the symplectic involutions and the other two are a subalgebra and a factor algebra of algebras of upper triangular matrices $UT_6(F)$ and $UT_3(F)$, respectively, with involution which transposes the matrices with respect to the other diagonal. Combining this description with the results in [41], Pipitone characterizes in Corollary 3.24 the finite dimensional algebras A with $*$-P.I. exponent 2 as well.

More details about $\mathrm{Exp}(A,*)$ are given in [23].

Giambruno and Regev [29] defined G-polynomials and G-polynomial identities for G a finite group of automorphisms and anti-automorphisms on an algebra R over a field F. In [6] Bahturin, Giambruno and Zaicev introduce essential G-polynomial identities. They characterize the G-codimensions of R and the polynomial identities for R when R satisfies an essential identity. In case $G = \{1,*\}$ where $*$ is an involution, we get $*$-polynomials, $*$-identities and $*$-codimensions. Thus it becomes possible to characterize the $*$-codimensions of an algebra with involution and to sharpen the results of Amitsur (Theorem 2.1 and Theorem 2.2) giving an upper bound on the degree of the polynomial identity.

We need the following definition:

Let $k = 2d+1$, $N = 2^{k2^{k+1}}$ and denote by $p_j, j \geq 3$, an integer for which $\underbrace{\log_N \cdots \log_N}_{j-2} p_j = p_2$ for $p_2 = 2^{k2^k}$. Then we set $f(2d,2) := \log_2 p_{2d}$.

Corollary 3.12 [6, Corollary 1]**.** *Let R be an algebra with involution $*$ over a field F of characteristic zero satisfying a non-trivial $*$-identity of degree d. Then for n sufficiently large we have $c_n(R,*) \leq 2^n(f(2d,2)-1)^{2n}$ and R satisfies a non-trivial polynomial identity whose degree is bounded by the function $f(2d,2)$.*

The $*$-codimension sequence affords a kind of measure on how "big" the variety is – the greater $c_n(\mathfrak{U},*)$ is, where $\mathfrak{U}$ is a variety of algebras with involution, the greater $\mathfrak{U}$ is itself as $T(\mathfrak{U},*)$ becomes smaller with respect to this "measure". The smallest varieties from this point of view are the varieties with *polynomial growth of $*$-codimensions*. These are the varieties $\mathfrak{U}$ for which there exist constants α, k with $c_n(\mathfrak{U},*) \leq \alpha n^k$ for all $n \in \mathbf{N}$. They are characterized in [25] and [27].

Algebras with polynomial growth of the codimensions are characterized in [27] as well.

Theorem 3.13 [27, Theorem 4]**.** *Let A be a finite dimensional algebra with involution $*$ over a field of characteristic 0 and J its Jacobson radical. Then*

$J^* = J$ and there exists a maximal semisimple subalgebra B such that $B = B^*$ and $A = B + J$.

The proof starts with the case $J^2 = 0$ using Wedderburn-Malcev theorem and then continues by induction on the degree of nilpotency of J. The theorem is issential in proving the main result in the paper, namely

Theorem 3.14 [27, Theorem 6]**.** *Let A be a finite dimensional algebra with involution over an algebraically closed field F of characteristic 0. Then the sequence of $*$-codimensions $\{c_n(A,*)\}_{n\geq 1}$ is polynomially bounded if and only if*
(i) *The sequence of codimensions $\{c_n(A)\}_{n\geq 1}$ is polynomially bounded and*
(ii) *$A = B + J$, where B is a maximal semisimple subalgebra of A and $b = b^*$ for all $b \in B$.*

Exploiting the representation theory of the hyperoctahedral group one gets

Theorem 3.15 [27, Theorem 7]**.** *Let A be a finite dimensional algebra with involution over a field F. Then the sequence of $*$-codimensions $\{c_n(A,*)\}_{n\geq 1}$ is polynomially bounded if and only if*

$$\chi_n(A,*) = \sum_{\substack{|\lambda|+|\mu|=n \\ n-\lambda_1<q}} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $J(A)^q = 0$.

The series of papers on growth of algebras includes [41], where Mishchenko and Valenti introduce a new finite dimensional algebra with involution, denoted M, and give a complete description of the ideal of $*$-identities of M through the representation theory of the hyperoctahedral group.

Let $A = F(e_{11} + e_{33}) \oplus Fe_{12} \oplus Fe_{13} \oplus Fe_{22} \oplus Fe_{23}$ and consider the involution $*$ obtained by reflecting a matrix along its secondary diagonal, namely

$$\begin{pmatrix} u & r & t \\ 0 & v & s \\ 0 & 0 & u \end{pmatrix}^* = \begin{pmatrix} u & s & t \\ 0 & v & r \\ 0 & 0 & u \end{pmatrix}.$$

Clearly $\dim \mathrm{Sym}(A,*) = 4$, $\dim \mathrm{Skew}(A,*) = 1$. The set $I = Fe_{13}$ is a two-sided ideal of A invariant under $*$ and let $M := A/I$. If we denote $e_{11} + e_{33} + I = a$, $e_{22} + I = b$, $e_{12} + I = c$, $e_{23} + I = c^*$, then $M = \mathrm{Span}_F\{a, b, c, c^*\}$ with the following multiplication table

$$aa = a, \quad bb = b, \quad ac = c, \quad bc^* = c^*, \quad cb = c, \quad c^*a = c^*,$$
$$ab = ba = ac^* = bc = ca = cc = cc^* = c^*b = c^*c = c^*c^* = 0.$$

Obviously $\mathrm{Sym}(M,*) = \mathrm{Span}\{a,b,c+c^*\}$ and $\mathrm{Skew}(M,*) = \mathrm{Span}\{c-c^*\}$.

The following result describes the B_n-cocharacters of $T(M,*)$.

Theorem 3.16 [41, Theorem 1]. *Let* $\chi_n(M,*) = \sum_{r=0}^{n} \sum_{\lambda\vdash r,\mu\vdash n-r} \mu_{\lambda,\mu}\chi_{\lambda,\mu}$ *be the n-th $*$-cocharacter of M. Then $\mu_{\lambda,\mu} = q+1$ if either*
(i) $\lambda = (p+q,p), \mu = (1)$ *for all* $p \geq 0, q \geq 0$*, or*
(ii) $\lambda = (p+q,p), \mu = \emptyset$ *for all* $p \geq 1, q \geq 0$*, or*
(iii) $\lambda = (p+q,p,1), \mu = \emptyset$ *for all* $p \geq 1, q \geq 0$.
In all other cases $\mu_{\lambda,\mu} = 0$ *except the case* $m_{(n),\emptyset} = 1$.

An immediate corollary is that $\mathrm{Exp}(M,*) = 2$. An alternative proof of this fact can be found in [44]q where the description of the proper subvarieties of the variety generated by M is given.

The decomposition of the $GL_r \times GL_r$-module of the proper polynomials $B_r(M,*)$ is the following.

Theorem 3.17 [43, Theorem 6]. *For $n \geq 2$, the n-th homogeneous component of $B_r(M,*) = B_r(*)/(T(M) \cap B_r(*)$ is*

$$B_r^{(n)}(M,*) = (W_r((n-1),1)) \otimes W_r(0)) \oplus (W_r((n-1)) \otimes W_r((1))).$$

Corollary 3.18 [43, Corollary 2]. *The n-th $*$-codimension of the variety generated by M, denoted* var $(M,*)$, $n \in \mathbf{N}$*, is*

$$c_n(\mathrm{var}(M),*) = 1 + \sum_{k=1}^{n} \binom{n}{k} k \ \geq 2^n.$$

Therefore var $(M,*)$ *does not have polynomial growth.*

Corollary 3.19 [43, Corollary 3]. *Every proper $*$-subvariety of the $*$-variety generated by M has polynomial growth.*

Let $G_2 = F \oplus F$ be the algebra with exchange involution $(a,b)^* = (b,a)$. In [25] it was proved that $\mathrm{var}(G_2,*)$ has almost polynomial growth. In [41] it is shown that $\mathrm{var}(G_2,*)$ and $\mathrm{var}(M,*)$ are the only two varieties with almost polynomial growth.

Theorem 3.20 [41, Theorem 4]. *Let A be a finite dimensional algebra with involution over an algebraically closed field of characteristic zero. Then* $\mathrm{var}(A,*)$ *does not have polynomial growth if and only if either* $G_2 \in \mathrm{var}(A,*)$ *or* $M \in \mathrm{var}(A,*)$.

Corollary 3.21 [41, Corollary 2]**.** *Let A be a finite dimensional algebra. Then the sequence of $*$-codimensions $\{c_n(A,*)\}_{n\geq 1}$ either has polynomial growth or exponential growth.*

Giambruno and Mishchenko [25] determine the $*$-cocharacters of the algebra G_2 getting that $\chi_n(G_2,*)=2^n$. They prove

Theorem 3.22 [25, Theorem 3]**.** *Let A be an algebra with involution. Then $c_n(A,*)\leq \alpha n^k$ for some α, k if and only if there exists a constant β such that*

$$\chi_n(A,*)=\sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

and $m_{\lambda,\mu}=0$ whenever either $|\lambda|>\beta$ or $|\mu|>\beta$.

Theorem 3.23 [25, Theorem 4]**.** *Let A be an algebra with involution such that*

$$T(A,*)\supset T(G_2,*).$$

Then $\{c_n(A,*)\}_{n\geq 1}$ is polynomially bounded.

The case $\mathrm{Exp}(A,*)=2$ is fully characterized in

Corollary 3.24 [46, Corollary]**.** *5 Let A be a finite dimensional algebra over a field of characteristic zero. Then* $\mathrm{Exp}(A,*)=2$ *if and only if either $T(A,*)\subseteq T(G_2,*)$ or $T(A,*)\subseteq T(M,*)$.*

The case when the multiplicities of the $*$-cocharacters are bounded by a constant is considered by Otera in [45].

Theorem 3.25 [45, Theorem 3.3.3]**.** *Let A be a finite dimensional algebra with involution over a field F of characteristic* 0 *such that its* $*$-T-*ideal is not contained in the* $*$-T-*ideal of $(M_2(F),s)$. Then the following conditions are equivalent:*

(i) *There exists a constant k such that for any $n\geq 1$ and $|\lambda|+|\mu|=n$ one has $m_{\lambda,\mu}\leq k$.*

(ii) *The* $*$-T-*ideal of A is not contained in the* $*$-T-*ideal of the algebra M and in the* $*$-T-*ideal of $(M_2(F),t)$.*

(iii) *$m_{\lambda,\mu}=0$ whenever $(|\lambda|-\lambda_1)+(|\mu|-\mu_1)\geq q$, where q is the index of nilpotency of the Jacobson radical of A.*

Now we pay attention to the varieties with exponential growth such that every proper subvariety has polynomial growth. Following Giambruno and MIshchenko [26] we call them *varieties with almost polynomial growth.* The paper [26] gives a complete description of the varieties $\mathfrak{V}$ with involution for which the sequence $c_n(\mathfrak{V}, *)$ is polynomially bounded. They prove that $\mathfrak{V}$ is such a variety if and only if $G_2, M \notin \mathfrak{V}$. It follows that G_2 and M generate the only two varieties with involution with almost polynomial growth and that there is no variety with intermediate growth.

More details on algebras with involution and growth are given in [24].

The codimensions of a concrete algebra are found by Anisimov in [5]. He considers the Grassmann algebra G (we need only characteristic 0). The algebra G is generated by $\{e_1, e_2, \ldots\}$ with defining relations $e_ie_j + e_je_i = 0$ and has a simply constructed involution. In [5] the author computes exactly the codimensions of G.

Theorem 3.26 [5, Theorem]. *Let G be a Grassmann algebra, φ_{id} - involution in G such that $\varphi_{\mathrm{id}}(e_i) = e_i$. Then $c_n(G, \varphi_{\mathrm{id}}) = 4^{n-\frac{1}{2}}$.*

The main idea of the proof uses the calculation of $c_n(G)$ done by Krakowski and Regev and an estimation of $c_n(G)$ mentioned before Theorem 3.3.

We end the survey with an application of the considered notions to the variety generated by $(M_2(F), t)$ done in [10].

The purpose of [10] is to describe the $*$-subvarieties of the variety $\mathrm{var}(M_2(F), t)$ or, equivalently, the $*$-T-ideals properly containing $T(M_2(F), *)$ by using the method due to Drensky in case of ordinary T-ideals. The authors construct two sequences of finite dimensional algebras with involution essential for this description.

We define Y-proper $*$-polynomials in $F\langle Y, Z\rangle$ as such in which the y's occur in commutators only. They are elements of the vector subspace $B_{(m)}(*)$ of $F_m(*)$ spanned by

$$\{z_1^{r_1} \cdots z_m^{r_m} u_1^{t_1} \cdots u_n^{t_n} | r_i, t_j \geq 0\},$$

where $u_1, u_2, \ldots$ are higher commutators.

Considering $W_{\lambda,\mu}$ as a representative of the corresponding isomorphism class of $GL_m \times GL_m$-modules it is generated by a non-zero element

$$w_{\lambda,\mu} := w_\lambda(y_1, \ldots, y_p) w_\mu(z_1, \ldots, z_q) \sum_{\sigma \in \mathrm{Sym}_n} \alpha_\sigma \sigma \ (\alpha_\sigma \in F),$$

where $w_\lambda(x_1, \ldots, x_p) := \prod_{j=1}^{\lambda_1} S_{p_j}(x_1, \ldots, x_{p_j})$, p_j is the height of the j-th column of the corresponding Young diagram, $p = p_1$ and the S_{p_j} is the standard polynomial of degree p_j.

The polynomial $w_{\lambda,\mu}$ is the so-called *highest weight vector* of the module $W_{\lambda,\mu}$.

Let $k \geq 1$ and $C_k = K[u]/(t^u)$ be the polynomial algebra modulo the ideal generated by u^k.

We define the following algebras with involution

$$\begin{aligned} R_p &= C_p e + uC_p a + uC_p b + C_p c \\ S_q &= C_q e + C_q a + uC_q b + uC_q c, \end{aligned}$$

where e, a, b, c are the matrices $e = e_{11} + e_{22}$, $a := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $b := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Lemma 3.27 [10, Lemma 4.2]**.** (i) *The following relations hold*

$$\begin{aligned} &a^2 = b^2 = -c^2 = e \\ &ab = c = -ba,\ ac = b = -ca,\ cb = a = -bc. \end{aligned}$$

(ii) *The previous relations yield*

$$[a, b] = 2c, [a, c] = 2b, [c, b] = 2a.$$

(iii) *For higher commutators, the following relations hold*

$$[c, \underbrace{a, \dots, a}_{p}] = 2^p ca^p;\ [b, \underbrace{a, \dots, a}_{p}] = 2^p ba^p.$$

Since the algebras $M_2(C_p)$ and $M_2(C_q)$ have the same $*$-polynomial identities as $M_2(F)$, and R_p and S_q are subalgebras of $M_2(C_p)$ and $M_2(C_q)$, respectively, we obtain that the $GL_m \times GL_m$ modules $B_{(m)}(R_p, *)$ and $B_{(m)}(S_q, *)$ are homomorphic images of $B_{(m)}(M_2(F), *)$. According to [17, Theorem 3.4] for finding the irreducible submodules in the decompositions of $B_{(m)}(R_p, *)$ and $B_{(m)}(S_q, *)$ it will suffice to work in $B_{(2)}(*)$ and consider Y-proper polynomials in which just one z occurs, i.e. Y-proper polynomials in y_1, y_2, z_1.

Lemma 3.28 [10, Lemma 4.3]**.** *Let W be the irreducible component of $B_{(2)}(M_2(F), *)$ associated to the pair $((\lambda_1, \lambda_2), k)$. Then*

$$\begin{aligned} w &= [z, \underbrace{y_1, \dots, y_1}_{\lambda_1 - \lambda_2}][y_2, y_1]^{\lambda_2} z^{k-1},\ \textit{if } k > 0, \\ w &= [[y_2, y_1], \underbrace{y_1, \dots, y_1}_{\lambda_1 - \lambda_2}][y_2, y_1]^{\lambda_2 - 1},\ \textit{if } k = 0 \end{aligned}$$

is the highest weight vector.

Lemma 3.29 [10, Lemma 4.4, Lemma 4.5]**.** *Let $k \geq 0$. Considering Lemma* 3.28

(i) *w is a $*$-polynomial identity for R_p if and only if $\lambda_1 + \lambda_2 \geq p$,*

(ii) *w is a $*$-polynomial identity for S_q if and only if $\lambda_2 + k \geq q$.*

Taking into account the consequences of the highest weight vectors the authors of [10] come to the following description of the $*$-T-ideals properly containing $T(M_2(F), *)$.

Definition 3.30 [10]**.** *Let $B^{(n)}(*)$ be the space of all Y-proper polynomials of degree n in $F\langle Y, Z\rangle$. The $*$-*T*-ideals of $F\langle Y, Z\rangle$, U_1 and U_2, are $*$-asymptotically equivalent if there exists $\nu \in N$ such that for all $n \geq \nu_0$ we have $U_1 \cap B^{(n)}(*) = U_2 \cap B^{(n)}(*)$, writing $U_1 \approx_* U_2$.*

The main result is

Theorem 3.31 [10, Theorem 6.1]**.** *Let $* = t$ be the transpose involution. If U is the $*$-*T*-ideal of $F\langle Y, Z\rangle$ of a proper subvariety of the variety $var(M_2(F), *)$, then $U \approx_* T(R_p, *) \cap T(S_q, *)$ for suitable p and q.*

The two main branches of investigations: giving a better description of involution algebras via their $*$-identities and finding the codimension series and estimating the algebras' growth, show the ways for further investigations considering algebras with involutions (of both types) using the rich background of ordinary P.I. algebras and methods applied to them.

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