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COHOMOLOGY OF THE
G-HILBERT SCHEME FOR \( \frac{1}{r}(1, 1, r - 1) \)

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Abstract. In this note we attempt to generalize a few statements drawn from the 3-dimensional McKay correspondence to the case of a cyclic group not in \( \text{SL}(3, \mathbb{C}) \). We construct a smooth, discrepant resolution of the cyclic, terminal quotient singularity of type \( \frac{1}{r}(1, 1, r - 1) \), which turns out to be isomorphic to Nakamura’s \( G \)-Hilbert scheme. Moreover we explicitly describe tautological bundles and use them to construct a dual basis to the integral cohomology on the resolution.

1. Introduction. In the case of a finite, abelian group \( G \subset \text{SL}(3, \mathbb{C}) \), Craw and Reid [2] construct explicitly a smooth, crepant toric resolution of the quotient singularity \( \mathbb{C}^3/G \). Moreover in [1] Craw shows that the integral cohomology of the resolution has rank equal to the order of the group \( G \) and

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constructs a dual basis using tautological bundles. For finite $G$ in $\text{GL}(2, \mathbb{C})$ the cohomology of the minimal resolution has rank smaller than the order of $G$ (compare [7]). Craw and Reid calculated $G$-Hilb for $G = \frac{1}{r}(1, a, r - a)$, and for most values of $a$ it is very discrepant and still singular, with ordinary double points $xy = zt$. We show that in the case of a cyclic, terminal, quotient singularity of type $\frac{1}{r}(1, 1, r - 1)$ the $G$-Hilbert scheme is a smooth, discrepant resolution and its integral cohomology has rank $2r - 1$. The dual basis to cohomology is constructed using tautological bundles introduced by Gonzalez–Springer and Verdier. We assume that the reader is familiar with basic toric geometry ([4], [9]).

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2. Toric resolution. Let us fix an integer $r \geq 2$ and the group $G$ generated by the element $\text{diag}(\varepsilon, \varepsilon, \varepsilon^{r-1})$, where $\varepsilon = e^{2\pi i / r}$. The group $G$ has $r$ characters which may be identified with $1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{r-1}$. To use toric geometry methods introduce the lattice

$$N = \mathbb{Z}^3 + \frac{1}{r}(1, 1, r - 1)\mathbb{Z},$$

and its dual $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Consider the cone $\sigma = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}e_3$ generated by non-negative combinations of the standard basis vectors of $\mathbb{Z}^3$ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ and define $X = \mathbb{C}^3/G$. Then it is easy to see that

$$X = \text{Spec } \mathbb{C}[x, y, z]^G \simeq \text{Spec } \mathbb{C}[\sigma^\vee \cap M],$$

where

$$\sigma^\vee = \{u \in M : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\},$$

and the functions $x, y, z$ are identified with the dual elements $e_1^*, e_2^*, e_3^*$ (see [4] p. 3–8 for more details). This identification will be used in the rest of the paper.

Definition 2.1. Let $p_i = \frac{1}{r}(r - i, r - i, i)$ for $i = 1, 2, \ldots, r$ be the points in the lattice $N$ (note that $p_r = e_3$). Define $Y$ as the toric variety given by the fan $\Delta$ obtained from the cone $\sigma$ by the sequence of successive star subdivisions along the rays $\mathbb{R}_{\geq 0}p_{r-1}, \ldots, \mathbb{R}_{\geq 0}p_1$. Denote by $f : Y \longrightarrow X$ the resulting proper, birational toric morphism given by the identity map on the lattice $N$, and let
Ex\( (f) \) be the exceptional set of \( f \) (see [4] p. 48 and picture below showing the fan \( \Delta \) intersected with the hyperplane \( e_1^* + e_2^* + 2e_3^* = 2 \)).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{figure}

**Lemma 2.1.** \( Y \) is a smooth toric variety.

**Proof.** Since the fan \( \Delta \) is simplicial it is enough to check that the primitive vectors along generating rays for every 3-dimensional cone in \( \Delta \) form a \( \mathbb{Z} \)-basis for the lattice \( N \). This follows easily as

\[
\det[e_1, e_2, p_1] = \det[e_j, p_i, p_{i+1}] = \frac{1}{r}
\]

for \( j = 1, 2, i = 1, \ldots, r - 1 \). \( \square \)

Denote by \( \tau_i = \mathbb{R}_{\geq 0}p_i \) the ray through \( p_i \) for \( i = 1, \ldots, r - 1 \). The irreducible components of exceptional set \( \text{Ex}(f) \) are in one-to-one correspondence with the rays \( \tau_i \). Each component is a compact toric surface defined by the fan \( \text{Star}(\tau_i) \) in the quotient lattice \( N(\tau_i) \) (details [4] p. 52). It is also useful to have dual coordinates for every 3-dimensional cone in the fan \( \Delta \). They are:

\[
\sigma^\vee_{e_1, e_2, p_1} = \sigma_{-e_1^* + (1-r)e_3^* + e_2^* + (1-r)e_3^*, re_3^*};
\]

\[
\sigma^\vee_{e_1, p_i, p_{i+1}} = \sigma_{e_1^* - e_2^*, ie_2^* + (i-r)e_3^* + (i+1)e_2^* + (i+1-r)e_3^*};
\]

\[
\sigma^\vee_{e_2, p_i, p_{i+1}} = \sigma_{-e_1^* + e_2^*, ie_1^* + (i-r)e_3^* + (i+1)e_1^* + (i+1-r)e_3^*};
\]

\[
\sigma^\vee_{e_3, p_i, p_{i+1}} = \sigma_{-e_1^* + e_3^*, ie_1^* + (i-r)e_3^* + (i+1)e_1^* + (i+1-r)e_3^*};
\]
for $i = 1, \ldots, r - 1$, where for example $\sigma_{e_1, e_2, p_1}$ denotes the cone generated by $\mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2$ and $\tau_1$.

**Definition 2.2.** Let $S_i$ be the $i$-th irreducible divisor in $\text{Ex}(f)$ defined by the fan $\text{Star}(\tau_i)$, that is

$$S_i = V(\tau_i).$$

**Lemma 2.2.** The exceptional irreducible divisors in $\text{Ex}(f)$ are $S_1 \simeq \mathbb{P}^2$ and $S_i \simeq \mathbb{F}_i$ for $i = 2, \ldots, r - 1$ where $\mathbb{F}_i$ is a Hirzebruch surface (see [4] p. 7).

**Proof.** For the surface $S_i$ pick two dual coordinates in an adjacent 3-dimensional cone in $\Delta$ vanishing on $\tau_i$. Evaluating them on primitive vectors along rays generating 2-dimensional cones containing $\tau_i$ gives generators of rays in the fan $\text{Star}(\tau_i)$. That is for the surface $S_1$ choose the cone $\sigma_{e_1, e_2, p_1}$ and set $X = e_1^* + (1-r)e_3^*$ and $Y = e_2^* + (1-r)e_3^*$. Then

$$(X(e_1), Y(e_1)) = (1, 0),$$

$$(X(e_2), Y(e_2)) = (0, 1),$$

$$(X(p_2), Y(p_2)) = (-1, -1),$$

so $S_1 \simeq \mathbb{P}^2$. Analogously from $\sigma^\vee_{e_2, p_i, p_{i+1}}$ pick $X = ie_1^* + (i-r)e_3^*$ and $Y = -e_1^* + e_2^*$. Then

$$(X(e_1), Y(e_1)) = (i, -1),$$

$$(X(p_{i-1}), Y(p_{i-1})) = (1, 0),$$

$$(X(e_2), Y(e_2)) = (0, 1),$$

$$(X(p_{i+1}), Y(p_{i+1})) = (-1, 0),$$

hence the lemma follows. \(\square\)

From the toric picture it is easy to see that $\text{Ex}(f)$ consists of a tower of $\mathbb{P}^2$ and Hirzebruch rational scrolls, that is $S_i \cap S_{i+1} = \mathbb{P}^1$ for $i = 1, \ldots, r - 2$, where $\mathbb{P}^1$ corresponds to the cone spanned by $\tau_i$ and $\tau_{i+1}$. Using homotopy $x \rightarrow tx$ of $\mathbb{C}^3$ we can contract $X$ to a singular point. The homotopy lifts via $f$ to $Y$. Since the exceptional set lies over the singularity on $X$ one sees that $Y$ is homotopic
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to a tubular neighborhood of $\text{Ex}(f)$ so that $H^*(Y, \mathbb{Z}) \simeq H^*(\text{Ex}(f), \mathbb{Z})$. The basis of $H^2(\mathbb{P}_i, \mathbb{Z})$ consists of rational curves $L_i$ and $M_i$ satisfying the relations $L_i^2 = 0$, $L_i M_i = 1$, and $M_i^2 = -i$ (see [10], Lemma 2.7). By induction on $r$ and using the Mayer-Vietoris sequence it is clear that the basis of $H^*(\text{Ex}(f), \mathbb{Z})$ is given by the class of a point in degree 0, the classes of the curves $L_i$ in degree 2 ($L_1$ stands for $\mathbb{P}_1$ in $S_1$) and by the classes of $S_i$ in degree 4.

**Definition 2.3** Nakamura. The G-Hilbert scheme $G\text{-Hilb} \mathbb{C}^3$ is the moduli space of G-clusters, that is 0-dimensional, $G$-invariant subschemes $Z \subset \mathbb{C}^3$ such that $H^0(Z, \mathcal{O}_Z)$ is the regular representation $\mathbb{C}[G]$ of the group $G$.

For working with $G\text{-Hilb} \mathbb{C}^3$ schemes following Nakamura [8] it is convenient to introduce the notion of a $G$-set.

**Definition 2.4.** A subset $\Gamma$ of monomials in $\mathbb{C}[x, y, z]$ is called a $G$-set if

1. it contains the constant monomial 1,
2. if $pq \in \Gamma$ then $p \in \Gamma$ and $q \in \Gamma$,
3. there is a 1–to–1 correspondence between $\Gamma$ and irreducible representations of $G$ with respect to the induced action of $G$ on $\mathbb{C}[x, y, z]$.

We can identify $G\text{-Hilb} \mathbb{C}^3$ with a moduli space for ideals $I$ in $\mathbb{C}[x, y, z]$ such that $\mathbb{C}[x, y, z]/I = \mathbb{C}[G]$. The monomials in a basis of $\mathbb{C}[x, y, z]/I$ give elements of a $G$-set.

**Lemma 2.3.** The only possible $G$-sets in the case of $\frac{1}{r}(1, 1, r - 1)$ are:

\[
\Gamma_i^x = \{z^i, z^{i-1}, \ldots, 1, x, x^2, \ldots, x^{r-i-1}\} \text{ for } i = 0, \ldots, r - 2, \\
\Gamma_i^y = \{z^i, z^{i-1}, \ldots, 1, y, y^2, \ldots, y^{r-i-1}\} \text{ for } i = 0, \ldots, r - 2, \\
\Gamma^z = \{z^{r-1}, z^{r-2}, \ldots, 1\}.
\]

**Proof.** If $\Gamma$ is a $G$-set, then $xz, yz \notin \Gamma$ since 1 already represents trivial character. Moreover $xy \notin \Gamma$ because $x$ and $y$ represent the same character $\varepsilon$, so $\Gamma$ contains only monomials in one variable. If $z^i$ is the maximal power of $z$ in $\Gamma$ then either $x^{r-i-1}$ or $y^{r-i-1}$ must be in $\Gamma$, and the result follows. □

**Lemma 2.4.** The morphism $f : Y \to X$ is a resolution of singularities and $Y \simeq G\text{-Hilb} \mathbb{C}^3$. 
Proof. After Lemma 2.1 it is enough to compute all $G$-sets (in the spirit of [8] or [1], Section 5.1) using dual coordinates for every cone in $\Delta$ and check if all possible are present. For the cone $\sigma_{e_1,e_2,p_i}$ the dual coordinates on the corresponding affine open chart $\mathbb{C}^3$ are $\alpha = \frac{x}{z^{r-1}}$, $\beta = \frac{y}{z^{r-1}}$, $\gamma = z^r$. They give generators $x - \alpha z^{r-1}, y - \beta z^{r-1}, z^r - \gamma$ of the ideal defining a $G$-cluster. In this case the corresponding $G$-set is given by $\Gamma^x$. Similarly for the cone $\sigma_{e_1,p_i,p_{i+1}}$ we get generators $x - \alpha y, y_i^{i+1} - \beta z^{r-i-1}, z^{r-i} - \gamma y^i$ and the $G$-set $\Gamma^y_{r-i-1}$, and for the cone $\sigma_{e_2,p_i,p_{i+1}}$ generators $y - \alpha x, x_i^{i+1} - \beta z^{r-i-1}, z^{r-i} - \gamma x^i$ and the $G$-set $\Gamma^z_{r-i-1}$. □

3. Tautological bundles. Tautological bundles on the resolutions of Kleinian singularities were defined by Gonzalez–Sprinberg and Verdier [5]. In the two dimensional case they define a basis of the $K$–group of the minimal resolution and have degree 1 on exactly one exceptional curve of the minimal resolution. In the toric case we adapt an equivalent definition (see [1] Def. 4.7, [11] Section 4 and [5] p. 417 for original treatment).

Definition 3.1. If $\rho_i : G \to \text{GL}(V_i)$ is an irreducible representation, let

$$R_i = \text{Hom}_{\mathbb{C}[G]}(V_i, \mathbb{C}[x, y, z])$$

be the $\mathcal{O}_X$-module generated by monomials in the $\varepsilon^i$-character space. Define tautological bundle $\mathcal{R}_i$ as

$$\mathcal{R}_i = f^*R_i / \text{Tors}_{\mathcal{O}_Y}.$$ i.e. pullback modulo torsion.

Each $R_i$ is generated by the monomials $x^i, y^i, z^{r-i} \in \mathbb{C}[x, y, z]$ as an $\mathcal{O}_X$-module. Multiplying by $z^i$ we see that it is isomorphic to the ideal sheaf $(x^i z^i, y^i z^i, z^r) \subset \mathcal{O}_X$. We claim that $\mathcal{R}_i$ is an invertible sheaf. Indeed on the toric picture it is represented as a Cartier divisor by the piecewise linear function on the fan $\Delta$ given by $i e_2^* + i e_3^*$ on the cone $\sigma_{e_2,e_3,p_i}$, $i e_2^* + i e_3^*$ on the cone $\sigma_{e_1,e_3,p_i}$ and by $r e_3^*$ on $\sigma_{p_i,e_1,e_2}$ (see [11] p. 5–8 and [2] Example 4.8). We note that this Cartier divisor is equivalent to the $\mathbb{Q}$-Cartier divisor corresponding to $(x^i, y^i, z^{r-i})$ and it is more convenient to expand it in terms of linear equivalence classes of exceptional surfaces:
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\[
\mathcal{R}_1 = -\frac{r-1}{r}S_1 - \frac{r-2}{r}S_2 - \cdots - \frac{2}{r}S_{r-2} - \frac{1}{r}S_{r-1},
\]
\[
\mathcal{R}_2 = -\frac{r-2}{r}S_1 - \frac{2(r-2)}{r}S_2 - \frac{2(r-3)}{r}S_3 - \cdots - \frac{2}{r}S_{r-2} - \frac{2}{r}S_{r-1},
\]
\[
\vdots
\]
\[
\mathcal{R}_i = -\frac{r-i}{r}S_1 - \frac{2(r-i)}{r}S_2 - \cdots - \frac{i(r-i)}{r}S_i - \frac{i(r-i-1)}{r}S_{i+1} - \cdots
\]
\[
\cdots - \frac{2i}{r}S_{r-2} - \frac{i}{r}S_{r-1},
\]
\[
\vdots
\]
\[
\mathcal{R}_{r-1} = -\frac{1}{r}S_1 - \frac{2}{r}S_2 - \cdots - \frac{r-2}{r}S_{r-2} - \frac{r-1}{r}S_{r-1}.
\]

Observe that as a $\mathbb{Q}$-Cartier divisor $\mathcal{R}_1$ is the discrepancy divisor for $f$ (see [12] p. 373–374), that is $f^*(K_X) = K_Y + \mathcal{R}_1$ and the Cartier divisor $r\mathcal{R}_1$ is linearly equivalent to $-rK_Y$ (the equivalence is given by linear function $re_1^* + re_2^* + re_3^*$). In fact $rK_X$ is linearly trivial.

4. Main result.

Definition 4.1. Define virtual sheaves

\[
\mathcal{V}_i = (\mathcal{R}_1 \oplus \mathcal{R}_i) \ominus ((\mathcal{R}_1 \otimes \mathcal{R}_i) \oplus \mathcal{O}_Y).
\]

These virtual sheaves will be used to construct the dual basis to cohomology. For any bundles $\mathcal{F}, \mathcal{G}$ define

\[
c(\mathcal{F} \oplus \mathcal{G}) = \frac{c(\mathcal{F})}{c(\mathcal{G})}.
\]

Theorem 4.1. The tautological bundles $\mathcal{R}_i$ form the dual basis of $H^2(Y, \mathbb{Z})$, that is $c_1(\mathcal{R}_i) \cdot L_j = \delta_{ij}$ and the virtual sheaves $\mathcal{V}_i$ form the dual basis of $H^4(Y, \mathbb{Z})$, that is $c_2(\mathcal{V}_i) \cdot S_j = \delta_{ij}$. 
Proof. The divisor $\mathcal{R}_i$ has degree 1 on the fiber $L_i$ of rational scroll $\mathbb{F}_i$ which corresponds to the line joining $e_1$ with $p_i$ in the toric picture. It has also degree 0 on $L_j$ for $i \neq j$. This proves the first part of the theorem. The second part is proven by inspecting the following table of first Chern classes computed on every compact surface:

<table>
<thead>
<tr>
<th></th>
<th>$c_1(\mathcal{R}_1)$</th>
<th>$c_1(\mathcal{R}_2)$</th>
<th>$c_1(\mathcal{R}_3)$</th>
<th>$\ldots$</th>
<th>$c_1(\mathcal{R}_{r-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{P}^2$</td>
<td>$L_1$</td>
<td>0</td>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathbb{F}_2$</td>
<td>$L_2$</td>
<td>$M_2 + 2L_2$</td>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathbb{F}_3$</td>
<td>$L_3$</td>
<td>$2L_3$</td>
<td>$M_3 + 3L_3$</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathbb{F}_4$</td>
<td>$L_4$</td>
<td>$2L_4$</td>
<td>$3L_4$</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\mathbb{F}_{r-1}$</td>
<td>$L_{r-1}$</td>
<td>$2L_{r-1}$</td>
<td>$3L_{r-1}$</td>
<td>$\ldots$</td>
<td>$M_{r-1} + (r - 1)L_{r-1}$</td>
</tr>
</tbody>
</table>

and by the equation $c_2(\mathcal{F} \oplus \mathcal{F}') = c_1(\mathcal{F})c_1(\mathcal{F}')$, which holds for any line bundles $\mathcal{F}, \mathcal{F}'$. The restriction of the bundle $\mathcal{R}_i$ to the surface $S_j$ is computed by choosing from the piecewise function for $\mathcal{R}_i$ a linear function on one of the 3-dimensional cones containing $\tau_j$ and subtracting it from the functions on all the other cones. Evaluating the resulting functions on primitive vectors in rays generating 2-dimensional cones containing $\tau_j$ gives minus coefficients for the desired torus invariant Cartier divisor on the fan $\text{Star}(\tau_j)$ (see [9] for more details). Observe also that $c_1(\mathcal{V}_i) = 0$, so the second Chern class of $\mathcal{V}_i$ is integral.

This result computes also

$$\text{rank } H^*(Y, \mathbb{Z}) = 2r - 1$$

($r - 1$ for the second and fourth cohomology and 1 for the zeroth). It would be also interesting to obtain similar results in the general case of $\frac{1}{r} (1, a, r - a)$ for the ‘economic’, smooth resolution (see [12], Section 5). We note also that this ‘economic’ resolution is isomorphic to the $G$-Hilbert scheme only for $a = 1$. □
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