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# COHOMOLOGY OF THE $G$-HILBERT SCHEME FOR $\frac{1}{r}(1,1,r-1)$

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Abstract. In this note we attempt to generalize a few statements drawn from the 3-dimensional McKay correspondence to the case of a cyclic group not in $\mathrm{SL}(3,\mathbb{C})$. We construct a smooth, discrepant resolution of the cyclic, terminal quotient singularity of type $\frac{1}{r}(1,1,r-1)$, which turns out to be isomorphic to Nakamura's $G$-Hilbert scheme. Moreover we explicitly describe tautological bundles and use them to construct a dual basis to the integral cohomology on the resolution.

**1. Introduction.** In the case of a finite, abelian group $G \subset \mathrm{SL}(3,\mathbb{C})$, Craw and Reid [2] construct explicitly a smooth, crepant toric resolution of the quotient singularity $\mathbb{C}^3/G$. Moreover in [1] Craw shows that the integral cohomology of the resolution has rank equal to the order of the group $G$ and

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constructs a dual basis using tautological bundles. For finite $G$ in $\mathrm{GL}(2,\mathbb{C})$ the cohomology of the minimal resolution has rank smaller than the order of $G$ (compare [7]). Craw and Reid calculated $G$-Hilb for $G=\frac{1}{r}(1,a,r-a)$, and for most values of $a$ it is very discrepant and still singular, with ordinary double points $xy=zt$. We show that in the case of a cyclic, terminal, quotient singularity of type $\frac{1}{r}(1,1,r-1)$ the $G$-Hilbert scheme is a smooth, discrepant resolution and its integral cohomology has rank $2r-1$. The dual basis to cohomology is constructed using tautological bundles introduced by Gonzalez–Sprinberg and Verdier. We assume that the reader is familiar with basic toric geometry ([4], [9]).

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**2. Toric resolution.** Let us fix an integer $r\geq 2$ and the group $G$ generated by the element $\mathrm{diag}(\varepsilon,\varepsilon,\varepsilon^{r-1})$, where $\varepsilon=e^{\frac{2\pi i}{r}}$. The group $G$ has $r$ characters which may be identified with $1,\varepsilon,\varepsilon^2,\dots,\varepsilon^{r-1}$. To use toric geometry methods introduce the lattice

$$N=\mathbb{Z}^3+\frac{1}{r}(1,1,r-1)\mathbb{Z},$$

and its dual $M=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$. Consider the cone $\sigma=\mathbb{R}_{\geq 0}e_1+\mathbb{R}_{\geq 0}e_2+\mathbb{R}_{\geq 0}e_3$ generated by non–negative combinations of the standard basis vectors of $\mathbb{Z}^3$ in $N\otimes_{\mathbb{Z}}\mathbb{R}$ and define $X=\mathbb{C}^3/G$. Then it is easy to see that

$$X=\operatorname{Spec}\mathbb{C}[x,y,z]^G\simeq\operatorname{Spec}\mathbb{C}[\sigma^\vee\cap M],$$

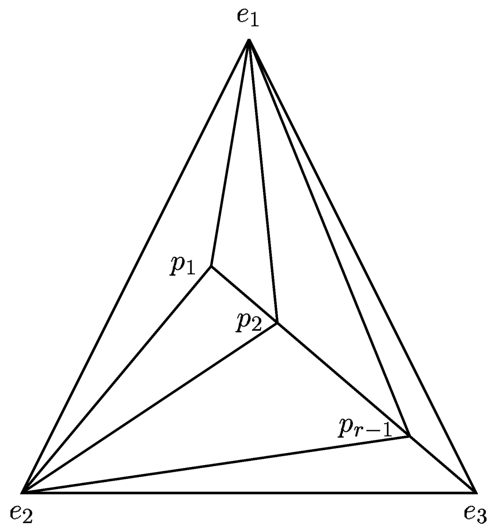
where

$$\sigma^\vee=\{u\in M\ :\ \langle u,v\rangle\ \geq\ 0 \text{ for all } v\in\sigma\},$$

and the functions $x,y,z$ are identified with the dual elements $e_1^*,e_2^*,e_3^*$ (see [4] p. 3–8 for more details). This identification will be used in the rest of the paper.

**Definition 2.1.** *Let $p_i=\frac{1}{r}(r-i,r-i,i)$ for $i=1,2,\dots,r$ be the points in the lattice $N$ (note that $p_r=e_3$). Define $Y$ as the toric variety given by the fan $\Delta$ obtained from the cone $\sigma$ by the sequence of successive star subdivisions along the rays $\mathbb{R}_{\geq 0}p_{r-1},\dots,\mathbb{R}_{\geq 0}p_1$. Denote by $f:Y\longrightarrow X$ the resulting proper, birational toric morphism given by the identity map on the lattice $N$, and let*

$\mathrm{Ex}(f)$ *be the exceptional set of* $f$ *(see* [4] *p. 48 and picture below showing the fan* $\Delta$ *intersected with the hyperplane* $e_1^* + e_2^* + 2e_3^* = 2$*).*




**Lemma 2.1.** *$Y$ is a smooth toric variety.*

Proof. Since the fan $\Delta$ is simplicial it is enough to check that the primitive vectors along generating rays for every 3-dimensional cone in $\Delta$ form a $\mathbb{Z}$-basis for the lattice $N$. This follows easily as

$$\det[e_1, e_2, p_1] = \det[e_j, p_i, p_{i+1}] = \frac{1}{r}$$

for $j = 1, 2$, $i = 1, \ldots, r-1$. $\square$

Denote by $\tau_i = \mathbb{R}_{\geq 0} p_i$ the ray through $p_i$ for $i = 1, \ldots, r-1$. The irreducible components of exceptional set $\mathrm{Ex}(f)$ are in one-to-one correspondence with the rays $\tau_i$. Each component is a compact toric surface defined by the fan $\mathrm{Star}(\tau_i)$ in the quotient lattice $N(\tau_i)$ (details [4] p. 52). It is also useful to have dual coordinates for every 3–dimensional cone in the fan $\Delta$. They are:

$$\sigma^{\vee}_{e_1,e_2,p_1} = \sigma_{e_1^*+(1-r)e_3^*,\, e_2^*+(1-r)e_3^*,\, re_3^*},$$

$$\sigma^{\vee}_{e_1,p_i,p_{i+1}} = \sigma_{e_1^*-e_2^*,\, ie_2^*+(i-r)e_3^*,\, (i+1)e_2^*+(i+1-r)e_3^*},$$

$$\sigma^{\vee}_{e_2,p_i,p_{i+1}} = \sigma_{-e_1^*+e_2^*,\, ie_1^*+(i-r)e_3^*,\, (i+1)e_1^*+(i+1-r)e_3^*}.$$

for $i = 1, \dots, r-1$, where for example $\sigma_{e_1,e_2,p_1}$ denotes the cone generated by $\mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2$ and $\tau_1$.

**Definition 2.2.** *Let $S_i$ be the $i$-th irreducible divisor in* $\mathrm{Ex}(f)$ *defined by the fan* $\mathrm{Star}(\tau_i)$*, that is*

$$S_i = \mathrm{V}(\tau_i).$$

**Lemma 2.2.** *The exceptional irreducible divisors in* $\mathrm{Ex}(f)$ *are $S_1 \simeq \mathbb{P}^2$ and $S_i \simeq \mathbb{F}_i$ for $i = 2, \dots, r-1$ where $\mathbb{F}_i$ is a Hirzebruch surface* (*see* [4] *p.* 7)*.*

Proof. For the surface $S_i$ pick two dual coordinates in an adjacent 3-dimensional cone in $\Delta$ vanishing on $\tau_i$. Evaluating them on primitive vectors along rays generating 2-dimensional cones containing $\tau_i$ gives generators of rays in the fan $\mathrm{Star}(\tau_i)$. That is for the surface $S_1$ choose the cone $\sigma_{e_1,e_2,p_1}$ and set $X = e_1^* + (1-r)e_3^*$ and $Y = e_2^* + (1-r)e_3^*$. Then

$$(X(e_1), Y(e_1)) = (1, 0),$$

$$(X(e_2), Y(e_2)) = (0, 1),$$

$$(X(p_2), Y(p_2)) = (-1, -1),$$

so $S_1 \simeq \mathbb{P}^2$. Analogously from $\sigma^\vee_{e_2,p_i,p_{i+1}}$ pick $X = ie_1^* + (i-r)e_3^*$ and $Y = -e_1^* + e_2^*$. Then

$$(X(e_1), Y(e_1)) = (i, -1),$$

$$(X(p_{i-1}), Y(p_{i-1})) = (1, 0),$$

$$(X(e_2), Y(e_2)) = (0, 1),$$

$$(X(p_{i+1}), Y(p_{i+1})) = (-1, 0),$$

hence the lemma follows. $\square$

From the toric picture it is easy to see that $\mathrm{Ex}(f)$ consists of a tower of $\mathbb{P}^2$ and Hirzebruch rational scrolls, that is $S_i \cap S_{i+1} = \mathbb{P}^1$ for $i = 1, \dots, r-2$, where $\mathbb{P}^1$ corresponds to the cone spanned by $\tau_i$ and $\tau_{i+1}$. Using homotopy $x \longrightarrow tx$ of $\mathbb{C}^3$ we can contract $X$ to a singular point. The homotopy lifts via $f$ to $Y$. Since the exceptional set lies over the singularity on $X$ one sees that $Y$ is homotopic to a tubular neighborhood of $\mathrm{Ex}(f)$ so that $\mathrm{H}^*(Y,\mathbb{Z}) \simeq \mathrm{H}^*(\mathrm{Ex}(f),\mathbb{Z})$. The basis of $\mathrm{H}^2(\mathbb{F}_i,\mathbb{Z})$ consists of rational curves $L_i$ and $M_i$ satisfying the relations $L_i^2 = 0$, $L_iM_i = 1$, and $M_i^2 = -i$ (see [10], Lemma 2.7). By induction on $r$ and using the Mayer-Vietoris sequence it is clear that the basis of $\mathrm{H}^*(\mathrm{Ex}(f),\mathbb{Z})$ is given by the class of a point in degree 0, the classes of the curves $L_i$ in degree 2 ($L_1$ stands for $\mathbb{P}^1$ in $S_1$) and by the classes of $S_i$ in degree 4.

**Definition 2.3** Nakamura**.** *The G-Hilbert scheme* $G$-$\mathrm{Hilb}\,\mathbb{C}^3$ *is the moduli space of G-clusters, that is* 0*-dimensional, G-invariant subschemes* $Z \subset \mathbb{C}^3$ *such that* $H^0(Z,\mathcal{O}_Z)$ *is the regular representation* $\mathbb{C}[G]$ *of the group* $G$*.*

For working with $G$-$\mathrm{Hilb}\,\mathbb{C}^3$ schemes following Nakamura [8] it is convenient to introduce the notion of a $G$-set.

**Definiton 2.4.** *A subset* $\Gamma$ *of monomials in* $\mathbb{C}[x,y,z]$ *is called a G-set if*

(1) *it contains the constant monomial* 1*,*

(2) *if* $pq \in \Gamma$ *then* $p \in \Gamma$ *and* $q \in \Gamma$*,*

(3) *there is a* 1*–to–*1 *correspondence between* $\Gamma$ *and irreducible representations of* $G$ *with respect to the induced action of* $G$ *on* $\mathbb{C}[x,y,z]$*.*

We can identify $G$-$\mathrm{Hilb}\,\mathbb{C}^3$ with a moduli space for ideals $I$ in $\mathbb{C}[x,y,z]$ such that $\mathbb{C}[x,y,z]/I = \mathbb{C}[G]$. The monomials in a basis of $\mathbb{C}[x,y,z]/I$ give elements of a $G$-set.

**Lemma 2.3.** *The only possible G-sets in the case of* $\dfrac{1}{r}(1,1,r-1)$ *are:*

$$\Gamma_i^x = \{z^i, z^{i-1}, \ldots, 1, x, x^2, \ldots, x^{r-i-1}\} \text{ for } i = 0,\ldots,r-2,$$

$$\Gamma_i^y = \{z^i, z^{i-1}, \ldots, 1, y, y^2, \ldots, y^{r-i-1}\} \text{ for } i = 0,\ldots,r-2,$$

$$\Gamma^z = \{z^{r-1}, z^{r-2}, \ldots, 1\}.$$

P r o o f. If $\Gamma$ is a $G$-set, then $xz, yz \notin \Gamma$ since 1 already represents trivial character. Moreover $xy \notin \Gamma$ because $x$ and $y$ represent the same character $\varepsilon$, so $\Gamma$ contains only monomials in one variable. If $z^i$ is the maximal power of $z$ in $\Gamma$ then either $x^{r-i-1}$ or $y^{r-i-1}$ must be in $\Gamma$, and the result follows. $\square$

**Lemma 2.4.** *The morphism* $f : Y \longrightarrow X$ *is a resolution of singularities and* $Y \simeq G$-$\mathrm{Hilb}\,\mathbb{C}^3$*.*

P r o o f. After Lemma 2.1 it is enough to compute all $G$-sets (in the spirit of [8] or [1], Section 5.1) using dual coordinates for every cone in $\Delta$ and check if all possible are present. For the cone $\sigma_{e_1,e_2,p_1}$ the dual coordinates on the corresponding affine open chart $\mathbb{C}^3$ are $\alpha = \frac{x}{z^{r-1}}$, $\beta = \frac{y}{z^{r-1}}$, $\gamma = z^r$. They give generators $x - \alpha z^{r-1}, y - \beta z^{r-1}, z^r - \gamma$ of the ideal defining a $G$-cluster. In this case the corresponding $G$-set is given by $\Gamma^z$. Similarly for the cone $\sigma_{e_1,p_i,p_{i+1}}$ we get generators $x - \alpha y, y^{i+1} - \beta z^{r-i-1}, z^{r-i} - \gamma y^i$ and the $G$-set $\Gamma^y_{r-i-1}$, and for the cone $\sigma_{e_2,p_i,p_{i+1}}$ generators $y - \alpha x, x^{i+1} - \beta z^{r-i-1}, z^{r-i} - \gamma x^i$ and the $G$-set $\Gamma^x_{r-i-1}$. $\square$

**3. Tautological bundles.** Tautological bundles on the resolutions of Kleinian singularities were defined by Gonzalez–Sprinberg and Verdier [5]. In the two dimensional case they define a basis of the K–group of the minimal resolution and have degree 1 on exactly one exceptional curve of the minimal resolution. In the toric case we adapt an equivalent definition (see [1] Def. 4.7, [11] Section 4 and [5] p. 417 for original treatment).

**Definiton 3.1.** *If $\rho_i : G \longrightarrow \mathrm{GL}(V_i)$ is an irreducible representation, let*

$$R_i = \mathrm{Hom}_{\mathbb{C}[G]}(V_i, \mathbb{C}[x, y, z])$$

*be the $\mathcal{O}_X$-module generated by monomials in the $\varepsilon^i$-character space. Define tautological bundle $\mathcal{R}_i$ as*

$$\mathcal{R}_i = f^* R_i / \mathrm{Tors}_{\mathcal{O}_Y}$$

*i.e. pullback modulo torsion.*

Each $R_i$ is generated by the monomials $x^i, y^i, z^{r-i} \in \mathbb{C}[x, y, z]$ as an $\mathcal{O}_X$-module. Multiplying by $z^i$ we see that it is isomorphic to the ideal sheaf $(x^i z^i, y^i z^i, z^r) \subset \mathcal{O}_X$. We claim that $\mathcal{R}_i$ is an invertible sheaf. Indeed on the toric picture it is represented as a Cartier divisor by the piecewise linear function on the fan $\Delta$ given by $ie_1^* + ie_3^*$ on the cone $\sigma_{e_2,e_3,p_i}$, $ie_2^* + ie_3^*$ on the cone $\sigma_{e_1,e_3,p_i}$ and by $re_3^*$ on $\sigma_{p_i,e_1,e_2}$ (see [11] p. 5–8 and [2] Example 4.8). We note that this Cartier divisor is equivalent to the $\mathbb{Q}$-Cartier divisor corresponding to $(x^i, y^i, z^{r-i})$ and it is more convenient to expand it in terms of linear equivalence classes of exceptional surfaces:

$$\mathcal{R}_1 = -\frac{r-1}{r}S_1 - \frac{r-2}{r}S_2 - \cdots - \frac{2}{r}S_{r-2} - \frac{1}{r}S_{r-1},$$

$$\mathcal{R}_2 = -\frac{r-2}{r}S_1 - \frac{2(r-2)}{r}S_2 - \frac{2(r-3)}{r}S_3 - \cdots - \frac{2\cdot 2}{r}S_{r-2} - \frac{2}{r}S_{r-1},$$

$$\vdots$$

$$\mathcal{R}_i = -\frac{r-i}{r}S_1 - \frac{2(r-i)}{r}S_2 - \cdots - \frac{i(r-i)}{r}S_i - \frac{i(r-i-1)}{r}S_{i+1} - \cdots$$

$$\cdots - \frac{2i}{r}S_{r-2} - \frac{i}{r}S_{r-1},$$

$$\vdots$$

$$\mathcal{R}_{r-1} = -\frac{1}{r}S_1 - \frac{2}{r}S_2 - \cdots - \frac{r-2}{r}S_{r-2} - \frac{r-1}{r}S_{r-1}.$$

Observe that as a $\mathbb{Q}$-Cartier divisor $\mathcal{R}_1$ is the discrepancy divisor for $f$ (see [12] p. 373–374), that is $f^*(K_X) = K_Y + \mathcal{R}_1$ and the Cartier divisor $r\mathcal{R}_1$ is linearly equivalent to $-rK_Y$ (the equivalence is given by linear function $re_1^* + re_2^* + re_3^*$). In fact $rK_X$ is linearly trivial.

## 4. Main result.

**Definition 4.1.** *Define virtual sheaves*

$$\mathcal{V}_i = (\mathcal{R}_1 \oplus \mathcal{R}_i) \ominus ((\mathcal{R}_1 \otimes \mathcal{R}_i) \oplus \mathcal{O}_Y).$$

These virtual sheaves will be used to construct the dual basis to cohomology. For any bundles $\mathcal{F}, \mathcal{G}$ define

$$c(\mathcal{F} \ominus \mathcal{G}) = \frac{c(\mathcal{F})}{c(\mathcal{G})}.$$

**Theorem 4.1.** *The tautological bundles $\mathcal{R}_i$ form the dual basis of $H^2(Y,\mathbb{Z})$, that is $c_1(\mathcal{R}_i)\cdot L_j = \delta_{ij}$ and the virtual sheaves $\mathcal{V}_i$ form the dual basis of $H^4(Y,\mathbb{Z})$, that is $c_2(\mathcal{V}_i)\cdot S_j = \delta_{ij}$.*

P r o o f. The divisor $\mathcal{R}_i$ has degree 1 on the fiber $L_i$ of rational scroll $\mathbb{F}_i$ which corresponds to the line joining $e_1$ with $p_i$ in the toric picture. It has also degree 0 on $L_j$ for $i \neq j$. This proves the first part of the theorem. The second part is proven by inspecting the following table of first Chern classes computed on every compact surface:

| | $c_1(\mathcal{R}_1)$ | $c_1(\mathcal{R}_2)$ | $c_1(\mathcal{R}_3)$ | $\ldots$ | $c_1(\mathcal{R}_{r-1})$ |
|---|---|---|---|---|---|
| $\mathbb{P}^2$ | $L_1$ | 0 | 0 | $\ldots$ | 0 |
| $\mathbb{F}_2$ | $L_2$ | $M_2 + 2L_2$ | 0 | $\ldots$ | 0 |
| $\mathbb{F}_3$ | $L_3$ | $2L_3$ | $M_3 + 3L_3$ | $\ldots$ | 0 |
| $\mathbb{F}_4$ | $L_4$ | $2L_4$ | $3L_4$ | $\ldots$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $\mathbb{F}_{r-1}$ | $L_{r-1}$ | $2L_{r-1}$ | $3L_{r-1}$ | $\ldots$ | $M_{r-1} + (r-1)L_{r-1}$ |

and by the equation $c_2(\mathcal{F} \oplus \mathcal{F}') = c_1(\mathcal{F})c_1(\mathcal{F}')$, which holds for any line bundles $\mathcal{F}, \mathcal{F}'$. The restriction of the bundle $\mathcal{R}_i$ to the surface $S_j$ is computed by choosing from the piecewise function for $\mathcal{R}_i$ a linear function on one of the 3-dimensional cones containing $\tau_j$ and subtracting it from the functions on all the other cones. Evaluating the resulting functions on primitive vectors in rays generating 2–dimensional cones containing $\tau_j$ gives minus coefficients for the desired torus invariant Cartier divisor on the fan $\mathrm{Star}(\tau_j)$ (see [9] for more details). Observe also that $c_1(\mathcal{V}_i) = 0$, so the second Chern class of $\mathcal{V}_i$ is integral.

This result computes also

$$\operatorname{rank} \mathrm{H}^*(Y, \mathbb{Z}) \;=\; 2r - 1$$

($r-1$ for the second and fourth cohomology and 1 for the zeroth). It would be also interesting to obtain similar results in the general case of $\frac{1}{r}(1, a, r-a)$ for the 'economic', smooth resolution (see [12], Section 5). We note also that this 'economic' resolution is isomorphic to the $G$-Hilbert scheme only for $a = 1$. $\square$

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