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KNESER AND HEREDITARILY KNESER SUBGROUPS OF A PROFINITE GROUP

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Abstract. Given a profinite group Γ acting continuously on a discrete quasi-cyclic group A, certain classes of closed subgroups of Γ (radical, hereditarily radical, Kneser, almost Kneser, and hereditarily Kneser) having natural field theoretic interpretations are defined and investigated. One proves that the hereditarily Kneser subgroups of Γ form a closed subspace of the irreducible spectral space of all closed subgroups of Γ, and a hereditarily Kneser criterion for hereditarily radical subgroups is provided.

Introduction. To any algebraic field extension E/F one can associate a torsion Abelian group, called the *Cogalois group* of the field extension E/F

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and denoted $\mathrm{Cog}(E/F)$, namely the torsion subgroup of the multiplicative factor group E^*/F^*. Thus $\mathrm{Cog}(E/F) = T(E/F)/F^*$, where

$$T(E/F) = \{x \in E^* \mid x^n \in F \text{ for some } n \in \mathbb{N}_{\geq 1}\}$$

The lattices $\mathbb{L}(E/F)$ and $\mathbb{L}(\mathrm{Cog}(E/F)) = \mathbb{L}(T(E/F) \mid F^*)$ of all intermediate subfields of the field extension E/F, resp. of all subgroups of $T(E/F)$ lying over F^*, are related through the natural maps $L \mapsto L \cap T(E/F), G \mapsto F(G)$. Roughly speaking, the aim of the Cogalois Theory consists in the study of the properties of these maps relating the lattices above. The roots of the Cogalois theory lie in some classical works of Siegel [17], Kneser [12], and Schinzel [14] devoted to particular classes of finite field extensions with Cogalois correspondence. A more general approach for arbitrary algebraic field extensions was developed in the 80'th by Greither-Harrison [10], Barrera-Mora, Rzedowski-Calderón, Villa-Salvador [8], and still more recently by Albu and Nicolae [1], [6]–[7]. For the actual state of art of the Cogalois Theory see Albu's monograph [2].

Now, assuming that E/F is a Galois (not necessarily finite) extension, and $\Gamma = \mathrm{Gal}(E/F)$ its Galois group with Krull's topology, the canonical morphism

$$\Psi : T(E/F) \longrightarrow Z^1(\Gamma, \mu_E), x \mapsto [\sigma \mapsto (\sigma x)x^{-1}]$$

where μ_E denotes the multiplicative group of the roots of unity in E, induces by Hilbert's Theorem 90 an isomorphism $\mathrm{Cog}(E/F) \cong Z^1(\Gamma, \mu_E)$.

Thus it seems natural to consider a pure group theoretic approach starting from an arbitrary profinite group Γ and a quasi-cyclic discrete group A, identified with a subgroup of $\mathbb{Q}/\mathbb{Z}$, on which Γ acts continuously. For any such pair (Γ, A), the objects to study are the lattices $\mathbb{L}(\Gamma)$ and $\mathbb{L}(Z^1(\Gamma, A))$ of all closed subgroups of the profinite group Γ, resp. of all subgroups of the Abelian torsion discrete group $Z^1(\Gamma, A)$ of continuous 1-cocycles from Γ to the discrete Γ-module A, which are related through the canonical reversing maps

$$\Delta \in \mathbb{L}(\Gamma) \mapsto \Delta^{\perp} = Z^1(\Gamma | \, \Delta, A) := \{g \in Z^1(\Gamma, A) \mid g\,|_{\Delta} = 0\} \in \mathbb{L}(Z^1(\Gamma, A))$$

and

$$G \in \mathbb{L}(Z^1(\Gamma, A)) \mapsto G^{\perp} := \{\sigma \in \Gamma \mid g(\sigma) = 0 \; \forall \, g \in G\} \in \mathbb{L}(\Gamma),$$

defining a *Galois connection*, i.e. $\Delta \leqslant \Delta^{\perp\perp}$ and $G \leqslant G^{\perp\perp}$ for all $\Delta \in \mathbb{L}(\Gamma), G \in \mathbb{L}(Z^1(\Gamma, A))$.

The lattices $\mathbb{L}(\Gamma)$ and $\mathbb{L}(Z^1(\Gamma, A))$ are equipped with natural topologies defined by the bases of quasi-compact open sets

$$\mathcal{U}_{\Delta} := \mathbb{L}(\Delta) = \{\Lambda \in \mathbb{L}(\Gamma) \mid \Lambda \leqslant \Delta\}$$

for Δ ranging over all open subgroups of Γ, resp.

$$\mathcal{U}_F := \{ G \in \mathbb{L}(Z^1(\Gamma, A)) \mid F \leqslant G \}$$

for F ranging over all finite subgroups of $Z^1(\Gamma, A)$.

Note that $\overline{\{ \Lambda \}} = \mathbb{L}(\Gamma \mid \Lambda) = \{ \Lambda' \in \mathbb{L}(\Gamma) \mid \Lambda \leqslant \Lambda' \}$ for all $\Lambda \in \mathbb{L}(\Gamma)$, and $\overline{\{G\}} = \mathbb{L}(G)$ for all $G \in \mathbb{L}(Z^1(\Gamma, A))$, so, w.r.t. the topologies above, $\mathbb{L}(\Gamma)$ and $\mathbb{L}(Z^1(\Gamma, A))$ are irreducible spectral spaces with the generic point $\{ 1 \}$, resp $Z^1(\Gamma, A)$, and the unique closed point Γ, resp. $\{ 0 \}$.

Moreover the both lattice operations on $\mathbb{L}(\Gamma)$ and $\mathbb{L}(Z^1(\Gamma, A))$ are continuous maps. However, in general, only the join $\Lambda_1 \vee \Lambda_2 := \overline{\Lambda_1 \cup \Lambda_2}$ in $\mathbb{L}(\Gamma)$, resp. the meet $G_1 \wedge G_2 := G_1 \cap G_2$ in $\mathbb{L}(Z^1(\Gamma, A))$, is a coherent map; a map $f : X \longrightarrow Y$ between spectral spaces is *coherent* if $f^{-1}(U)$ is a quasi-compact open subset of X for all quasi-compact open subsets U of Y. Note also that the canonical actions of the profinite group Γ on the topological lattices $\mathbb{L}(\Gamma)$ and $\mathbb{L}(Z^1(\Gamma, A))$

$$(\sigma, \Lambda) \mapsto \sigma \Lambda \sigma^{-1},$$

resp.

$$(\sigma, G) \mapsto \sigma G := \{ \sigma g \mid g \in G \},$$

where

$$(\sigma g)(\tau) = \sigma g(\sigma^{-1} \tau \sigma) \,\text{for}\, \sigma, \tau \in \Gamma, g \in Z^1(\Gamma, A),$$

are coherent maps, in particular, continuous maps.

Some remarkable closed subspaces of the spectral space $\mathbb{L}(Z^1(\Gamma, A))$ consisting of the so called *Kneser* and *Cogalois groups of cocycles* are introduced and investigated in [4, 5]. In the present work the accent will be moved on the spectral space $\mathbb{L}(\Gamma)$, more precisely on some of its subspaces consisting of closed subgroups of the profinite group Γ with interesting algebraic and topological properties. Thus, the following classes of closed subgroups of Γ having natural field theoretic interpretations are defined and investigated : *radical, hereditarily radical, Kneser, almost Kneser, and hereditarily Kneser*. The main results of the paper are Corollary 2.15, stating that the hereditarily Kneser subgroups of Γ form a closed subspace of the spectral space $\mathbb{L}(\Gamma)$, and Theorem 3.2, providing a hereditarily Kneser criterion for hereditarily radical subgroups of Γ. A forthcoming paper will be devoted to a particularly interesting subclass of hereditarily Kneser subgroups – the *Cogalois subgroups* –, and to some applications of the group theoretic approach from [4, 5] and the present paper to the field theoretic Cogalois theory.

1. Notation and preliminaries. Let Γ be a profinite group acting continuously on a discrete quasi-cyclic group A identified with a subgroup of $\mathbb{Q}/\mathbb{Z}$. The lattices $\mathbb{L}(\Gamma)$ and $\mathbb{L}(Z^1(\Gamma, A))$ of all closed subgroups of Γ, resp. of all subgroups of $Z^1(\Gamma, A)$, are equipped with natural spectral topologies as defined in Introduction. For $\Lambda \in \mathbb{L}(\Gamma), G \in \mathbb{L}(Z^1(\Gamma, A))$, we denote by $\mathbb{L}(\Gamma \,|\, \Lambda)_o$ the sublattice of $\mathbb{L}(\Gamma \,|\, \Lambda)$ consisting of all open subgroups of Γ lying over Λ, and by $\mathbb{L}(G)_f$ the sublattice of $\mathbb{L}(G)$ consisting of the finite subgroups of G.

Recall that a topological space X is called *spectral* (or *coherent*) if the family of quasi-compact open subsets of X is closed under finite intersections (in particular, X itself is quasi-compact) and forms a base for the topology on X, and every irreducible closed subset of X is the closure of a unique point of X. A spectral space X becomes a *profinite* (or *boolean* or *Stone*) space, i.e. a compact totally disconnected space, by taking the boolean lattice generated by the distributive lattice of all quasi-compact open subsets of X as a base of clopen sets for a finer topology on X. For more details concerning the spectral and profinite spaces, which are duals by the Stone's Representation Theorem to (bounded) distributive lattices and boolean lattices (algebras), respectively, the reader may consult [11, 13], and/or [9].

The natural reversing maps $(-)^{\perp} : \mathbb{L}(\Gamma) \longrightarrow \mathbb{L}(Z^1(\Gamma, A))$ and $(-)^{\perp} : \mathbb{L}(Z^1(\Gamma, A)) \longrightarrow \mathbb{L}(\Gamma)$ as defined in Introduction have the following properties ([4, Propositions 0.1 and 0.3])

(i) The map $\Lambda \mapsto \Lambda^{\perp}$ is a semi-lattice morphism $(\mathbb{L}(\Gamma), \vee) \longrightarrow (\mathbb{L}(Z^1(\Gamma, A)), \wedge)$, i.e. $(\Lambda_1 \cup \Lambda_2)^{\perp} = \Lambda_1^{\perp} \cap \Lambda_2^{\perp}$ for $\Lambda_i \in \mathbb{L}(\Gamma), i = 1, 2$. It is also a Γ-equivariant coherent map, in particular, a continuous map.

(ii) The map $G \mapsto G^{\perp}$ is a semi-lattice morphism $(\mathbb{L}(Z^1(\Gamma, A)), \vee) \longrightarrow (\mathbb{L}(\Gamma), \wedge)$, i.e. $(G_1 + G_2)^{\perp} = G_1^{\perp} \cap G_2^{\perp}$ for $G_i \in \mathbb{L}(Z^1(\Gamma, A)), i = 1, 2$. It is also a Γ-equivariant continuous map.

However, in general, the map $G \mapsto G^{\perp}$ from (ii) is not coherent, as we can see from the following simple example. Let $\Gamma = \widehat{\mathbb{Z}}, A = \sum_{p \in \mathcal{P}'} (1/p)\mathbb{Z}/\mathbb{Z}$, where $\mathcal{P}'$ consists of all odd prime numbers p for which the order $f_p|(p-1)$ of the element $2 \bmod p \in \mathbb{F}_p^*$ is even. Consider the continuous action $\Gamma \times A \longrightarrow A, (\sigma, a) \mapsto 2^{\sigma} a$. Setting $\Delta = 2\widehat{\mathbb{Z}}$, it follows that $Z^1(\Gamma, (1/p)\mathbb{Z}/\mathbb{Z})^{\perp} = f_p \widehat{\mathbb{Z}} \leqslant \Delta$ for all $p \in \mathcal{P}'$. Thus the subgroups $Z^1(\Gamma, (1/p)\mathbb{Z}/\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ of $Z^1(\Gamma, A)$, for p ranging over the infinite set $\mathcal{P}'$, are the minimal elements of the poset $((-)^{\perp}))^{-1}(\mathbb{L}(\Delta))$, and hence the open set $((-)^{\perp}))^{-1}(\mathbb{L}(\Delta))$ is not quasi-compact.

For $G \in \mathbb{L}(Z^1(\Gamma, A))$, set $\check{G} := \mathrm{Hom}\,(G, A) = \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z})$. $\check{G}$, the Pontryagin dual of the discrete Abelian torsion group G, is an Abelian profinite group. Moreover $\check{G}$ is a topological Γ-module w.r.t. the action defined by $(\sigma\chi)(g) = \sigma\chi(g)$ for $\sigma \in \Gamma, \chi \in \check{G}, g \in G$. The canonical continuous map $\eta_G : \Gamma \longrightarrow \check{G}, \sigma \mapsto (g \mapsto g(\sigma))$, is a 1-cocycle, inducing an injective continuous map $\Gamma/G^\perp \longrightarrow \check{G}$, in particular, $(\Gamma : G^\perp) \le |G|$ for all $G \in \mathbb{L}(Z^1(\Gamma, A))_f$.

Definition 1.1 ([4, Definition 1.2]). *$G \in \mathbb{L}(Z^1(\Gamma, A))$ is called a* Kneser subgroup *of $Z^1(\Gamma, A)$ if the continuous cocycle $\eta_G : \Gamma \longrightarrow \check{G}$ is onto, i.e.* $(\Gamma : G^\perp) = |\check{G}|$ *as supernatural numbers* (cf. [15, Ch. I, 1.4]).

Remark 1.2. One checks easily that the following statements are equivalent for $G \in Z^1(\Gamma, A)$.

(i) G is a Kneser subgroup of $Z^1(\Gamma, A)$.
(ii) $\eta_G(G)$ is a subgroup of $\check{G}$.
(iii) $\eta_G(G)$ is a Γ-subspace of the Γ-space $\check{G}$.
(iv) $\eta_G(G)$ is a Γ-submodule of $\check{G}$.

Denote by $\mathcal{K}(Z^1(\Gamma, A))$ the subset of $\mathbb{L}(Z^1(\Gamma, A))$ consisting of all Kneser subgroups of $Z^1(\Gamma, A)$. According to [4], Corollary 1.8, $\mathcal{K}(Z^1(\Gamma, A))$ is a closed subspace of the spectral space $\mathbb{L}(Z^1(\Gamma, A))$. To obtain a criterion for a subgroup G of $Z^1(\Gamma, A)$ to be Kneser it suffices to describe the minimal members w.r.t. inclusion of the open subset $\mathbb{L}(Z^1(\Gamma, A)) \setminus \mathcal{K}(Z^1(\Gamma, A))$. To do that, we introduced in [4] some basic notation which will be also used in the sequel.

$\mathbb{P}$ denotes the set of prime natural numbers;

$\mathcal{P} = \{p \in \mathbb{P} \,|\, p \neq 2\} \cup \{4\}$;

$\mathcal{P}_G = \{p \in \mathcal{P} \,|p|\, |G|\}$ for $G \in \mathbb{L}(Z^1(\Gamma, A))$;

$\widehat{r} \in \mathbb{Q}/\mathbb{Z}$ denotes the class of $r \in \mathbb{Q}$;

$\mathcal{P}(\Gamma, A) = \{\, p \in \mathcal{P} \,|\, \widehat{1/p} \in A \setminus A^\Gamma\}$;

For $n \in \mathbb{N}_{\ge 1}$ such that $\widehat{1/n} \in A$, $\varepsilon_n \in B^1(\Gamma, A)$

denotes the coboundary associated to $\widehat{1/n}$;

$g^\perp := G^\perp$ for $g \in Z^1(\Gamma, A), G = \langle g \rangle$;

If $\widehat{1/4} \in A \setminus A^\Gamma$, define $\varepsilon_4' \in Z^1(\Gamma, A)$ by

$$\varepsilon_4'(\sigma) = \begin{cases} \widehat{1/4} & \text{if } \ \sigma\,\widehat{1/4} = -\,\widehat{1/4}, \\ \widehat{0} & \text{if } \ \sigma\,\widehat{1/4} = \widehat{1/4}. \end{cases}$$

The abstract version of the field theoretic Kneser criterion [12] reads as follows. Note that the place of the primitive roots of unity ζ_p, p odd prime, from the Kneser criterion is taken in its abstract version by the coboundary ε_p, while the cocycle ε'_4 corresponds to $1-\zeta_4$.

Theorem 1.3 ([4, Theorem 1.20]). *The following assertions are equivalent for a subgroup G of $Z^1(\Gamma, A)$.*

(1) $G \in \mathcal{K}(\Gamma, A)$.

(2) $\varepsilon_p \notin G$ *whenever* $4 \neq p \in \mathcal{P}(\Gamma, A)$ *and* $\varepsilon'_4 \notin G$ *whenever* $4 \in \mathcal{P}(\Gamma, A)$.

From a logical point of view, the statement above can be interpreted as a quantifier elimination result: the property of a subgroup $G \leqslant Z^1(\Gamma, A)$ to be Kneser, described by a sentence (in a suitable language) involving quantifiers, turns out to be equivalent with a (possible infinite) conjunction of very simple quantifier-free sentences.

A particularly interesting subclass of Kneser groups of cocycles, introduced and studied in [5], is defined below.

Definition 1.4 ([5, Definition 2.1]). *A subgroup G of $Z^1(\Gamma, A)$ is said to be a* Cogalois subgroup *of $Z^1(\Gamma, A)$ if it is a Kneser subgroup of $Z^1(\Gamma, A)$ and the maps $(-)^\perp : \mathbb{L}(G) \longrightarrow \mathbb{L}(\Gamma \,|\, G^\perp)$ and $G \cap (-)^\perp : \mathbb{L}(\Gamma \,|\, G^\perp) \longrightarrow \mathbb{L}(G)$ are lattice anti-isomorphisms, inverse to one another.*

Denote by $\mathcal{C}(Z^1(\Gamma, A))$ the subset of $\mathcal{K}(Z^1(\Gamma, A))$ consisting of all Cogalois subgroups of $Z^1(\Gamma, A)$. According to [5], Corollary 2.7, $\mathcal{C}(Z^1(\Gamma, A))$ is a closed subspace of the spectral space $\mathcal{K}(Z^1(\Gamma, A))$. One of the various equivalent characterizations for the Cogalois groups of cocycles proved in [5] is mentioned below.

Theorem 1.5 ([5, Theorem 2.5]). *The following statements are equivalent for a subgroup G of $Z^1(\Gamma, A)$.*

(1) $G \in \mathcal{C}(Z^1(\Gamma, A))$.

(2) $G^\perp \not\subseteq \varepsilon_p^\perp$ *for all* $p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)$.

2. Radical, Kneser and hereditarily Kneser subgroups. In this section we study subgroups of a profinite group Γ acting continuously on a discrete subgroup A of $\mathbb{Q}/\mathbb{Z}$, which are both closed in the topology of Γ and closed under the closure operator $\Delta \mapsto \Delta^{\perp\perp}$. The abstract versions of the field theoretic notions of radical, G-Kneser, and Kneser field extensions (cf. [2, Ch. 11]) are introduced, and their main properties are investigated. On the other hand, natural topological arguments are used to put in evidence new classes of subgroups of Γ (hereditarily radical, almost Kneser and hereditarily Kneser) with suitable field theoretic interpretations.

The concept defined below is the abstract version of the concept of radical field extension.

Definition 2.1. *A subgroup* Δ *of* Γ *is said to be* G-radical *if* $G \in \mathbb{L}(Z^1(\Gamma, A))$ *and* $\Delta = G^{\perp}$. *A* radical subgroup *of* Γ *is a subgroup which is* G*-radical for some* $G \leqslant Z^1(\Gamma, A)$.

Since $Z^1(\Gamma, A)$ is a torsion group and $(\Gamma : g^{\perp}) \leqslant \operatorname{ord}(g)$ for all $g \in Z^1(\Gamma, A)$, it follows that for all $G \in \mathbb{L}(Z^1(\Gamma, A))$, $G^{\perp} = \bigcap_{g \in G} g^{\perp}$ is closed in Γ as intersection of open subgroups, so any radical subgroup of Γ is necessarily closed.

The next obvious lemma provides equivalent descriptions for radical subgroups.

Lemma 2.2. *The following statements are equivalent for a subgroup* Δ *of* Γ.

(1) Δ *is radical.*

(2) $\Delta = \Delta^{\perp\perp}$.

(3) Δ *is* $\Delta^{\perp}$*-radical.*

We shall denote by $\mathcal{R}(\Gamma)$ the poset of all radical subgroups of Γ. Since for any family $(\Delta_i)_{i \in I}$ of radical subgroups of Γ, $\bigcap\limits_{i \in I} \Delta_i = \big(\sum\limits_{i \in I} \Delta_i^{\perp}\big)^{\perp}$, it follows that $\mathcal{R}(\Gamma)$ is a meet-subsemilattice of $\mathbb{L}(\Gamma)$. Observe that Γ is the last element of $\mathcal{R}(\Gamma)$, while the closed normal subgroup $\{1\}^{\perp\perp} = Z^1(\Gamma, A)^{\perp}$ of Γ is the least element of $\mathcal{R}(\Gamma)$. Also notice that the kernel Δ of the action of Γ on A belongs to $\mathcal{R}(\Gamma)$ since $\Delta = B^1(\Gamma, A)^{\perp}$.

Remark 2.3. If $\Delta \in \mathcal{R}(\Gamma)$ and $\Lambda \in \mathbb{L}(\Gamma|\Delta)$, then Λ is not necessarily a radical subgroup of Γ, in other words, in general, $\mathcal{R}(\Gamma)$ is not an *upper subset* of $\mathbb{L}(\Gamma)$, and hence, in general, $\mathcal{R}(\Gamma)$ is not a closed subset of the spectral space $\mathbb{L}(\Gamma)$, i.e., $\mathcal{R}(\Gamma) \subsetneq \overline{\mathcal{R}(\Gamma)} = \mathbb{L}(\Gamma|Z^1(\Gamma, A)^{\perp})$.

To see that, consider the following simple example: let $A = (1/2^n)\mathbb{Z}/\mathbb{Z}$, $n \geqslant 4$, and $\Gamma = (\mathbb{Z}/2^n\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2}\mathbb{Z}$, with the (faithful) canonical action given by multiplication. If we consider the elements $\sigma = -1 \bmod 2^n$ and $\tau = 5 \bmod 2^n$ of Γ, then we obtain the following presentation $\Gamma = \langle\, \sigma, \tau \,|\, \sigma^2 = \tau^{2^{n-2}} = [\sigma, \tau] = 1 \,\rangle$. The morphism $Z^1(\Gamma, A) \longrightarrow A \times A$, $g \mapsto (g(\sigma), g(\tau) +$

$2g(\sigma))$, maps isomorphically $Z^1(\Gamma, A)$ onto $((1/2^{n-1})\mathbb{Z}/\mathbb{Z}) \times ((1/2)\mathbb{Z}/\mathbb{Z})$, sending $B^1(\Gamma, A)$ onto $(1/2^{n-1})\mathbb{Z}/\mathbb{Z}$. It follows that $\Delta := \{1\} = B^1(\Gamma, A)^{\perp} = Z^1(\Gamma, A)^{\perp} \in \mathcal{R}(\Gamma)$. But $\Lambda_i := \langle \sigma, \tau^{2^i} \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2-i}\mathbb{Z}, 2 \le i \le n-2$, is not a radical subgroup of Γ; indeed, since $\Lambda_i^{\perp} = \langle \alpha \rangle \cong \mathbb{Z}/2\mathbb{Z}$, where α is the morphism of order 2 defined by $\alpha(\sigma) = 0$, $\alpha(\tau) = \widehat{1/2}$, we have $\Lambda_i^{\perp\perp} = \alpha^{\perp} = \langle \sigma, \tau^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-3}\mathbb{Z}$, and hence $\Lambda_i^{\perp\perp} \neq \Lambda_i$ as $n-3 > n-2-i$ by assumption. $\square$

The following notion is justified by Remark 2.3.

Definition 2.4. *A closed subgroup Δ of Γ is said to be* hereditarily radical *(abbreviated* h-radical*) if Λ is radical for any $\Lambda \in \mathbb{L}(\Gamma|\Delta)$.*

Note that $\Delta \in \mathbb{L}(\Gamma)$ is h-radical iff the canonical map $\mathbb{L}(\Delta^{\perp}) \longrightarrow \mathbb{L}(\Gamma|\Delta)$, $G \mapsto G^{\perp}$, is onto. In the sequel we shall denote by $\mathcal{HR}(\Gamma)$ the poset of all h-radical subgroups of Γ. Thus $\mathcal{HR}(\Gamma)$ is an upper subset of $\mathbb{L}(\Gamma), \Gamma \in \mathcal{HR}(\Gamma)$, and, in general, $\mathcal{HR}(\Gamma) \subsetneq \mathcal{R}(\Gamma)$ by Remark 2.3. Moreover, it follows easily that $\Delta \in \mathcal{HR}(\Gamma)$ iff Λ is radical for any open subgroup of Γ lying over Δ, and hence $\mathcal{HR}(\Gamma)$ is a closed Γ-subspace of the spectral space $\mathbb{L}(\Gamma)$. The maximal elements w.r.t. inclusion of the open set $\mathbb{L}(\Gamma) \setminus \mathcal{HR}(\Gamma)$ are exactly the (open) subgroups Δ for which $\Delta \neq \Delta^{\perp\perp}$ and the canonical map $\mathbb{L}(\Delta^{\perp}) \longrightarrow \mathbb{L}(\Gamma|\Delta) \setminus \{\Delta\}, G \mapsto G^{\perp}$, is onto. In particular, the (possibly empty) set of the maximal proper open subgroups Δ satisfying $\Delta^{\perp} = \{0\}$ is a subset of the set above. Note that in the situation described in Remark 2.3, $\mathbb{L}(\Gamma) \setminus \mathcal{HR}(\Gamma) = \mathbb{L}(\langle \sigma, \tau^4 \rangle)$.

Definition 2.5. *A subgroup Δ of Γ is said to be G-*Kneser *if Δ is G-radical and G is a Kneser subgroup of $Z^1(\Gamma, A)$. Δ is said to be a* Kneser *subgroup of Γ if Δ is G-Kneser for some $G \leqslant Z^1(\Gamma, A)$.*

Clearly, any Kneser subgroup Δ of Γ is the intersection of all open Kneser subgroups (and hence of all Kneser subgroups of type $g^{\perp}$) of Γ lying over Δ. In the sequel we shall denote by $\mathcal{K}(\Gamma)$ the poset of the Kneser subgroups of Γ. Observe that $\Gamma \in \mathcal{K}(\Gamma) \subseteq \mathcal{R}(\Gamma)$.

Remarks 2.6. (1) If $\Delta \in \mathbb{L}(\Gamma)$ is simultaneously G-Kneser and H-Kneser, then G and H are not necessarily isomorphic. For instance, let $\Gamma = \langle \sigma, \tau \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $A = (1/4)\mathbb{Z}/\mathbb{Z}$, with the action given by $\sigma\,\widehat{1/4} = -\widehat{1/4}$ and $\tau\,\widehat{1/4} = \widehat{1/4}$. The morphism $Z^1(\Gamma, A) \longrightarrow A \times A$, $g \mapsto (g(\sigma), g(\tau))$ maps isomorphically $Z^1(\Gamma, A)$ onto $(1/4)\mathbb{Z}/\mathbb{Z} \times (1/2)\mathbb{Z}/\mathbb{Z}$. The trivial subgroup $\{1\}$ of Γ is simultaneously G-Kneser and H-Kneser, where $G = \langle \alpha \rangle \cong \mathbb{Z}/4\mathbb{Z}$, with $\alpha(\sigma) = \widehat{1/4}$, $\alpha(\tau) = \widehat{1/2}$, and $H = \langle \varepsilon_4, \beta \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, with $\beta(\sigma) = 0$, $\beta(\tau) = \widehat{1/2}$.

(2) Note that, in general, $\mathcal{K}(\Gamma)$ is not an upper subset of $\mathbb{L}(\Gamma)$, and hence not necessarily a closed subset of the spectral space $\mathbb{L}(\Gamma)$. Indeed let Γ and A be as defined in Remark 2.3. Then $\Delta = \{1\}$ is $B^1(\Gamma, A)$-Kneser, while $\langle\sigma\rangle \notin \mathcal{R}(\Gamma)$ as we have already seen, and hence $\langle\sigma\rangle \notin \mathcal{K}(\Gamma)$. However $\langle\sigma\rangle^{\perp\perp} = \langle\sigma, \tau^2\rangle \in \mathcal{K}(\Gamma)$. □

Though, in general, $\mathcal{K}(\Gamma)$ is not a closed subspace of the spectral space $\mathbb{L}(\Gamma)$, it is closed w.r.t. the profinite topology of $\mathbb{L}(\Gamma)$ as image of the profinite space $\mathcal{K}(Z^1(\Gamma, A))$ through the continuous map $\mathbb{L}(Z^1(\Gamma, A)) \longrightarrow \mathbb{L}(\Gamma)$, $G \mapsto G^{\perp}$. As a consequence, we obtain the following characterisation of the Kneser subgroups of Γ.

Lemma 2.7. *A necessary and sufficient condition for a closed subgroup Λ of Γ to be Kneser is that $\mathbb{L}(\Gamma|\Lambda)_o \cap \mathcal{K}(\Gamma)$ is cofinal in the poset $\mathbb{L}(\Gamma|\Lambda)_o$ of all open subgroups of $\mathbb{L}(\Gamma|\Lambda)$.*

Proof. The "only if" part is obvious since, assuming $\Lambda = G^{\perp}$ for some $G \in \mathcal{K}(Z^1(\Gamma, A))$, $\{F^{\perp} \mid F \in \mathbb{L}(G)_f\} \subseteq \mathcal{K}(\Gamma) \cap \mathbb{L}(\Gamma|\Lambda)_o$ is cofinal in $\mathbb{L}(\Gamma|\Lambda)_o$. Conversely, assuming $\Lambda \notin \mathcal{K}(\Gamma)$, since $\mathcal{K}(\Gamma)$ is closed in the profinite space $\mathbb{L}(\Gamma)$, it follows that there exists an open normal subgroup Δ of Γ such that $\{\Lambda' \in \mathbb{L}(\Gamma) \mid \Lambda'\Delta = \Lambda\Delta\} \cap \mathcal{K}(\Gamma) = \emptyset$, so $\Lambda\Delta \in \mathbb{L}(\Gamma|\Lambda)_o$ and $\Delta' \notin \mathcal{K}(\Gamma)$ for all $\Delta' \in \mathbb{L}(\Lambda\Delta|\Lambda)_o$, since $\Delta'\Delta = \Lambda\Delta$. □

Definition 2.8. *A closed subgroup Λ of Γ is said to be* almost Kneser*, abbreviated* a-Kneser*, if Λ belongs to $\overline{\mathcal{K}(\Gamma)}$, the closure of $\mathcal{K}(\Gamma)$ in the spectral space $\mathbb{L}(\Gamma)$.*

The next lemma provides a characterisation of the a-Kneser subgroups of Γ.

Lemma 2.9. *The following statements are equivalent for a closed subgroup Λ of Γ.*

(1) *Λ is a-Kneser.*

(2) $\mathbb{L}(\Lambda) \cap \mathcal{K}(\Gamma) \neq \emptyset$.

Proof. (1) $\Longrightarrow$ (2): Assuming that Λ is a-Kneser, it follows that $\mathbb{L}(\Delta) \cap \mathcal{K}(\Gamma) \neq \emptyset$ for all $\Delta \in \mathbb{L}(\Gamma|\Lambda)_o$. As for any such Δ, $\mathbb{L}(\Delta)$ is clopen and $\mathcal{K}(\Gamma)$ is closed in the profinite space $\mathbb{L}(\Gamma)$, it follows that the non-empty set $X_\Delta := \mathbb{L}(\Delta) \cap \mathcal{K}(\Gamma)$ is closed too. Since the family $(X_\Delta)_{\Delta \in \mathbb{L}(\Gamma|\Lambda)_o}$ has finite intersection property, it follows by compactness that $\mathbb{L}(\Lambda) \cap \mathcal{K}(\Gamma) = \bigcap_{\Delta \in \mathbb{L}(\Gamma|\Lambda)_o} X_\Delta \neq \emptyset$, as required.

The implication (2) $\Longrightarrow$ (1) is obvious. $\square$

Corollary 2.10. *The posets $\mathcal{K}(\Gamma)$ and $\overline{\mathcal{K}(\Gamma)}$ have the same minimal members.*

Similarly with the notion of h-radical, we define a subclass of Kneser subgroups of Γ as follows:

Definition 2.11. *A closed subgroup Δ of Γ is said to be* hereditarily Kneser*, abbreviated* h-Kneser*, if any closed subgroup Λ lying over Δ, in particular, Δ itself, is Kneser.*

Remark 2.6, (2) provides examples of Kneser subgroups which are not h-Kneser. On the other hand, basic examples of h-Kneser subgroups are provided by the next result.

Lemma 2.12.

(1) *Let $A = (1/p^n r)\mathbb{Z}/\mathbb{Z}$, where p is an odd prime number, $n \geq 1$, and $2 \leq r|(p-1)$. Let $\Gamma = \mathbb{Z}/p^k\mathbb{Z} \rtimes_u \mathbb{Z}/r\mathbb{Z} = \langle \sigma, \tau \mid \sigma^r = \tau^{p^k} = \sigma\tau\sigma^{-1}\tau^{-u} = 1 \rangle$, where $0 \leq k \leq n$ and $u \in (\mathbb{Z}/p^n r\mathbb{Z})^*$ satisfies : $u \bmod p^n \in (\mathbb{Z}/p^n\mathbb{Z})^*$ has order r, and $u \equiv 1 \bmod l$ for $l \in \mathcal{P}$, $l|r$. Consider the action of Γ on A given by $\sigma a = ua, \tau a = a$ for $a \in A$. Then, $\Delta := \{1\}$ is a h-Kneser subgroup of Γ.*

(2) *Let $A = (1/2^n)\mathbb{Z}/\mathbb{Z}, n \geq 3, \Gamma = \langle \sigma, \tau | \sigma^2 = \tau^{2^{k-1}} = (\sigma\tau)^2 = 1\rangle \cong \mathbb{D}_{2^k}, 1 \leq k \leq n-2, \sigma\widehat{1/2^{n-1}} = -\widehat{1/2^{n-1}}, \tau\widehat{1/2^n} = \widehat{1/2^n}$. Then, $\Delta := \{1\}$ is a h-Kneser subgroup of Γ.*

Proof. (1) By assumption it follows that $B^1(\Gamma, A)^\perp = \varepsilon_p^\perp = \langle\tau\rangle \leqslant B^1(\Gamma, (1/r)\mathbb{Z}/\mathbb{Z})^\perp$, $\Gamma \cong \varepsilon_p^\perp \rtimes \Gamma/\varepsilon_p^\perp$, and $\sum_{i=0}^{r-1} u^i \equiv 0 \bmod p^n r$. The morphism $Z^1(\Gamma, A) \longrightarrow A \times A$, $g \mapsto (g(\sigma), g(\tau))$, maps isomorphically $Z^1(\Gamma, A)$ onto $A \times p^{n-k}rA \cong \mathbb{Z}/p^n r\mathbb{Z} \times \mathbb{Z}/p^k\mathbb{Z} \cong \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^k r\mathbb{Z}$, and hence the maximal Kneser subgroups of $Z^1(\Gamma, A)$ are the direct summands of the cyclic subgroup $B^1(\Gamma, (1/p^n)\mathbb{Z}/\mathbb{Z}) = \langle\alpha\rangle \cong \mathbb{Z}/p^n\mathbb{Z}$, where $\alpha(\sigma) = \widehat{1/p^n}, \alpha(\tau) = 0$. Setting $\beta(\sigma) = \widehat{1/r}, \beta(\tau) = \widehat{1/p^k}$, it follows that there are exactly p^k maximal Kneser subgroups of $Z^1(\Gamma, A)$, namely the conjugates $\langle\tau^i\beta\rangle, i \in \mathbb{Z}/p^k\mathbb{Z}$, of the cyclic group $\langle\beta\rangle \cong \mathbb{Z}/p^k r\mathbb{Z}$ through the canonical action of Γ on $Z^1(\Gamma, A)$. Let $\Lambda \in \mathbb{L}(\Gamma)$. If $\Lambda \leqslant \varepsilon_p^\perp$, i.e., $\Lambda = \langle\tau^{p^j}\rangle, 0 \leq j \leq k$, then $\Lambda = (p^{k-j}r\beta)^\perp \in \mathcal{K}(\Gamma)$ as required. If $\Lambda \not\leqslant \varepsilon_p^\perp$, then $\Lambda \cong (\Lambda \cap \varepsilon_p^\perp) \rtimes (\Lambda/(\Lambda \cap \varepsilon_p^\perp))$, so $\Lambda = \langle\tau^{p^j}, \tau^t\sigma^s\rangle$ for some

$0 \le j \le k, 0 \le t < p^k, s|r, s \ne r$. Note that $\tau^t\sigma^s = \tau^i\sigma^s\tau^{-i}$, where $i \in \mathbb{Z}/p^k\mathbb{Z}$ is uniquely determined by the condition $i(1-u^s) \equiv t \bmod p^k$, since $u^s \not\equiv 1 \bmod p$. As $\sum_{\mu=0}^{s-1} u^\mu \equiv 0 \bmod s$ since $u \equiv 1 \bmod l$ for $l \in \mathcal{P}, l|s$, by assumption, it follows that $\Lambda = (p^{k-j}(r/s)\tau^i\beta)^\perp \in \mathcal{K}(\Gamma)$ as desired.

(2) We distinguish the following two cases:

(i) $\sigma\widehat{1/2^n} = -\widehat{1/2^n}$: The map $Z^1(\Gamma, A) \longrightarrow A \times A, g \mapsto (g(\sigma), g(\tau))$, maps isomorphically $Z^1(\Gamma, A)$ onto $(1/2^n)\mathbb{Z}/\mathbb{Z} \oplus (1/2^{k-1})\mathbb{Z}/\mathbb{Z}$. Define $\alpha \in Z^1(\Gamma, A)$ by $\alpha(\sigma) = 0, \alpha(\tau) = \widehat{1/2^{k-1}}$. Let $\Lambda \in \mathbb{L}(\Gamma)$. If $\Lambda \leqslant \langle\tau\rangle$, then $\Lambda = \langle\tau^{2^i}\rangle, 0 \le i \le k-1$, and hence $\Lambda = H_i^\perp$, where $H_i = \langle\varepsilon_4, 2^i\alpha\rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{k-i-1}\mathbb{Z} \in \mathcal{K}(Z^1(\Gamma, A))$. If $\Lambda \not\leqslant \langle\tau\rangle$, then $\tau^j\sigma \in \Lambda$ for some $0 \le j \le 2^{k-1}-1$, and hence $\Lambda \in \mathbb{L}(\Gamma|\beta^\perp)$, where $\beta \in Z^1(\Gamma, A)$ is defined by $\beta(\sigma) = -j\widehat{1/2^{k-1}}, \beta(\tau) = \widehat{1/2^{k-1}}$. As $\beta^\perp = \langle\tau^j\sigma\rangle \not\leqslant \langle\tau\rangle = \varepsilon_4^\perp$, it follows by Theorem 1.5 that $\langle\beta\rangle \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$ is a Cogalois subgroup of $Z^1(\Gamma, A)$, and hence $\Lambda = (2^i\beta)^\perp$ for some $0 \le i \le k-1$, so Λ is a Kneser subgroup of Γ as required. Note that the result remains also true in the case $k = n-1$.

(ii) $\sigma\widehat{1/2^n} = -(1+2^{n-1})\widehat{1/2^n} = -\widehat{1/2^n} + \widehat{1/2}$: We are reduced to the case (1) since $Z^1(\Gamma, A) = Z^1(\Gamma, (1/2^{n-1})\mathbb{Z}/\mathbb{Z})$ as $0 = g(\sigma^2) = 2^{n-1}g(\sigma)$ and $0 = g(\tau^{2^{k-1}}) = 2^{k-1}g(\tau)$ for all $g \in Z^1(\Gamma, A)$. $\square$

In the sequel we shall denote by $\mathcal{HK}(\Gamma)$ the poset of all h-Kneser subgroups of Γ. Thus $\mathcal{HK}(\Gamma)$ is un upper subset of $\mathcal{HR}(\Gamma)$, $\Gamma \in \mathcal{HK}(\Gamma)$, and, in general, $\mathcal{HK}(\Gamma) \subsetneq \mathcal{K}(\Gamma)$ by Remarks 2.6, (2). One may ask whether for arbitrary pairs (Γ, A), $\mathcal{HK}(\Gamma)$ is a closed subspace of the spectral space $\mathcal{HR}(\Gamma)$, or, in other words, does any $\Delta \in \mathbb{L}(\Gamma)$ belong to $\mathcal{K}(\Gamma)$ whenever $\Lambda \in \mathcal{K}(\Gamma)$ for all open subgroups of Γ lying over Δ ? The affirmative answer to the question above will be an immediate consequence of the following result.

Theorem 2.13. *The canonical map $(-)^\perp : \mathcal{K}(Z^1(\Gamma, A)) \longrightarrow \mathbb{L}(\Gamma), G \mapsto G^\perp$, is coherent.*

Proof. By [4], Corollary 1.8, $\mathcal{K}(Z^1(\Gamma, A))$ is a spectral space as a closed subspace of the spectral space $\mathbb{L}(Z^1(\Gamma, A))$. For any open subgroup Δ of Γ, let $\mathcal{W}_\Delta := \{G \in \mathcal{K}(Z^1(\Gamma, A))\,|G^\perp \leqslant \Delta\}$ denote the inverse image of the basic quasi-compact open set $\mathcal{U}_\Delta$ of the spectral space $\mathbb{L}(\Gamma)$. We may assume that $\mathcal{W}_\Delta$ is non-empty since otherwise we have nothing to prove. Let W_Δ denote the (non-empty) subset of $\mathcal{W}_\Delta$ consisting of its minimal members w.r.t. inclusion. We have to show that the set W_Δ is finite and all its members are finite Kneser subgroups of $Z^1(\Gamma, A)$, since then, by Zorn's lemma, it follows that $\mathcal{W}_\Delta = \bigcup_{F\in W_\Delta} \mathcal{U}_F$, so $\mathcal{W}_\Delta$ is quasi-compact open as a finite union of basic quasi-compact open subsets of the spectral space $\mathcal{K}(Z^1(\Gamma, A))$.

Assuming that some $G \in W_\Delta$ is infinite, we deduce by minimality that $F^\perp \not\subseteq \Delta$ for all finite subgroups F of G, which are Kneser too as subgroups of $G \in \mathcal{K}(Z^1(\Gamma, A))$. Since the family of non-empty closed subsets $F^\perp \setminus \Delta$ of Γ for F ranging over all finite subgroups of G has finite intersection property, it follows by compactness of Γ that $G^\perp \setminus \Delta = \bigcap_F (F^\perp \setminus \Delta) \neq \emptyset$, i.e. $G^\perp \not\subseteq \Delta$, which is a contradiction. Thus it remains only to show that the set W_Δ is finite.

Let T_Δ denote the set of all Kneser subgroups of $Z^1(\Gamma, A)$ which are contained in $\Delta^\perp$. The set T_Δ is finite and all its members are finite groups. Indeed, for any $H \in T_\Delta$, we obtain $\Delta \leqslant \Delta^{\perp\perp} \leqslant H^\perp$, and hence $|H| = (\Gamma : H^\perp)|(\Gamma : \Delta) < \infty$, so H is a subgroup of the finite group $\Delta^\perp[n] = Z^1(\Gamma|\Delta, A[n])$, where $n = (\Gamma : \Delta)$.

Thus it suffices to show that for any $H \in T_\Delta$, the set $W_{\Delta,H} := \{G \in W_\Delta |\, G \cap \Delta^\perp = H\}$ is finite. Moreover we claim that it suffices to show that for any pair (Γ, A) and any open subgroup Δ of Γ, the set $W_{\Delta,0}$ is finite. Indeed, assuming that $H \in T_\Delta$ and $G \in W_{\Delta,H}$, let $\widetilde{G} = \mathrm{res}^\Gamma_{H^\perp}(G)$. As $\Delta \leqslant H^\perp$, we obtain $H \leqslant G \cap H^{\perp\perp} \leqslant G \cap \Delta^\perp = H$, and hence $\widetilde{G} \cong G/H$ is a Kneser group of $Z^1(H^\perp, A)$ by [4], Corollary 1.12, and $\widetilde{G}^\perp = G^\perp \cap H^\perp = G^\perp \leqslant \Delta \leqslant H^\perp$. On the other hand, it follows that $\widetilde{G} \cap Z^1(H^\perp|\Delta, A) = \mathrm{res}^\Gamma_{H^\perp}(G \cap \Delta^\perp) = \mathrm{res}^\Gamma_{H^\perp}(H) = \{0\}$ and $\widetilde{G} \in \mathcal{K}(Z^1(H^\perp, A))$ is minimal with the property that $\widetilde{G} \leqslant \Delta$ since for any proper subgroup $\widetilde{G'}$ of $\widetilde{G}$ its inverse image G' through the canonical projection $G \longrightarrow \widetilde{G}$ is a proper subgroup of G lying over H, so $\widetilde{G'}^\perp = G'^\perp \cap H^\perp = G'^\perp$ is not contained in Δ by the minimality property of G. As $G^\perp = \widetilde{G}^\perp$ for all $G \in W_{\Delta,H}$, the fibers of the canonical map $W_{\Delta(\leqslant\Gamma),H} \longrightarrow W_{\Delta(\leqslant H^\perp),0}, G \mapsto \widetilde{G} = \mathrm{res}^\Gamma_{H^\perp}(G)$, are finite sets, and hence the proof of the finiteness of $W_{\Delta(\leqslant\Gamma),H}$ is reduced to the proof of the finiteness of the set $W_{\Delta(\leqslant H^\perp),0}$, as claimed.

Thus it remains to show that for any pair (Γ, A) and any open subgroup Δ of Γ, the set $W_{\Delta,0}$ as defined above is finite. We shall proceed by induction on the index $n := (\Gamma : \Delta)$. The case $n = 1$, i.e., $\Delta = \Gamma$, is trivial as $W_\Gamma = \{\{0\}\}$. Given Γ, A, Δ, assume $n > 1$, i.e., $\Delta \neq \Gamma$, and let $G \in W_{\Delta,0}$. Setting $\widetilde{G} = \mathrm{res}^\Gamma_\Delta(G)$, it follows by [4], Corollary 1.12 that $\widetilde{G} \cong G/(G \cap \Delta^\perp) \cong G \notin \mathcal{K}(Z^1(\Delta, A))$ since $G^\perp \leqslant \Delta$ and $(\Delta^\perp \cap G)^\perp = \{0\}^\perp = \Gamma \neq \Delta$. Consequently, by Theorem 1.3, there exists $p \in \mathcal{P}(\Delta, A) \subseteq \mathcal{P}(\Gamma, A)$ such that $\varepsilon|_\Delta \in \widetilde{G}$, where

$$\varepsilon = \begin{cases} \varepsilon_p & \text{if} \quad p \neq 4 \\ \varepsilon'_4 & \text{if} \quad p = 4 \end{cases}$$

As $G \cong \widetilde{G}$, it follows that there exists a unique element $g \in G$ such that $g|_\Delta = \varepsilon|_\Delta$, in particular $\operatorname{ord}(g) = \operatorname{ord}(\varepsilon|_\Delta) = p$ since $p \in \mathcal{P}(\Delta, A)$. Thus $G^\perp \leqslant \Delta \cap g^\perp = \Delta \cap \varepsilon^\perp = \Delta \cap \varepsilon_p^\perp \neq \Delta$. Clearly, $G \in W_{\Delta_p, H}$, where $\Delta_p = \Delta \cap \varepsilon_p^\perp$ and $H = G \cap \Delta_p^\perp$. Thus $W_{\Delta,0}$ is covered by the union of the sets $W_{\Delta_p,H}$ as above, so it remains to show that any such set $W_{\Delta_p,H}$ is finite and the set of possible pairs (p, H) is finite too. As for any p, H belongs to the finite set T_{Δ_p}, we have only to show that the p' s range over a finite subset of $\mathcal{P}(\Delta, A)$.

First let us show that the set $W_{\Delta_p,H}$ above is finite. By the reduction step above and the induction hypothesis we have to show that $(H^\perp : \Delta_p) < n = (\Gamma : \Delta)$. As $H \leqslant \Delta_p^\perp$, we obtain $\Delta_p \leqslant \Delta_p^{\perp\perp} \leqslant H^\perp$, and hence $\Delta_p \leqslant H^\perp \cap \Delta$. On the other hand, since $g \in H$, we obtain $H^\perp \cap \Delta \leqslant g^\perp \cap \Delta = \varepsilon_p^\perp \cap \Delta = \Delta_p$, so $H^\perp \cap \Delta = \Delta_p$, i.e., the set $H^\perp/\Delta_p$ is identified with a subset of the finite set Γ/Δ, and hence $(H^\perp : \Delta_p) \leq (\Gamma : \Delta)$. Since $g \in H$ and $g|_\Delta = \varepsilon|_\Delta$, it follows that $\operatorname{res}^\Gamma_\Delta(H) \notin \mathcal{K}(Z^1(\Delta, A))$. As $H \cap \Delta^\perp \leqslant G \cap \Delta^\perp = \{0\}$, it follows by [4], Proposition 1.11 that $H^\perp\Delta \neq \Gamma$, and hence $(H^\perp : \Delta_p) < (\Gamma : \Delta)$ as required.

Finally observe that the subgroup D generated by the element $g - \varepsilon$ belonging to $\Delta^\perp[p]$ is Kneser since otherwise $\varepsilon \in D$ by Theorem 1.3, and hence $\langle\varepsilon\rangle = \langle g\rangle \leqslant G$ as $\operatorname{ord}(g) = p$, contrary to the assumption that $G \in \mathcal{K}(Z^1(\Gamma, A))$. Consequently, $\operatorname{ord}(g - \varepsilon) = (\Gamma : D) \,|\, (\Gamma : \Delta)$. As $\operatorname{ord}(g - \varepsilon) = p$ if $p \neq 4$, it follows that the set of possible p's is finite as desired. $\square$

Corollary 2.14 *A closed subgroup Δ of Γ is Kneser whenever any open subgroup of Γ lying over Δ is Kneser. In particular, the following statements are equivalent for a closed subgroup Δ of Γ.*

(1) *Δ is h-Kneser.*

(2) *Any open subgroup of Γ lying over Δ is Kneser.*

Proof. Let $\Delta \in \mathbb{L}(\Gamma)$ be such that any open subgroup $\Lambda \in \mathbb{L}(\Gamma|\Delta)$ is Kneser. Since $\overline{\{\Delta\}} = \mathbb{L}(\Gamma|\Delta)$ is a closed subset of the spectral space $\mathbb{L}(\Gamma)$, its inverse image $X := \{G \in \mathcal{K}(Z^1(\Gamma, A)) \,|\, \Delta \leqslant G^\perp\}$ through the continuous map $(-)^\perp : \mathcal{K}(Z^1(\Gamma, A)) \longrightarrow \mathbb{L}(\Gamma)$, $G \mapsto G^\perp$, is a closed subset of the spectral space $\mathcal{K}(Z^1(\Gamma, A))$, and hence also closed w.r.t. the profinite topology of $\mathcal{K}(Z^1(\Gamma, A))$. As the map $(-)^\perp$ is coherent by Theorem 2.13, the image $(-)^\perp(X) \subseteq \mathbb{L}(\Gamma|\Delta)$ is closed w.r.t. the profinite topology of $\mathbb{L}(\Gamma)$. Consequently, $\Delta \in (-)^\perp(X)$, i.e., $\Delta \in \mathcal{K}(\Gamma)$, since by assumption $\mathbb{L}(\Gamma|\Delta)_o \subseteq (-)^\perp(X)$ and $\mathbb{L}(\Gamma|\Delta)$ is obviously the closure of $\mathbb{L}(\Gamma|\Delta)_o$ w.r.t. the profinite topology of $\mathbb{L}(\Gamma)$. $\square$

Corollary 2.15. *$\mathcal{HK}(\Gamma)$ is a closed Γ-subspace of the spectral Γ-space $\mathcal{HR}(\Gamma)$.*

Remark 2.16. Let E/F be a Galois extension with $\Gamma := \mathrm{Gal}(E/F)$ acting continuously on the discrete multiplicative quasi-cyclic group $A := \mu_E$ of all roots of unity in E. For $L \in \mathbb{L}(E/F)$, set $\Delta := \mathrm{Gal}(E/L)$. We obtain

(i) $\Delta \in \mathcal{R}(\Gamma)$ iff L/F is radical (cf. [2, Ch. 2]).

(ii) $\Delta \in \mathcal{HR}(\Gamma)$ iff K/F is radical for all (finite) subextensions of L/F.

(iii) $\Delta \in \mathcal{K}(\Gamma)$ iff L/F is Kneser (cf. [2, Ch. 11]).

(iv) Δ is a-Kneser iff there exists a Kneser subextension K of E/F such that $L \subseteq K$.

(v) $\Delta \in \mathcal{HK}(\Gamma)$ iff every subextension K of L/F is Kneser iff every finite subextension K of L/F is Kneser.

3. A criterion for hereditarily Kneser subgroups. To obtain an analogue of Theorem 1.3 (Abstract Kneser Criterion) providing a characterisation of the h-Kneser groups inside $\mathcal{HR}(\Gamma)$, we have to describe the set $(\mathcal{HR}(\Gamma) \setminus \mathcal{HK}(\Gamma))_{\max}$ of the maximal elements w.r.t. inclusion of the open subset $\mathcal{HR}(\Gamma) \setminus \mathcal{HK}(\Gamma)$ of the spectral space $\mathcal{HR}(\Gamma)$. Note that the set above consists of all open radical subgroups $\Delta \leqslant \Gamma$ which are not Kneser but any $\Lambda \in \mathbb{L}(\Gamma|\Delta) \setminus \{\Delta\}$ is Kneser.

With this aim, we introduce the following four types of open subgroups Δ of Γ:

(A) $\Delta = \varepsilon_p^{\perp}$, where $p \in \mathcal{P}(\Gamma, A) \setminus \{4\}$, $(\Gamma : \Delta) = l^m \,|(p-1)$, l is a prime number, $m \geq 1$, $A^{\Delta}(l) = (1/l^{m-1})\mathbb{Z}/\mathbb{Z}$ (in particular, $\widehat{1/2} \notin A$ if $l = 2, m = 1$), and $\widehat{1/4} \in A^{\Gamma}$ if $l = 2, m \geq 3$.

(B) The normalizer $N_{\Gamma}(\Delta)$ of Δ in Γ is $\varepsilon_p^{\perp}$, where $p \in \mathcal{P}(\Gamma, A) \setminus \{4\}, \varepsilon_p^{\perp}/\Delta \cong \mathbb{Z}/p^k\mathbb{Z}, k \geq 1, \widehat{1/p^n} \in A$ for some $n \geq k+1 \geq 2$, $(1/(p^{n-1}r))\mathbb{Z}/\mathbb{Z} \leqslant A^{\varepsilon_p^{\perp}}$, where $r = (\Gamma : \varepsilon_p^{\perp}) \,|\, (p-1)$, $\widehat{1/l} \in A^{\Gamma}$ for $l \in \mathcal{P}, l \,|\, r$, $\Delta' := \Delta \cap \varepsilon_{p^n}^{\perp} \lhd \Gamma$, and

$$\Gamma/\Delta' \cong (\varepsilon_{p^n}^{\perp}/\Delta') \rtimes (\Gamma/\varepsilon_{p^n}^{\perp}).$$

(C) $\Delta \lhd \Gamma, 4 \in \mathcal{P}(\Gamma, A), \Delta \leqslant \varepsilon_4^{\perp}, \varepsilon_4^{\perp}/\Delta \cong \mathbb{Z}/2^k\mathbb{Z}$, $k \geq 1, \widehat{1/2^n} \in A^{\varepsilon_4^{\perp}}$ for some $n \geq k+2 \geq 3, \sigma\, \widehat{1/2^n} = -\,\widehat{1/2^n} + \widehat{1/2}$ for $\sigma \in \Gamma \setminus \varepsilon_4^{\perp}$, and

$$\Gamma/\Delta \cong \langle \sigma, \tau \,|\, \sigma^4 = 1, \sigma^2 = \tau^{2^{k-1}}, \sigma\tau\sigma^{-1} = \tau^{-1} \rangle.$$

(D) $4 \in \mathcal{P}(\Gamma, A), N_\Gamma(\Delta) = \varepsilon_4^\perp, \varepsilon_4^\perp/\Delta \cong \mathbb{Z}/2^k\mathbb{Z}, k \geq 1, \widehat{1/2^n} \in A$ for some $n \geq k+2 \geq 3, \widehat{1/2^{n-1}} \in A^{\varepsilon_4^\perp}, \Delta' := \Delta \cap \varepsilon_{2^n}^\perp \lhd \Gamma$, and $\Gamma/\Delta' \cong (\varepsilon_4^\perp/\Delta') \rtimes (\Gamma/\varepsilon_4^\perp)$ has the presentation

$$\Gamma/\Delta' \cong \langle \sigma, \tau, \delta \,|\, \sigma^2 = \tau^{2^k} = \delta^2 = 1, \delta\tau = \tau\delta, \sigma\tau\sigma^{-1} = \tau^{-1}, \sigma\delta\sigma^{-1} = \delta\tau^{2^{k-1}} \rangle,$$

with the action of Γ/Δ' on $(1/2^n)\mathbb{Z}/\mathbb{Z}$ given by $\sigma\widehat{1/2^n} = -\widehat{1/2^n}, \tau\widehat{1/2^n} = \widehat{1/2^n}$, $\delta\widehat{1/2^n} = \widehat{1/2^n} + \widehat{1/2}$.

Lemma 3.1. *The necessary and sufficient condition for an open subgroup $\Delta \leqslant \Gamma$ to belong to $(\mathcal{HR}(\Gamma) \setminus \mathcal{HK}(\Gamma))_{\max}$ is that Δ is of one of the types* (A)–(D) *above.*

Proof. Let $\Delta \in (\mathcal{HR}(\Gamma) \setminus \mathcal{HK}(\Gamma))_{\max}$. Since Δ is an open radical subgroup of Γ, there exists a finite subgroup $G \in \mathbb{L}(Z^1(\Gamma, A)) \setminus \mathcal{K}(Z^1(\Gamma, A))$ such that $\Delta = G^\perp$. Choose such a subgroup $G \leqslant Z^1(\Gamma, A)$ of minimal order $|G|$. Assuming that G has proper direct summands, say $G = G_1 \oplus G_2$, with $0 \neq G_i \leqslant G, i = 1, 2$, it follows that $G_i \in \mathcal{K}(Z^1(\Gamma, A)), i = 1, 2$. Indeed, assuming $G_1 \notin \mathcal{K}(Z^1(\Gamma, A))$, i.e. $(\Gamma : G_1^\perp) < |G_1|$, it follows by the minimality of $|G|$ that $G_1^\perp \in \mathbb{L}(\Gamma \,|\, \Delta) \setminus \{\,\Delta\,\}$, so $G_1^\perp \in \mathcal{K}(\Gamma)$ by assumption, i.e. there exists $G' \in \mathcal{K}(Z^1(\Gamma, A))$ such that $G_1^\perp = G'^\perp$. Consequently, $\Delta = G^\perp = G_1^\perp \cap G_2^\perp = G'^\perp \cap G_2^\perp = (G' + G_2)^\perp$, and $|G' + G_2| \leq |G'||G_2| = (\Gamma : G'^\perp)|G_2| = (\Gamma : G_1^\perp)|G_2| < |G_1||G_2| = |G|$, contrary to the minimality of $|G|$.

In particular, G is a p-group for some prime number p, since otherwise it follows by the fact above and [4], Corollary 1.16 (the local-global principle for Kneser groups of cocycles) that $G \in \mathcal{K}(Z^1(\Gamma, A))$, and hence $\Delta = G^\perp \in \mathcal{K}(\Gamma)$, which is a contradiction. Set $\exp(G) = p^n$, so $\widehat{1/p^n} \in A$.

We distinguish the following two cases:

Case 1 : $p \neq 2$.

As $G \notin \mathcal{K}(Z^1(\Gamma, A))$, it follows by Theorem 1.3, that $\varepsilon_p \in G$, and hence $\Delta = G^\perp \leqslant \varepsilon_p^\perp$. Set $2 \leq r := (\Gamma : \varepsilon_p^\perp) \,|\, (p-1)$.

We claim that G is cyclic of order $p^n, n \geq 1$. If $\Delta = \varepsilon_p^\perp$, then $G = \langle \varepsilon_p \rangle \cong \mathbb{Z}/p\mathbb{Z}$, by the minimality of $|G|$. Thus we may assume $\Delta \neq \varepsilon_p^\perp$ and hence $G \neq \langle \varepsilon_p \rangle$. Let $\widetilde{G} := \mathrm{res}^\Gamma_{\varepsilon_p^\perp}(G) \leqslant Z^1(\varepsilon_p^\perp, (1/p^n)\mathbb{Z}/\mathbb{Z})$. As a p-group, $\widetilde{G}$ is a Kneser subgroup of $Z^1(\varepsilon_p^\perp, (1/p^n)\mathbb{Z}/\mathbb{Z})$ by Theorem 1.3, and hence Cogalois by [5], Corollary 2.9, since $p \neq 2$. In particular, the canonical map $\mathbb{L}(\widetilde{G}) \longrightarrow \mathbb{L}(\varepsilon_p^\perp|\Delta), U \mapsto U^\perp$, is a lattice anti-isomorphism.

First let us show that the p-group $\widetilde{G}$ is cyclic. Assuming $\widetilde{G} = \widetilde{G}_1 \oplus \widetilde{G}_2$, with $0 \neq \widetilde{G}_i \leqslant \widetilde{G}, i = 1, 2$, we obtain $\widetilde{G}_i^{\perp} \in \mathbb{L}(\varepsilon_p^{\perp}|\Delta) \setminus \{\, \Delta, \varepsilon_p^{\perp} \,\}, i = 1, 2$, and $\widetilde{G}_1^{\perp} \cap \widetilde{G}_2^{\perp} = \Delta$. On the other hand, $\widetilde{G}_1^{\perp}$ and $\widetilde{G}_2^{\perp}$ are Kneser subgroups of Γ as proper overgroups of Δ. Let $G_i \in \mathcal{K}(Z^1(\Gamma, A)), i = 1, 2$, be such that $\widetilde{G}_i^{\perp} = G_i^{\perp}, i = 1, 2$. Since $|G_i| = (\Gamma : G_i^{\perp}) = (\Gamma : \varepsilon_p^{\perp})(\varepsilon_p^{\perp} : G_i^{\perp}) = r|\widetilde{G}_i|$, and $(r, |\widetilde{G}_i|) = 1$, it follows that $|G_i(p)| = |\widetilde{G}_i|, i = 1, 2$. As $r = (\Gamma : \varepsilon_p^{\perp}) \,|\, (\Gamma : (\varepsilon_p^{\perp} \cap G_i(p)^{\perp}))$, $|\widetilde{G}_i| = (\Gamma : G_i(p)^{\perp}) \,|\, (\Gamma : (\varepsilon_p^{\perp} \cap G_i(p)^{\perp}))$, and $(r, |\widetilde{G}_i|) = 1$, we obtain $r|\widetilde{G}_i| \,|\, (\Gamma : (\varepsilon_p^{\perp} \cap G_i(p)^{\perp})) \,|\, (\Gamma : G_i^{\perp}) = r|\widetilde{G}_i|$, and hence $G_i^{\perp} = G_i(p)^{\perp} \cap \varepsilon_p^{\perp}, i = 1, 2$. As a proper overgroup of $\Delta\,, \varepsilon_p^{\perp} \in \mathcal{K}(\Gamma)$, so there exists $H \in \mathcal{K}(Z^1(\Gamma, A))$ such that $\varepsilon_p^{\perp} = H^{\perp}$, in particular, $|H| = r$. Consequently, $G_i^{\perp} = (G_i(p) \oplus H)^{\perp}, i = 1, 2$, and $\Delta = G^{\perp} = (G_1(p) + G_2(p) + H)^{\perp}$. By the minimality of $|G|$, we obtain $|G_1(p)|\,|G_2(p)|\,|H| \geq |G_1(p) + G_2(p) + H| \geq |G| > (\Gamma : G^{\perp}) =$
$(\Gamma : \varepsilon_p^{\perp})(\varepsilon_p^{\perp} : \widetilde{G}^{\perp}) = |H|\,|\widetilde{G}| = |H|\,|\widetilde{G}_1|\,|\widetilde{G}_2| = |H|\,|G_1(p)|\,|G_2(p)|$, which is a contradiction. Thus $\widetilde{G}$ is cyclic, as required.

Choose some $g \in G$ such that $\widetilde{G} = \langle g|_{\varepsilon_p^{\perp}} \rangle$, so $G^{\perp} = \widetilde{G}^{\perp} = g^{\perp} \cap \varepsilon_p^{\perp} = \langle g, \varepsilon_p \rangle^{\perp}$, and hence $G = \langle g, \varepsilon_p \rangle$ by the minimality of $|G|$. Assuming $\langle g \rangle \cap \langle \varepsilon_p \rangle = 0$, i.e. $G = \langle g \rangle \oplus \langle \varepsilon_p \rangle$, it follows that $\langle \varepsilon_p \rangle \in \mathcal{K}(Z^1(\Gamma, A))$ as a proper direct summand of G, which is a contradiction. Consequently, the cocycle ε_p of prime order p belongs to $\langle g \rangle$, so $G = \langle g \rangle \cong \mathbb{Z}/p^n\mathbb{Z}$, as claimed.

We may assume that $p^{n-1}g = \varepsilon_p$. Note also that $\operatorname{ord}(g(\sigma)) = p^n$ for all $\sigma \in \Gamma \setminus \varepsilon_p^{\perp}$, since for any such σ, $p^{n-1}g(\sigma) = \varepsilon_p(\sigma) \neq 0$.

We distinguish the following subcases:

Subcase 1.1: $n = 1$, i.e. $\Delta = \varepsilon_p^{\perp} \notin \mathcal{K}(\Gamma)$, but $\Lambda \in \mathcal{K}(\Gamma)$ for $\Lambda \in \mathbb{L}(\Gamma|\Delta) \setminus \{\Delta\}$.

First let us show that $r = (\Gamma : \Delta) = l^m \,|\, (p-1)$ for some prime number l and some $m \geq 1$. Assuming the contrary, let $r = \prod_{i=1}^{k} l_i^{m_i}, k \geq 2$, l_i pairwise distinct prime numbers, and $m_i \geq 1$. Let $\Lambda_i, i = 1, .., k$, denote the unique subgroup of Γ of index $l_i^{m_i}$ lying over Δ, so $\Gamma/\Delta \cong \prod_{i=1}^{k} \Gamma/\Lambda_i \cong \mathbb{Z}/r\mathbb{Z}$. As $\Lambda_i \neq \Delta$, $\Lambda_i \in \mathcal{K}(\Gamma)$, so $\Lambda_i = G_i^{\perp}$ for some $G_i \in \mathcal{K}(Z^1(\Gamma, A))$, i.e. $|G_i| = (\Gamma : \Lambda_i) = l_i^{m_i}$. Thus $\Delta = \bigcap_{i=1}^{k} \Lambda_i = \bigcap_{i=1}^{k} G_i^{\perp} = G^{\perp}$, where $G = (\bigoplus_{i=1}^{k} G_i)^{\perp}$. As $(|G_i|, |G_j|) = 1$ for $i \neq j$, it follows by [4], Corollary 1.16, that $G \in \mathcal{K}(Z^1(\Gamma, A))$ and hence $\Delta \in \mathcal{K}(\Gamma)$, which is a contradiction.

To conclude that Δ is of type (A), it remains to show that $A^{\Delta}(l) = (1/l^{m-1})\mathbb{Z}/\mathbb{Z}$ and $\widehat{1/4} \in A^{\Gamma}$ if $l = 2, m \geq 3$.

Let Λ denote the unique subgroup of Γ lying over Δ such that $(\Gamma : \Lambda) = l^{m-1}$. By assumption, $\Lambda \in \mathcal{K}(\Gamma)$, so $\Lambda = H^{\perp}$ for some $H \in \mathcal{K}(Z^1(\Gamma, A))$, i.e. $|H| = (\Gamma : \Lambda) = l^{m-1}$.
As $\Gamma/\Lambda \cong \mathbb{Z}/l^{m-1}\mathbb{Z}$ is cyclic, it follows by [5], Theorem 2.19, that H is a Cogalois subgroup of $Z^1(\Gamma, A)$, and hence $H = \langle h \rangle \cong \mathbb{Z}/l^{m-1}\mathbb{Z}$ for some $h \in Z^1(\Gamma, A)$. Consequently, $\widehat{1/l^{m-1}} \in A^{\Lambda} \leqslant A^{\Delta}$ as $h^{\perp} = \Lambda \lhd \Gamma$. Assuming $\widehat{1/l^{m}} \in A^{\Delta}$ too, it follows by Lemma 2.12, (1), that Δ is h-Kneser, which is a contradiction. Thus $A^{\Delta}(l) = (1/l^{m-1})\mathbb{Z}/\mathbb{Z}$, as desired. On the other hand, assuming $l = 2, m \geq 3$ and $\widehat{1/4} \notin A^{\Gamma}$, it follows that $4 \in \mathcal{P}(\Gamma, A) \cap \mathcal{P}_H$, and hence $\Lambda = H^{\perp} \nleqslant \varepsilon_4^{\perp}$ by Theorem 1.5, since $H = \langle h \rangle \cong \mathbb{Z}/2^{m-1}\mathbb{Z}$ is a Cogalois subgroup of $Z^1(\Gamma, A)$. However we have seen above that $\widehat{1/4} \in (1/2^{m-1})\mathbb{Z}/\mathbb{Z} \leqslant A^{\Lambda}$, so $\Lambda \leqslant \varepsilon_4^{\perp}$, which is a contradiction.

Subcase 1.2: $n \geq 2$, i.e. $\Delta = G^{\perp} \subsetneq \varepsilon_p^{\perp}, \mathbb{L}(\Gamma|\Delta) \setminus \mathcal{K}(\Gamma) = \{\Delta\}, G = \langle\, g \,\rangle \cong \mathbb{Z}/p^n\mathbb{Z}$, and $p^{n-1}g = \varepsilon_p$.

Let $\widetilde{G} := \mathrm{res}^{\Gamma}_{\varepsilon_p^{\perp}}(G), \widetilde{g} := g|_{\varepsilon_p^{\perp}}$, and $p^k, 1 \leq k \leq n-1$, be its order, so $(\varepsilon_p^{\perp} : \Delta) = p^k$, as $\Delta = \widetilde{G}^{\perp}$ and $\widetilde{G}$ is a Cogalois subgroup of $Z^1(\varepsilon_p^{\perp}, A)$. Recall that $\mathrm{ord}\,(g(\sigma)) = p^n$ for all $\sigma \in \Gamma \setminus \varepsilon_p^{\perp}$. As $pG = \langle\, pg \,\rangle \neq G$, it follows by the minimality of $|G|$ that $(pG)^{\perp}$ is a proper overgroup of Δ, so $(pG)^{\perp} \in \mathcal{K}(\Gamma)$. Note that $(pG)^{\perp} \leqslant \varepsilon_p^{\perp}$ since $\varepsilon_p = p^{n-2}(pg) \in pG$. As $\widetilde{G} \cong \mathbb{Z}/p^k\mathbb{Z}$ is Cogalois, it follows that $((pG)^{\perp} : \varepsilon_p^{\perp}) = p^{k-1}$, so $(\Gamma : (pG)^{\perp}) = p^{k-1}r$. Let $H \in \mathcal{K}(Z^1(\Gamma, A))$ be such that $(pG)^{\perp} = H^{\perp}$ and hence $|H| = p^{k-1}r$. Since $p\widetilde{G}$ and $\widetilde{H} := \mathrm{res}^{\Gamma}_{\varepsilon_p^{\perp}}(H)$ are Cogalois subgroups of $Z^1(\varepsilon_p^{\perp}, A)$, and $\widetilde{H}^{\perp} = (p\widetilde{G})^{\perp} = H^{\perp}$, it follows by [5], Corollary 2.12, that $\widetilde{H} = p\widetilde{G} \cong \mathbb{Z}/p^{k-1}\mathbb{Z}$. Consequently, $|H \cap \varepsilon_p^{\perp\perp}| = r \,|\, (p-1)$, so $H = (H \cap \varepsilon_p^{\perp\perp}) \oplus H(p)$. As $H \in \mathcal{K}(Z^1(\Gamma, A))$, its subgroup $H' := H \cap \varepsilon_p^{\perp\perp}$ is also Kneser, and hence $H'^{\perp} = \varepsilon_p^{\perp}$ since $\varepsilon_p^{\perp} \leqslant H'^{\perp}$ and $(\Gamma : H'^{\perp}) = |H'| = r = (\Gamma : \varepsilon_p^{\perp})$. Moreover H' is a Cogalois subgroup of $Z^1(\Gamma, A)$ by [5], Theorem 2.19, since $\Gamma/H'^{\perp} = \Gamma/\varepsilon_p^{\perp} \cong \mathbb{Z}/r\mathbb{Z}$ is cyclic, so $H' \cong \mathbb{Z}/r\mathbb{Z}$, in particular $\widehat{1/r} \in A^{\varepsilon_p^{\perp}}$, as $H'^{\perp} = \varepsilon_p^{\perp} \lhd \Gamma$, and hence $\widehat{1/l} \in A^{\Gamma}$ for $l \in \mathcal{P}, l|r$. By [5], Corollary 2.12, H' is the unique Cogalois subgroup of $Z^1(\Gamma, A)$ satisfying $H'^{\perp} = \varepsilon_p^{\perp}$.

Choose a generator h of $H(p) \cong \widetilde{H} = p\widetilde{G} \cong \mathbb{Z}/p^{k-1}\mathbb{Z}$ such that $\widetilde{h} := h|_{\varepsilon_p^{\perp}} = p\widetilde{g}$, so $h - pg \in \varepsilon_p^{\perp\perp}$ and hence $\varepsilon_p^{\perp} \leqslant (h - pg)^{\perp}$. Moreover $(h - pg)^{\perp} = \varepsilon_p^{\perp}$. Indeed, assuming $\tau \in (h - pg)^{\perp}$, we obtain $p^k g(\tau) = p^{k-1}h(\tau) = 0$, i.e. $\tau \in (p^k G)^{\perp} = (G \cap \varepsilon_p^{\perp\perp})^{\perp} = \varepsilon_p^{\perp}$, as $\varepsilon_p \in G$.

On the other hand, since $\mathrm{ord}\,(h) = p^{k-1} < p^{n-1} = \mathrm{ord}\,((pg)(\sigma))$, $\sigma \in \Gamma \setminus \varepsilon_p^\perp$, it follows that $\mathrm{ord}\,((h-pg)(\sigma)) = p^{n-1}$ for all $\sigma \in \Gamma \setminus \varepsilon_p^\perp$. Consequently, $\widehat{1/p^{n-1}} \in A^{\varepsilon_p^\perp}$, i.e. $\varepsilon_{p^{n-1}}^\perp = \varepsilon_p^\perp$, since $(h-pg)^\perp = \varepsilon_p^\perp \lhd \Gamma$. As $\mathrm{ord}\,(\widetilde{g}) = p^k \leq p^{n-1}$, it follows that $\widetilde{g} = g|_{\varepsilon_p^\perp} \in \mathrm{Hom}\,(\varepsilon_p^\perp, (1/p^k)\mathbb{Z}/\mathbb{Z})$, so $\Delta = g^\perp = \mathrm{Ker}\,(\widetilde{g}) \lhd \varepsilon_p^\perp$ and $\varepsilon_p^\perp/\Delta \cong \mathbb{Z}/p^k\mathbb{Z}$. Thus $\Delta' := \Delta \cap \varepsilon_{p^n}^\perp \lhd \varepsilon_p^\perp$. Moreover $\Delta' \lhd \Gamma$ since $g(\sigma\delta\sigma^{-1}) = g(\sigma) - \delta g(\sigma) = 0$ for all $\sigma \in \Gamma, \delta \in \Delta'$.

It remains to consider the following three situations:

1.2.1: $\varepsilon_{p^n}^\perp = \varepsilon_p^\perp$, i.e. $\widehat{1/p^n} \in A^{\varepsilon_p^\perp}$.

It follows that $\Delta = \Delta' \lhd \Gamma$, so we may assume without loss that $\Delta = \{1\}$, $|\Gamma| = p^k r$, $\varepsilon_p^\perp \cong \mathbb{Z}/p^k\mathbb{Z}$, and $A = A^{\varepsilon_p^\perp} = (1/p^n r)\mathbb{Z}/\mathbb{Z}$. Thus $\Gamma \cong \varepsilon_p^\perp \rtimes \Gamma/\varepsilon_p^\perp \cong \mathbb{Z}/p^k\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$, and hence $\Delta = \{1\}$ is h-Kneser by Lemma 2.12, (1), which is a contradiction. Consequently, the situation 1.2.1 cannot occur.

1.2.2: $\Delta \leqslant \varepsilon_{p^n}^\perp \neq \varepsilon_{p^{n-1}}^\perp = \varepsilon_p^\perp$.

Thus $\Delta \lhd \Gamma$, so we may assume $\Delta = \{1\}, A = (1/p^n r)\mathbb{Z}/\mathbb{Z}, \varepsilon_p^\perp \cong \mathbb{Z}/p^k\mathbb{Z}$, and $\Gamma/\varepsilon_{p^n}^\perp \cong \varepsilon_p^\perp/\varepsilon_{p^n}^\perp \times \Gamma/\varepsilon_p^\perp \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z} \cong \mathbb{Z}/pr\mathbb{Z}$. Let $\sigma \in \Gamma$ be such that $\Gamma/\varepsilon_{p^n}^\perp = \langle \sigma\varepsilon_{p^n}^\perp \rangle$, and let $u = u_\sigma \in (\mathbb{Z}/p^n r\mathbb{Z})^*$ defining the action of σ. It follows that $\sigma^r \in \varepsilon_p^\perp \setminus \varepsilon_{p^n}^\perp$, so $\langle \sigma^r \rangle = \varepsilon_p^\perp \cong \mathbb{Z}/p^k\mathbb{Z}$. On the other hand, $g(\sigma^{pr}) = \frac{u^{pr}-1}{u-1} g(\sigma) = 0$ since $\mathrm{ord}\,(g) = p^n, u^{pr} \equiv 1 \mod p^n$ but $u \not\equiv 1 \bmod p$. As $g^\perp = \Delta = \{1\}$, it follows that $\sigma^{pr} = 1$, so $k = 1$ and $\Gamma = \langle \sigma \rangle \cong \mathbb{Z}/pr\mathbb{Z}$. As we have seen above, there exists a unique Cogalois subgroup $H' \cong \mathbb{Z}/r\mathbb{Z}$ of $Z^1(\Gamma, A)$ such that $\varepsilon_p^\perp = H'^\perp$, so the monomorphism $Z^1(\Gamma, A) \longrightarrow A = (1/p^n r)\mathbb{Z}/\mathbb{Z}, \alpha \mapsto \alpha(\sigma)$, is onto, and $Z^1(\Gamma, A) = G \oplus H' \cong \mathbb{Z}/p^n r\mathbb{Z}$. In particular, H' is the maximal Kneser subgroup of $Z^1(\Gamma, A)$, and hence $\varepsilon_p^\perp = \langle \sigma^r \rangle$ is the minimal Kneser subgroup of Γ. Consequently, the proper subgroup $\langle \sigma^p \rangle \cong \mathbb{Z}/r\mathbb{Z}$ of Γ is not Kneser, which is a contradiction. Moreover note that the subgroup above is not radical since $\langle \sigma^p \rangle^\perp = 0$, so $\langle \sigma^p \rangle^{\perp\perp} = \Gamma \neq \langle \sigma^p \rangle$. Thus the situation 1.2.2 cannot occur.

1.2.3: $\Delta \nleqslant \varepsilon_{p^n}^\perp$.

To conclude that Δ is of type (B) we have only to check that $N_\Gamma(\Delta) = \varepsilon_p^\perp$ and $\Gamma/\Delta' \cong (\varepsilon_{p^n}^\perp/\Delta') \rtimes (\Gamma/\varepsilon_{p^n}^\perp)$. As $\Delta \lhd \varepsilon_p^\perp$ and $\Delta' \lhd \Gamma$ it remains to show that $\sigma\delta\sigma^{-1} \notin \Delta = g^\perp$ whenever $\sigma \in \Gamma \setminus \varepsilon_p^\perp$ and $\delta \in \Delta \setminus \Delta' = \Delta \setminus \varepsilon_{p^n}^\perp$. For σ and δ as above, we obtain $g(\sigma\delta\sigma^{-1}) = g(\sigma) - \delta\, g(\sigma) \neq 0$, as required, since $\mathrm{ord}\,(g(\sigma)) = p^n$. On the other hand, choose $\sigma \in \Gamma$ such that $\Gamma/\varepsilon_{p^n}^\perp = \langle \sigma\varepsilon_{p^n}^\perp \rangle \cong \mathbb{Z}/pr\mathbb{Z}$, and let $u := u_\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^*$ defining the action of σ on $(1/p^n)\mathbb{Z}/\mathbb{Z}$. It follows that $g(\sigma^{pr}) = \frac{u^{pr}-1}{u-1}\, g(\sigma) = 0$ since $\mathrm{ord}\,(u) = pr$, in particular, $u \not\equiv 1 \mod p$, and

$g(\sigma) \in (1/p^n)\mathbb{Z}/\mathbb{Z}$. Thus $\sigma^{pr} \in \Delta \cap \varepsilon_{p^n}^{\perp} = \Delta'$, and hence $\Gamma/\Delta' \cong (\varepsilon_{p^n}^{\perp}/\Delta') \rtimes (\Gamma/\varepsilon_{p^n}^{\perp})$ as desired.

Case 2: $p = 2$.

As $G \notin \mathcal{K}(Z^1(\Gamma, A))$, it follows by Theorem 1.3, that $\varepsilon_4' \in G$, and hence $\Delta = G^{\perp} \leqslant \varepsilon_4'^{\perp} = \varepsilon_4^{\perp}$. Note that $\langle \varepsilon_4' \rangle \neq G$, since otherwise $\Delta = \varepsilon_4^{\perp} \in \mathcal{K}(\Gamma)$, which is a contradiction. In particular, $|G| \geq 8$.

We claim that G is cyclic of order 2^n, $n \geq 3$. Let $\widetilde{G} := \operatorname{res}^{\Gamma}_{\varepsilon_4^{\perp}}(G) \leqslant Z^1(\varepsilon_4^{\perp}, A)$. As $\mathcal{P}(\varepsilon_4^{\perp}, A) \cap \mathcal{P}_{\widetilde{G}} = \emptyset$, $\widetilde{G}$ is Cogalois by Theorem 1.5.

First let us show that the 2-group $\widetilde{G}$ is cyclic. Assuming $\widetilde{G} = G_1' \oplus G_2'$, with $0 \neq G_i' \leqslant \widetilde{G}, i = 1, 2$, we obtain $G_i'^{\perp} \in \mathbb{L}(\varepsilon_4^{\perp} \,|\, \Delta) \setminus \{\Delta, \varepsilon_4^{\perp}\}, i = 1, 2$, and $G_1'^{\perp} \cap G_2'^{\perp} = \Delta$. As proper overgroups of Δ, $G_1'^{\perp}$ and $G_2'^{\perp}$ are Kneser subgroups of Γ, so $G_i'^{\perp} = G_i^{\perp}$ for some $G_i \in \mathcal{K}(Z^1(\Gamma, A)), i = 1, 2$. As $(G_1 + G_2)^{\perp} = G_1^{\perp} \cap G_2^{\perp} = G_1'^{\perp} \cap G_2'^{\perp} = \Delta$, we obtain $|G| \leq |G_1 + G_2|$ by the minimality of $|G|$. Thus $|G_i| = (\Gamma : G_i^{\perp}) = (\Gamma : \varepsilon_4^{\perp})(\varepsilon_4^{\perp} : G_i'^{\perp}) = 2\,|G_i'|, i = 1, 2$, and hence the G_i 's are 2-groups. Consequently, $\widetilde{G}_i := \operatorname{res}^{\Gamma}_{\varepsilon_4^{\perp}}(G_i) \in \mathcal{K}(Z^1(\varepsilon_4^{\perp}, A)), i = 1, 2$, so $(G_i \cap \varepsilon_4^{\perp\perp})^{\perp} = \varepsilon_4^{\perp}$ by [4], Corollary 1.12, in particular, $G_i \cap \varepsilon_4^{\perp\perp}$ is a non-trivial 2-group, $i = 1, 2$. Note that $\varepsilon_4 \in G_i \cap \varepsilon_4^{\perp\perp}, i = 1, 2$ since $\varepsilon_4^{\perp\perp}[2] = \langle \varepsilon_4 \rangle$. Thus $\varepsilon_4 \in G_1 \cap G_2$, and hence $2\,|G_1'|\,|G_2'| = 2\,|\widetilde{G}| = (\Gamma : \varepsilon_4^{\perp})\,(\varepsilon_4^{\perp} : \widetilde{G}^{\perp}) = (\Gamma : G^{\perp}) < |G| \leq |\,G_1 + G_2\,| \leq \frac{|G_1|\,|G_2|}{2} = 2\,|G_1'|\,|G_2'|$, which is a contradiction. Consequently, $\widetilde{G}$ is cyclic, as required.

Let $g \in G$ be such that $\widetilde{G} = \langle g|_{\varepsilon_4^{\perp}} \rangle$, so $G^{\perp} = \widetilde{G}^{\perp} = g^{\perp} \cap \varepsilon_4^{\perp} = \langle g, \varepsilon_4 \rangle^{\perp}$, and hence $G = \langle g, \varepsilon_4 \rangle$ by the minimality of $|G|$. As $\varepsilon_4' \in G$ and $2\,\varepsilon_4' = \varepsilon_4$, we obtain $G = \langle g \rangle \cong \mathbb{Z}/2^n\mathbb{Z}, n \geq 3$, as claimed.

Thus $\Delta = g^{\perp} \subsetneq \varepsilon_4^{\perp}$, and we may assume that $2^{n-2}g = \varepsilon_4'$. Setting $\widetilde{g} = g|_{\varepsilon_4^{\perp}}$, it follows that $\operatorname{ord}(\widetilde{g}) = (\varepsilon_4^{\perp} : g^{\perp}) = 2^k$ for some k satisfying $1 \leq k \leq n-2$, so $(\Gamma : g^{\perp}) = 2^{k+1}$. Note also that $\operatorname{ord}(g(\sigma)) = 2^n$ for all $\sigma \in \Gamma \setminus \varepsilon_4^{\perp}$ since for any such σ, $2^{n-1}g(\sigma) = \varepsilon_4(\sigma) = \widehat{1/2} \neq 0$.

As $2\,G = \langle 2\,g \rangle \neq G$, it follows by the minimality of $|G|$ that $(2G)^{\perp}$ is a proper overgroup of Δ, so $(2\,G)^{\perp} \in \mathcal{K}(\Gamma)$. Note also that $(2\,G)^{\perp} \leqslant \varepsilon_4'^{\perp} = \varepsilon_4^{\perp}$ since $\varepsilon_4' = 2^{n-3}(2\,g) \in 2\,G$. As $\widetilde{G} \cong \mathbb{Z}/2^k\mathbb{Z}$ is Cogalois, it follows that $(\,(2\,G)^{\perp} : \varepsilon_4^{\perp}) = 2^{k-1}$, so $(\Gamma : (2G)^{\perp}) = 2^k$. Let $H \in \mathcal{K}(Z^1(\Gamma, A))$ be such that $(2G)^{\perp} = H^{\perp}$, in particular $|H| = 2^k$. Since $\widetilde{H} := \operatorname{res}^{\Gamma}_{\varepsilon_4^{\perp}}(H)$ and $2\,\widetilde{G}$ are Cogalois subgroups of $Z^1(\varepsilon_4^{\perp}, A)$, and $\widetilde{H}^{\perp} = (2\,\widetilde{G})^{\perp} = H^{\perp}$, it follows by [5], Corollary 2.12, that $\widetilde{H} = 2\,\widetilde{G} \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$, and hence $H \cap \varepsilon_4^{\perp\perp} = \ker(\operatorname{res}^{\Gamma}_{\varepsilon_4^{\perp}} : H \longrightarrow \widetilde{H}) = \langle \varepsilon_4 \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Let $h \in H$ be such that $\widetilde{h} := h|_{\varepsilon_4^{\perp}} = 2\,\widetilde{g}$, so $\operatorname{ord}(h) \in \{2^{k-1}, 2^k\}$. Thus

$h-2g \in \varepsilon_4^{\perp\perp}$, and hence $\varepsilon_4^{\perp} \leqslant (h-2\,g)^{\perp}$. Since $\mathrm{ord}\,(h) \le 2^k < 2^{n-1} = \mathrm{ord}\,(2\,g)$ and $(\Gamma : \varepsilon_4^{\perp}) = 2$, it follows that $h \neq 2\,g$, so $\Gamma \neq (h-2\,g)^{\perp} = \varepsilon_4^{\perp}$.

On the other hand, since for all $\sigma \in \Gamma \setminus \varepsilon_4^{\perp}$, $\mathrm{ord}\,((h-2\,g)(\sigma)) = 2^{n-1}$ and $\sigma^2 \in \varepsilon_4^{\perp}$, it follows that $\sigma\widehat{1/2^{n-1}} = -\,\widehat{1/2^{n-1}}$ for any such σ, and hence $\widehat{1/2^{n-1}} \in A^{\varepsilon_4^{\perp}}$, i.e. $\varepsilon_{2^{n-1}}^{\perp} = \varepsilon_4^{\perp}$. As $\mathrm{ord}\,(\widetilde{g}) = 2^k < 2^{n-1}$, it follows that $\widetilde{g} = g|_{\varepsilon_4^{\perp}} \in \mathrm{Hom}\,(\varepsilon_4^{\perp}, (1/2^k)\mathbb{Z}/\mathbb{Z})$, so $\Delta = g^{\perp} = \ker(\widetilde{g}) \lhd \varepsilon_4^{\perp}$ and $\varepsilon_4^{\perp}/\Delta \cong \mathbb{Z}/2^k\mathbb{Z}$. Note also that $\Delta' := \Delta \cap \varepsilon_{2^n}^{\perp} \lhd \Gamma$ since $g(\sigma\delta\sigma^{-1}) = g(\sigma) - \delta g(\sigma) = 0$ for all $\sigma \in \Gamma, \delta \in \Delta'$.

We distinguish the following three situations :

2.1: $\varepsilon_{2^n}^{\perp} = \varepsilon_4^{\perp}$, i.e. $\widehat{1/2^n} \in A^{\varepsilon_4^{\perp}}$.

It follows that $\Delta = \Delta' \lhd \Gamma$, so we may assume without loss that $\Delta = \{1\}, |\Gamma| = 2^{k+1}, \varepsilon_4^{\perp} \cong \mathbb{Z}/2^k\mathbb{Z}$, and $A = A^{\varepsilon_4^{\perp}} = (1/2^n)\mathbb{Z}/\mathbb{Z}$. Thus $\varepsilon_4^{\perp} = B^1(\Gamma, A)^{\perp}$ is the kernel of the action of Γ on A, and hence there are only two possibilities:

2.1.1: $\sigma a = -a$ for $\sigma \in \Gamma \setminus \varepsilon_4^{\perp}, a \in A$.

In this case $\Gamma \cong \varepsilon_4^{\perp} \rtimes (\Gamma/\varepsilon_4^{\perp}) \cong (\mathbb{Z}/2^k\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{D}_{2^{k+1}}$, and hence $\Delta = \{1\}$ is h-Kneser by Lemma 2.12, (2), contrary to our assumption. Thus the situation 2.1.1 cannot occur.

2.1.2: $\sigma a = -(1+2^{n-1})a$ for $\sigma \in \Gamma \setminus \varepsilon_4^{\perp}, a \in A$.

As $\mathrm{ord}\,(g(\sigma)) = 2^n$, we obtain $g(\sigma^2) = 2^{n-1}g(\sigma) = \widehat{1/2}$, and hence $g(\sigma^4) = 0$, so $\mathrm{ord}\,(\sigma) = 4$ since $\Delta = g^{\perp} = \{1\}$. Choosing a generator τ of $\varepsilon_4^{\perp} \cong \mathbb{Z}/2^k\mathbb{Z}$, we obtain the presentation $\Gamma \cong \langle \sigma, \tau \,|\, \sigma^4 = 1, \sigma^2 = \tau^{2^{k-1}}, \sigma\tau\sigma^{-1} = \tau^{-1}\rangle$, concluding that Δ is of type (C).

2.2: $\Delta \leqslant \varepsilon_{2^n}^{\perp} \neq \varepsilon_{2^{n-1}}^{\perp} = \varepsilon_4^{\perp}$.

Thus $\Delta \lhd \Gamma$, so we may assume that $\Delta = \{1\}, A = (1/2^n)\mathbb{Z}/\mathbb{Z}, \varepsilon_4^{\perp} \cong \mathbb{Z}/2^k\mathbb{Z}$, and $\Gamma/\varepsilon_{2^n}^{\perp} \cong (\varepsilon_4^{\perp}/\varepsilon_{2^n}^{\perp}) \times (\Gamma/\varepsilon_4^{\perp}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\sigma \in \Gamma \setminus \varepsilon_4^{\perp}$ be such that $\sigma a = -a$ for $a \in A$, and let $\tau \in \varepsilon_4^{\perp} \setminus \varepsilon_{2^n}^{\perp}$, so τ generates $\varepsilon_4^{\perp}$ and $\tau a = (1+2^{n-1})a$ for $a \in A$. As $\mathrm{ord}\,(g(\sigma)) = 2^n$ and $\mathrm{ord}\,(g(\tau)) = 2^k$, it follows that $g(\sigma^2) = 0$ and $g(\sigma\tau\sigma^{-1}) = (1-\tau)g(\sigma) + \sigma g(\tau) = 2^{n-1}g(\sigma) - g(\tau) = \widehat{1/2} - g(\tau) = g(\tau^{2^{k-1}-1})$. Consequently, $k \ge 2$, i.e. $\Delta \neq \varepsilon_{2^n}^{\perp}$, so $n \ge k+2 \ge 4$, and $\Gamma \cong \langle \sigma, \tau \,|\, \sigma^2 = \tau^{2^k} = 1, \sigma\tau\sigma^{-1} = \tau^{2^{k-1}-1}\rangle \cong \varepsilon_4^{\perp} \rtimes (\Gamma/\varepsilon_4^{\perp}) \ncong \mathbb{D}_{2^{k+1}}$. The monomorphism $Z^1(\Gamma, A) \longrightarrow A \times A, \alpha \mapsto (\alpha(\sigma), \alpha(\tau) - 2^{n-k}\alpha(\sigma))$, maps isomorphically $Z^1(\Gamma, A)$ onto $((1/2^n)\mathbb{Z}/\mathbb{Z}) \times ((1/2^{k-1})\mathbb{Z}/\mathbb{Z})$, and hence $\langle\sigma\rangle \neq \langle\sigma\rangle^{\perp\perp} = \langle \sigma, \tau^{2^{k-1}}\rangle$, i.e. $\langle\sigma\rangle \notin \mathcal{R}(\Gamma)$, which is a contradiction. Consequently, the situation 2.2 cannot occur.

2.3: $\Delta \nleqslant \varepsilon_{2^n}^{\perp}$.

We may assume $\Delta' = \{1\}, A = (1/2^n)\mathbb{Z}/\mathbb{Z}$, so $\Delta = \langle\delta\rangle \cong \mathbb{Z}/2\mathbb{Z}, \varepsilon_{2^n}^{\perp} = \langle\tau\rangle \cong \mathbb{Z}/2^k\mathbb{Z}$, and $\varepsilon_4^{\perp} = \varepsilon_{2^{n-1}}^{\perp} = \Delta\,\varepsilon_{2^n}^{\perp} = \langle\delta,\tau\rangle \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2^k\mathbb{Z}$. We obtain $\delta a = (1+2^{n-1})a, \tau a = a$, for $a \in A$. Choose $\sigma \in \Gamma\setminus\varepsilon_4^{\perp}$ satisfying $\sigma a = -a$ for $a \in A$, so $\sigma^2 \in g^{\perp}\cap\varepsilon_{2^n}^{\perp} = \{1\}$, i.e. $\operatorname{ord}(\sigma) = 2$. Thus $\Gamma \cong \varepsilon_4^{\perp} \rtimes (\Gamma/\varepsilon_4^{\perp})$. As $g(\sigma\tau\sigma^{-1}) = -g(\tau) = g(\tau^{-1})$, and $g(\sigma\delta\sigma^{-1}) = (1-\delta)g(\sigma) = 2^{n-1}g(\sigma) = \widehat{1/2} = g(\tau^{2^{k-1}}\delta)$, we obtain the presentation

$$\Gamma \cong \langle \sigma,\delta,\tau \mid \sigma^2 = \tau^{2^k} = \delta^2 = (\sigma\tau)^2 = [\delta,\tau] = (\sigma\delta)^2\tau^{2^{k-1}} = 1\rangle,$$

so Δ is of type (D), as required.

Conversely, we have to show that $\Delta \in (\mathcal{HR}(\Gamma)\setminus\mathcal{HK}(\Gamma))_{\max}$ whenever Δ is an open subgroup of Γ of one of the types (A) – (D).

(A): Assume $\Delta = \varepsilon_p^{\perp}, p \in \mathcal{P}(\Gamma, A)\setminus\{4\}, (\Gamma:\Delta) = l^m$, l a prime number, $m \geq 1$, $A^{\Delta}(l) = (1/l^{m-1})\mathbb{Z}/\mathbb{Z}$, and $\widehat{1/4} \in A^{\Gamma}$ for $l = 2, m \geq 3$. First we have to show that $\Delta \notin \mathcal{K}(\Gamma)$. Assuming the contrary, let $G \in \mathcal{K}(Z^1(\Gamma, A))$ be such that $\Delta = G^{\perp}$, so $|G| = (\Gamma:\Delta) = l^m$. If G is not cyclic, let $0 \neq G_i \leqslant G, i = 1,2$ be such that $G = G_1 \oplus G_2$. As $\Gamma/\Delta \cong \mathbb{Z}/l^m\mathbb{Z}$, and l is a prime number, $\mathbb{L}(\Gamma\,|\,\Delta)$ is totally ordered, so we may assume $G_1^{\perp} \leqslant G_2^{\perp}$, and hence $\Delta = (G_1+G_2)^{\perp} = G_1^{\perp}\cap G_2^{\perp} = G_1^{\perp}$. Consequently, $(\Gamma : G_1^{\perp}) = (\Gamma:\Delta) = |G| > |G_1|$, which is a contradiction. Thus $G = \langle g\rangle \cong \mathbb{Z}/l^m\mathbb{Z}$, so $\operatorname{ord}(g(\sigma)) = l^m$ for some $\sigma \in \Gamma\setminus\Delta$, in particular $(1/l^m)\mathbb{Z}/\mathbb{Z} \leqslant A$. Since $\Delta \lhd \Gamma$, it follows that for any such σ, $\tau g(\sigma) = g(\tau\sigma) = g(\sigma(\sigma^{-1}\tau\sigma)) = g(\sigma)$ for all $\tau \in \Delta = g^{\perp}$, so $\widehat{1/l^m} \in A^{\Delta}$, contrary to the assumption that $A^{\Delta}(l) = (1/l^{m-1})\mathbb{Z}/\mathbb{Z}$.

It remains to check that $\Lambda \in \mathcal{K}(\Gamma)$ whenever Λ is a proper overgroup of Δ. For any such Λ, we obtain $\Lambda \lhd \Gamma$ and $\Gamma/\Lambda \cong \mathbb{Z}/l^k\mathbb{Z}, 0 \leq k \leq m-1$. We may assume $k \geq 1$ for $l \neq 2$, resp. $k \geq 2$ for $l = 2$, since otherwise either $\Lambda = \Gamma$ or $(\Gamma:\Lambda) = 2$ and $\widehat{1/2} \in A^{\Gamma} \leqslant A^{\Lambda}$. Let $\sigma \in \Gamma$ be such that $\sigma\Delta$ is a generator of $\Gamma/\Delta \cong \mathbb{Z}/l^m\mathbb{Z}$. Since $\Delta \leqslant \varepsilon_{l^{m-1}}^{\perp} \leqslant \varepsilon_{l^k}^{\perp}$, Γ/Δ acts on $(1/l^k)\mathbb{Z}/\mathbb{Z}$. Let $u := u_{\sigma} \in (\mathbb{Z}/l^k\mathbb{Z})^*$ be such that $\sigma\widehat{1/l^k} = u\widehat{1/l^k}$. Obviously, $u \equiv 1 \bmod l$ if $l \neq 2$, and $u \equiv 1 \bmod 4$ if $l = 2$, as, by assumption, $\widehat{1/4} \in A^{\Gamma}$ for $l = 2, m \geq 3$. Consequently, $\Lambda = g^{\perp} \in \mathcal{K}(\Gamma)$, where the cocycle $g \in Z^1(\Gamma\,|\,\Delta, A^{\Delta})$ is defined by $g(\sigma) = \widehat{1/l^k}$.

(B): Let Δ be an open subgroup of type (B). We may assume without loss that $\Delta' := \Delta\cap\varepsilon_{p^n}^{\perp} = \{1\}$ and $A = (1/(p^n r)\mathbb{Z}/\mathbb{Z}$. Thus $\Delta = \langle\delta\rangle \cong \mathbb{Z}/p\mathbb{Z}, \varepsilon_{p^n}^{\perp} = \varepsilon_{p^n r}^{\perp} = \langle\tau\rangle \cong \mathbb{Z}/p^k\mathbb{Z}$, and $\varepsilon_p^{\perp} = \varepsilon_{p^{n-1}r}^{\perp} = \Delta\,\varepsilon_{p^n}^{\perp} \cong \mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p^k\mathbb{Z}$. Let $\sigma \in \Gamma$ be such that $\sigma^{pr} = 1$ and $\Gamma/\varepsilon_{p^n}^{\perp} = \langle\sigma\,\varepsilon_{p^n}^{\perp}\rangle$. Note that such a σ exists since

$\Gamma \cong \varepsilon_{p^n}^{\perp} \rtimes (\Gamma/\varepsilon_{p^n}^{\perp})$ by assumption. As $\sigma^r \in \varepsilon_p^{\perp} \setminus \varepsilon_{p^n}^{\perp}$, it follows that $\varepsilon_p^{\perp} = \langle \delta\,, \tau \rangle = \langle \sigma^r, \tau \rangle$ and $\langle \sigma^r \rangle \lhd \Gamma$. As $\Delta \ntriangleleft \Gamma$ and $\mathrm{ord}\,(\delta) = p$, we may assume that $\delta = \sigma^r \tau^{p^{k-1}}$. Let $u := u_\sigma \in A^*$ be such that $\sigma a = ua$ for $a \in A$. By assumption, it follows that $\mathrm{ord}\,(u \mod p^n) = pr$, $\mathrm{ord}\,(u \mod p^j) = r$ for $1 \le j \le n-1$, and $u \equiv 1 \bmod l$ for $l \in \mathcal{P}, l \,|\, r$. Setting $\sigma\tau\sigma^{-1} = \tau^v$, with $v \in (\mathbb{Z}/p^k\mathbb{Z})^*$, it follows that $\mathrm{ord}\,(v \bmod p^k) \,|\, r = \mathrm{ord}\,(v \bmod p)$ since $\sigma^r \tau = \tau \sigma^r$ and the conjugates $\sigma^i \delta\, \sigma^{-i}$, $0 \le i < r$, of the element $\delta = \sigma^r \tau^{p^{k-1}}$ are pairwise distinct, as $N_\Gamma(\Delta) = \varepsilon_p^{\perp}$ by assumption. Thus we may assume without loss that $v \equiv u \bmod p^k$, obtaining the presentation

$$\Gamma \cong \langle \sigma, \tau \,|\, \sigma^{pr} = \tau^{p^k} = \sigma\tau\sigma^{-1}\tau^{-u} = 1 \rangle$$

The monomorphism $Z^1(\Gamma, A) \longrightarrow A \times A, \alpha \mapsto (\alpha(\sigma), \alpha(\tau))$, maps isomorphically $Z^1(\Gamma, A)$ onto $((1/(p^n r))\mathbb{Z}/\mathbb{Z}) \times ((1/p^k)\mathbb{Z}/\mathbb{Z})$. It follows that $\Delta = \langle \delta \rangle = g^{\perp}$ for a convenient $g \in Z^1(\Gamma, A)$ with $\mathrm{ord}\,(g(\sigma)) = p^n$ and $\mathrm{ord}\,(g(\tau)) = p^k$, so Δ is a radical subgroup of Γ. However $\Delta \notin \mathcal{K}(\Gamma)$ since all the maximal Kneser subgroups of $Z^1(\Gamma, A)$ are isomorphic to $(\mathbb{Z}/r\mathbb{Z}) \times (\mathbb{Z}/p^k\mathbb{Z}) \cong \mathbb{Z}/p^k r\mathbb{Z}$, so the normal subgroup $\langle \sigma^r \rangle \cong \mathbb{Z}/p\mathbb{Z}$ is the minimal Kneser subgroup of Γ, in particular the unique Kneser subgroup of order p of Γ.

It remains to show that all proper overgroups of Δ are Kneser. Let $\Lambda := \langle \delta, \sigma^r \rangle = \langle \tau^{p^{k-1}}, \sigma^r \rangle \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Note that $\Lambda \lhd \Gamma$ and $\mathbb{L}(\varepsilon_p^{\perp} \,|\, \Delta) \setminus \{\Delta\} \subseteq \mathbb{L}(\Gamma \,|\, \Lambda)$ since $\varepsilon_p^{\perp}/\Delta \cong \mathbb{Z}/p^k\mathbb{Z}$. Moreover $\mathbb{L}(\Gamma \,|\, \Delta) \setminus \{\Delta\} = \mathbb{L}(\Gamma \,|\, \Lambda)$. Indeed, for any $\gamma \in \Gamma \setminus \varepsilon_p^{\perp}, \gamma\delta\gamma^{-1} \in \varepsilon_p^{\perp} \setminus \Delta$ since $N_\Gamma(\Delta) = \varepsilon_p^{\perp}$, and hence $\Delta \ne \langle \delta, \gamma \rangle \cap \varepsilon_p^{\perp}$, as required. Applying Lemma 2.12, (1), to the induced action of $\Gamma/\Lambda \cong (\varepsilon_p^{\perp}/\Lambda) \rtimes (\Gamma/\varepsilon_p^{\perp}) \cong (\mathbb{Z}/p^{k-1}\mathbb{Z}) \rtimes_u (\mathbb{Z}/r\mathbb{Z})$ on $(1/p^{n-1}r)\mathbb{Z}/\mathbb{Z}$, we conclude that $\Lambda \in \mathcal{HK}(\Gamma)$, so $\mathbb{L}(\Gamma \,|\, \Delta) \setminus \{\Delta\} \subseteq \mathcal{K}(\Gamma)$, as desired.

(C): Let Δ be an open subgroup of Γ of type (C). As $\Delta \lhd \Gamma$, we may assume that $\Delta = \{1\}$ and $A = A^{\varepsilon_4^{\perp}} = (1/2^n)\mathbb{Z}/\mathbb{Z}$, so $\varepsilon_4^{\perp} = \varepsilon_{2^n}^{\perp} = \langle \tau \rangle \cong \mathbb{Z}/2^k\mathbb{Z}$, $1 \le k \le n-2, \Gamma \cong \langle \sigma, \tau \,|\, \sigma^4 = 1, \sigma^2 = \tau^{2^{k-1}}, \sigma\tau\sigma^{-1} = \tau^{-1} \rangle$, in particular, $\Gamma = \langle \sigma \rangle \cong \mathbb{Z}/4\mathbb{Z}$ and $\tau = \sigma^2$ if $k = 1$, and $\sigma a = -(1 + 2^{n-1})a, \tau a = a$ for $a \in A$. Note that $\Lambda := \langle \sigma^2 \rangle \lhd \Gamma$, Λ is the center of Γ if $k \ge 2$, and $\Gamma/\Lambda \cong (\mathbb{Z}/2^{k-1}\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{D}_{2^k}$, and hence $\Lambda \in \mathcal{HK}(\Gamma)$ by Lemma 2.12, (2). On the other hand, the monomorphism $Z^1(\Gamma, A) \longrightarrow A \times A, \alpha \mapsto (\alpha(\sigma), \alpha(\tau) - 2^{n-k}\alpha(\sigma))$ maps isomorphically $Z^1(\Gamma, A)$ onto $((1/2^n)\mathbb{Z}/\mathbb{Z}) \times ((1/2^{k-1})\mathbb{Z}/\mathbb{Z})$. Setting $g(\sigma) = \widehat{1/2^n}, g(\tau) = \widehat{1/2^k}$, we obtain $g \in Z^1(\Gamma, A)$ and $\Delta = \{1\} = g^{\perp}$, so Δ is a radical subgroup of Γ. However $\Delta \notin \mathcal{K}(\Gamma)$ since the maximal Kneser subgroups K of $Z^1(\Gamma, A)$ are all isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{k-1}\mathbb{Z})$, and hence $K^{\perp} = \Lambda = \langle \sigma^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$ is the minimal Kneser subgroup of Γ. It remains to observe that $\mathbb{L}(\Gamma) \setminus \{1\} = \mathbb{L}(\Gamma \,|\, \Lambda) = \mathcal{K}(\Gamma)$ as desired.

(D): Let Δ be an open subgroup of Γ of type (D). Since $\Delta' = \Delta \cap \varepsilon_4^{\perp} \lhd \Gamma$, we may assume that $\Delta' = \{1\}$ and $A = (1/2^n)\mathbb{Z}/\mathbb{Z}$. By assumption, $\Gamma \cong \varepsilon_4^{\perp} \rtimes (\Gamma/\varepsilon_4^{\perp}) \cong \langle \sigma, \tau, \delta \,|\, \sigma^2 = \tau^{2^k} = \delta^2 = 1, \delta\tau = \tau\delta, \sigma\tau\sigma^{-1} = \tau^{-1}, \sigma\delta\sigma^{-1} = \delta\tau^{2^{k-1}} \rangle$, and $\sigma a = -a, \tau a = a, \delta a = (1 + 2^{n-1})a$ for $a \in A$. In particular, $N_\Gamma(\Delta) = \varepsilon_4^{\perp} = \varepsilon_{2^{n-1}}^{\perp}$, and the center $Z(\Gamma) = \langle (\sigma\delta)^2 = \tau^{2^{k-1}} \rangle \cong \mathbb{Z}/2\mathbb{Z}$. The monomorphism $Z^1(\Gamma, A) \longrightarrow A \times A \times A, \alpha \mapsto (\alpha(\sigma), \alpha(\delta), \alpha(\tau) - 2^{n-k}\alpha(\sigma))$, maps isomorphically $Z^1(\Gamma, A)$ onto $((1/2^n)\mathbb{Z}/\mathbb{Z}) \times ((1/2)\mathbb{Z}/\mathbb{Z}) \times ((1/2^{k-1})\mathbb{Z}/\mathbb{Z})$. It follows that $\Delta = \langle \delta \rangle = g^{\perp}$, where $g \in Z^1(\Gamma, A)$ is defined by $g(\sigma) = \widehat{1/2^n}, g(\tau) = \widehat{1/2^k}, g(\delta) = 0$, so $\Delta \in \mathcal{R}(\Gamma)$. On the other hand, the maximal Kneser subgroups K of $Z^1(\Gamma, A)$ are all isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2^{k-1}\mathbb{Z})$, and hence $K^{\perp} = Z(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ is the (unique) minimal Kneser subgroup of Γ. Consequently, the normal subgroup $\Lambda := \langle \delta, \tau^{2^{k-1}} \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the (unique) minimal Kneser subgroup of Γ lying over Δ, and $\Gamma/\Lambda \cong \mathbb{D}_{2^k}$, so $\Lambda \in \mathcal{HK}(\Gamma)$ by Lemma 2.12, (2). It remains to observe that $\mathbb{L}(\Gamma \,|\, \Delta) \setminus \{\Delta\} = \mathbb{L}(\Gamma \,|\, \Lambda)$ as required. $\square$

As a consequence of Lemma 3.1, we obtain the following h-Kneser criterion for h-radical subgroups.

Theorem 3.2. *The following assertions are equivalent for* $\Lambda \in \mathcal{HR}(\Gamma)$.

(1) $\Lambda \in \mathcal{HK}(\Gamma)$.

(2) $\Lambda \not\leq \Delta$ *whenever* Δ *is an open subgroup of* Γ *of one of the types* (A) - (D).

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