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REMARKS ON THE NAGATA CONJECTURE

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ABSTRACT. The famous Nagata Conjecture predicts the lowest degree of
a plane curve passing with prescribed multiplicities through given points
in general position. We explain how this conjecture extends naturally via
multiple point Seshadri constants to ample line bundles on arbitrary surfaces.
We show that if there exist curves of unpredictable low degree, then they
must have equal multiplicities in all but possibly one of the given points.
We use this restriction in order to obtain lower bounds on multiple point
Seshadri constants on a surface. We discuss also briefly a seemingly new
point of view on the Nagata Conjecture via the bigness of the involved
linear series.

0. Introduction. The aim of this survey paper is to give an introduction
to questions revolving around the Nagata Conjecture. This circle of problems is
subject to intensive current investigations and it is pretty difficult to give an
up to date account on the state of matters, we merely restrict to presenting a

sample of ideas which emerged recently and which, in our opinion, allow to look at the Nagata Conjecture from a new perspective. In particular statements in the spirit of Proposition 2.7 brought together the Nagata Conjecture and the Harbourne-Hirschowitz Conjecture (see problems A and D below). Whereas the latter Conjecture concerns mainly linear series on the projective plane, the Nagata Conjecture can be generalized to arbitrary surfaces (see 2.1). Again, via Seshadri constants, it motivates some expectations in the spirit of Harbourne-Hirschowitz for arbitrary surfaces.

The Nagata Conjecture itself arouse in connection with his studies [27] on the existence problem of plane algebraic curves of given degree with singularities of prescribed order in points in general position (which in turn was motivated by the 14th problem of Hilbert). By a special position construction followed by a degeneration argument he showed that for any $s \geq 4$ given s^2 points $P_1, \ldots, P_{s^2}$ in general position in $\mathbb{P}^2$ and given non-negative integers $m_1, \ldots, m_{s^2}$, the degree d of a curve passing through these points with multiplicities at least $m_1, \ldots, m_{s^2}$ is subject to restriction:

$$d > \frac{1}{s} \sum_{i=1}^{s^2} m_i.$$

Nagata conjectured that the same is true for any number of points bigger than 9.

It is natural to consider the problem on the blowing up of $\mathbb{P}^2$. So let $\pi : X \longrightarrow \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ in r points $P_1, \ldots, P_r$ with exceptional divisors $E_1, \ldots, E_r$ and let $H = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$. Line bundles of the form

$$(1) \qquad\qquad L = dH - \sum_{i=1}^{r} k_i E_i$$

have been studied by many authors with respect to different properties. This is an area of vivid current research. There are a lot of natural questions one might ask about line bundles of the form (1). The following short list gives only a sample of possibilities.

A. When is the linear series $|L|$ non empty?

B. When does $|L|$ define a morphism to a projective space, i.e. when is the line bundle L base point free?

C. When is the morphism defined by $|L|$ an embedding, i.e. when is L very ample (more generally: when does $|L|$ separate 0-dimensional subschemes of given length)?

D. A stable version of Question C is when is L ample?

In the simplest case, when all the coefficients $k_1, \ldots, k_r$ are equal 1, there are optimal answers to all these questions. Whereas Problem A is trivial in this case, the other three problems were solved only recently. The answers are:

A. The linear system $|L|$ is non empty, i.e. $h^0(dH - \sum_{i=1}^r E_i) \neq 0$ if the number r is restricted by $r \leq h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 1$.

B. Coppens showed in [11, Section 3.3] that if $d \geq 7$, then L is globally generated provided the number r of points blown up satisfies $r \leq h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 3$, i.e. if there are at least three independent sections (which is the smallest possible number as the self-intersection of an effective L is positive for $d \geq 4$).

C. This problem has a long story, see e.g. [4], [5], the ultimate result being proved by D'Almeida and Hirschowitz [12]. They showed that φ_L is an embedding if X is obtained from $\mathbb{P}^2$ by blowing up at most $r \leq h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 6$ general points and $d \geq 5$.

D. Küchle [22] and independently Xu [40] showed that L is ample, provided L has a positive self-intersection $L^2 > 0$ (equivalently $r \leq L^2 - 1$) and $d \geq 3$. This follows also already by a repeated use of results in [16].

If the coefficients $k_1, \ldots, k_r$ are arbitrary, then the problem breaks up into two parts depending on the number r of points blown up. If $r \leq 9$, then again pretty much is known due in particular to recent works of Di Rocco [14], see also Example 2.4. On the other hand, if $r \geq 10$ there are a lot of partial and conjectural results around but the picture seems still far from being complete. Note that the breaking point is related to the positivity of the anti-canonical divisor on the blowup X.

Note also that the Nagata Conjecture is a special case of Problem A, see Remark 2.6. It states that if $|L|$ is non-empty then d needs to be sufficiently large. There is another conjecture due to Harbourne and Hirschowitz [17], [21] which predicts the dimension of $|L|$ more exactly. Counting the conditions in a naive way one arrives to the number

$$e(|L|) = e(d; k_1, \ldots, k_r) := \max \left\{ \binom{d+2}{2} - \sum_{i=1}^r \binom{k_i+1}{2} - 1, -1 \right\}$$

which is called the expected dimension of $|L|$. If $e(|L|)$ is non-negative then certainly $|L|$ is non-empty. If $e(|L|)$ is equal to the actual dimension of $|L|$ then the linear series $|L|$ is said to be *non-special*. Otherwise, in particular if $|L|$ is non-empty and $e(|L|)$ is negative, the linear series $|L|$ is called *special*. The Conjecture of Harbourne and Hirschowitz relates the speciality of $|L|$ to the existence of certain (-1)-curves. More exactly they predict that any special linear system is (-1)-special, i.e. there exist smooth irreducible curves $A_1, \ldots, A_t$ in X with selfintersection $A_i^2 = -1$ such that

- $L.A_1 \leq -2$,

- $L.A_j \leq -1$ for $j = 2, \ldots, t$,

- the residual system $M := L + \sum_{i=1}^{t}(L.A_i) \cdot A_i$ has non-negative expected dimension $\nu(|M|)$.

Thus Harbourne-Hirschowitz Conjecture provides a very clear picture of the structure of special linear series on blowups of $\mathbb{P}^2$. One cannot hope for this phenomenon to hold on arbitrary surfaces. On the other hand the Nagata Conjecture while being considerably less exact, extends in a convincing way to arbitrary surfaces. This was in part suggested by recent results of Biran on symplectic packings [7]. It is convenient to express this generalization in the language of Seshadri constants. Some 15 years ago Demailly [13] introduced them motivated partly by attempts to prove another famous conjecture of Fujita in arbitrary dimension. In a sense Seshadri constants capture the concept of the local positivity of a line bundle. Roughly speaking the Seshadri constant at a point measures the rate of growth of the number of jets generated by tensor powers of a line bundle at the given point (or, more generally, along a subvariety). Whereas originally Seshadri constants were viewed as a useful tool to produce sections of adjoint line bundles, they quickly became a subject of indepeden interest quite on its own e.g. [2], [3], [18], [28], [32]. These invariants turned out to be very hard to control. Apart from abelian surfaces [2] their exact value is known only in few examples. Even providing bounds on these numbers is an interesting but simultaneously a hard problem.

We recall basic properties of Seshadri constants in the next section and then explain how Seshadri constants are related to the Nagata Conjecture and its generalizations.

Notation. We work throughout over the field $\mathbb{C}$ of complex numbers. If X is a variety we denote by K_X the canonical divisor of X. A polarized variety is

a pair (X, L) consisting of a smooth variety X and an ample line bundle L on X. For a coherent sheaf $\mathcal{F}$ on X we denote by $H^i(X, \mathcal{F}) = H^i(\mathcal{F})$ the cohomology groups of $\mathcal{F}$ and by $h^i(\mathcal{F})$ their dimensions. For divisors and invertible sheaves we use rather additive than the tensor product notation but we stick to tensor notation for arbitrary sheaves. This gives rise to hybrids like $L + K_X \otimes \mathfrak{m}_x$ but we hope that this will cause no confusion. The numerical equivalence of divisors is denoted by $\equiv$. For a given real number α we denote by $\lceil \alpha \rceil$ its round-up, i.e. the least integer greater or equal α and by $\lfloor \alpha \rfloor$ its round-down, i.e. $-\lceil -\alpha \rceil$. By very general points on a variety X we mean points lying in the complement of a possibly countable sum of Zariski closed proper subsets of the parameter space.

1. Seshadri constants. In this part we recall basic definitions and properties of Seshadri constants and show some general facts on their behaviour.

We start with three (of course equivalent) definitions of the Seshadri constant of a line bundle at a point, each of them exhibiting a different flavor of information encoded in this invariant. Let X be a smooth projective variety, L a nef line bundle on X and $x \in X$ a fixed point. The *Seshadri constant of L at x* is the real number

$$(2) \qquad \varepsilon(L, x) = \inf_{C \ni x} \frac{L.C}{\operatorname{mult}_x C},$$

where the infimum is taken over all curves C passing through x. Note that it is enough to consider the irreducible ones as

$$\min\left\{ \frac{L.C_1}{\operatorname{mult}_x C_1}, \frac{L.C_2}{\operatorname{mult}_x C_2} \right\} \leq \frac{L.(C_1 + C_2)}{\operatorname{mult}_x(C_1 + C_2)}$$

holds. In the sequel we shall need the complimentary inequality

$$(3) \qquad \frac{L.(C_1 + C_2)}{\operatorname{mult}_x(C_1 + C_2)} \leq \max\left\{ \frac{L.C_1}{\operatorname{mult}_x C_1}, \frac{L.C_2}{\operatorname{mult}_x C_2} \right\}.$$

We define the *global Seshadri constant of L* as the number

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x).$$

A line bundle is ample if and only if $\varepsilon(L) > 0$. This is the Seshadri criterion of ampleness [20] and this also explains the name of the constants.

Now, let $f : Y \longrightarrow X$ be the blowing up of X at x with the exceptional divisor E. The definition (2) can be reformulated as follows:

$$(4) \qquad \varepsilon(L, x) = \sup\{\lambda \in \mathbb{R} : \quad f^*L - \lambda E \text{ is nef}\}.$$

Roughly speaking $\varepsilon(L, x)$ measures the length of the ray from the nef point f^*L in the direction of $-E$, lying in the range of the nef cone of X, or equivalently, the slope of the nef cone restricted to the plane generated by f^*L and E. Using vanishing theorems the same can be expressed in terms of the number of jets generated by a line bundle at a given point defined as the maximal integer $s = s(L, x)$ such that the evaluation mapping

$$(5) \qquad H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_X / \mathfrak{m}_x^{s+1})$$

is surjective. We say also that L is s-generated at x in this situation. If L is ample then

$$(6) \qquad \varepsilon(L, x) = \limsup_{k \to \infty} \frac{s(kL, x)}{k}.$$

Thus if L itself is s-generated at x then $\varepsilon(L, x) \geq s$, in particular $\varepsilon(L, x) \geq 1$ for L very ample. It is somewhat surprising that the converse statement is false, i.e. a line bundle need not to be even effective no matter how big its Seshadri constant at *every* point of X is.

Example 1.1. For any given positive integer N there exists a polarized variety (X, L) such that $\varepsilon(L) \geq N$ and L is not even effective.

Proof. For $i = 1, 2$ let $g_i \geq N + 1$ be given and let C_i be curves of genus g_i such that there are no correspondences between C_1 and C_2. Let $X = C_1 \times C_2$ and $\pi_i : X \longrightarrow C_i$ be the canonical projections with fibers F_i. Then $\mathrm{Num}(X) \cong \mathbb{Z} \cdot F_1 \oplus \mathbb{Z} \cdot F_2$.

Let L_i be a general line bundle on C_i of degree $g_i - 1$. Then $h^0(L_i) = 0$. Let $L = \pi_1^* L_1 \otimes \pi_2^* L_2$. Since F_i are clearly nef it follows that every effective curve C satisfies $C \equiv aF_1 + bF_2$ with $a, b \geq 0$. Then Nakai-Moishezon criterion implies that L is ample. On the other hand $h^0(L) = 0$ by Künneth formula.

Let $x \in X$ be fixed and let $C \equiv aF_1 + bF_2$ be an irreducible curve through x. Let $F_{1,x} = \pi_1^{-1}\pi_1(x)$ be the fiber through x. Then either $C = F_{1,x}$ and $\mathrm{mult}_x C = 1$ or C intersects $F_{1,x}$ properly and $\mathrm{mult}_x C \leq b$. Repeating the

reasoning with the second projection we get $\mathrm{mult}_x\, C \leq \min\{a, b\}$. Thus in both cases

$$\frac{L.C}{\mathrm{mult}_x\, C} \geq \frac{a(g_2 - 1) + b(g_1 - 1)}{\min\{a, b\}} \geq N,$$

which implies $\varepsilon(L, x) \geq N$. $\square$

The above example shows a truly asymptotic nature of the equality (6). On the other hand a large Seshadri constant of L implies some positivity of the adjoint line bundle $K_X + L$. This property was one of the original reasons for interest in Seshadri constants. Lazarsfeld proved the following result for surfaces in [24, Proposition 5.7].

Proposition 1.2. *Let X be a smooth projective variety of dimension n, L an ample line bundle on X and $x \in X$ a fixed point. If $\varepsilon(L, x) > s + n$ or $\varepsilon(L, n) = s + n$ and $L^n > (s+n)^n$ for some integer s, then $K_X + L$ is s-generated at x.*

P r o o f. It is a rather straightforward application of Kodaira vanishing (for nef and big line bundles). The following standard exact sequence

$$0 \longrightarrow (K_X + L) \otimes \mathrm{m}_x^{s+1} \longrightarrow K_X + L \longrightarrow (K_X + L) \otimes \mathcal{O}_X / \mathrm{m}_x^{s+1} \longrightarrow 0$$

says that to prove the assertion it suffices to show the vanishing of the cohomology group $H^1((K_X + L) \otimes \mathrm{m}_x^{s+1})$. Let $f : Y \longrightarrow X$ be the blowing up of X at x with the exceptional divisor E. The projection formula and the Leray spectral sequence imply that there is an isomorphism

$$H^1((K_X + L) \otimes \mathrm{m}_x^{s+1}) \cong H^1(f^*(K_X + L) - (s+1)E) = H^1(K_Y + f^*L - (s+n)E).$$

Now, by the assumptions and (4) the line bundle $f^*L - (s + n)E$ is nef and big and the vanishing of the group on the right follows. $\square$

Taking $X = \mathbb{P}^n$ and $L = \mathcal{O}_X(k)$ we have $\varepsilon(L) = k$. This shows that the assumptions of the above proposition cannot be weakened, at least not in the second case.

To conclude this paragraph we note that it makes sense to define Seshadri constants for $\mathbb{Q}$ or even $\mathbb{R}$-divisors. Obviously, for $\alpha \geq 0$ one has $\varepsilon(\alpha L; x) = \alpha\varepsilon(L; x)$ (the same holds for the global Sesahdri constant). Thus $\varepsilon(L; x)$ determines values at x for the whole ray αL in the nef cone. We have the following easy fact relating the values for distinct rays. This shows that $\varepsilon(\cdot, x)$ considered as a function on the nef cone of X is convex.

Lemma 1.3. *Let L_1, L_2 be nef line bundles on a smooth projective variety X and let $x \in X$ be a fixed point. Then for $0 \le s \le 1$ we have*

$$\varepsilon(sL_1 + (1-s)L_2; x) \ge s\varepsilon(L_1; x) + (1-s)\varepsilon(L_2; x).$$

P r o o f. We use definition (2). For every curve $C \subset X$ passing through x we have

$$\frac{(sL_1 + (1-s)L_2).C}{\text{mult}_x C} = s\frac{L_1.C}{\text{mult}_x C} + (1-s)\frac{L_2.C}{\text{mult}_x C} \ge s\varepsilon(L_1; x) + (1-s)\varepsilon(L_2; x).$$

Taking the infimum on the left hand side of the above inequality we get the assertion. $\square$

Now we turn to bounds on Seshadri constants. First, there exists a uniform upper bound:

(7) $$\varepsilon(L, x) \le \sqrt[n]{L^n}.$$

which follows easily from (4) and Kleiman's nefness criterion. This inequality motivates the following terminology. An effective curve C on X will be called *L-submaximal* at a point $x \in X$, if

$$\sqrt{\frac{L.C}{\text{mult}_x C}} < \sqrt{L^2} \ .$$

We will say that C *computes the Seshadri constant of L at x, if* $\varepsilon(L, x) = \sqrt{\frac{L.C}{\text{mult}_x C}}$.

For a polarized algebraic surface (X, L) the following lemma characterizes submaximal curves in linear systems given by tensor powers of L.

Lemma 1.4. *Let X be an algebraic surface and L an ample line bundle on X. If for some positive integer m there exists a submaximal curve $C \in |mL|$ at $x \in X$, then its component computes $\varepsilon(L, x)$.*

P r o o f. Since $\varepsilon(L, x)$ is submaximal the real-valued Nakai-Moishezon criterion [8] implies that there exists an irreducible curve D such that

$$\varepsilon(L, x) = \frac{L.D}{\text{mult}_x D}.$$

If C intersects D properly, then we have

$$mL.D = C.D \geq \mathrm{mult}_x C \cdot \mathrm{mult}_x D > \frac{L.C}{\sqrt{L^2}} \cdot \frac{L.D}{\varepsilon(L,x)} = \frac{\sqrt{L^2}}{\varepsilon(L,x)} \cdot mL.D$$

which is impossible as $\dfrac{\sqrt{L^2}}{\varepsilon(L,x)} > 1$ by the assumption. This shows that D is a component of C. $\square$

The following observation is an immediate consequence of the above lemma. We state it here as a toy case of Proposition 2.7. It was in fact the motivation for the general statement.

Corollary 1.5. *Let X be a smooth projective surface with Picard number $\rho(X) = 1$ and L an ample generator of the Néron-Severi group. Let $x \in X$ be an arbitrary point. Then either $\varepsilon(L,x) = \sqrt{L^2}$ is maximal, or there exists exactly one irreducible and reduced curve computing $\varepsilon(L,x)$.*

P r o o f. Suppose that $\varepsilon(L,x)$ is submaximal and computed by irreducible and reduced curves C_1 and C_2. Then Lemma 1.4 implies immediately $C_1 = C_2$. Note that if $\varepsilon(L,x)$ is equal to $\sqrt{L^2}$ there could be infinitely many curves computing its value, the simplest example being $X = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(1)$. Indeed, then any line in the pencil through $P \in \mathbb{P}^2$ computes $\varepsilon(L,P)$.

Also if the Picard number is greater than 1 the result fails. This can be seen on X a smooth quadric in $\mathbb{P}^3$ and $L = \mathcal{O}_X(1)$. For $x \in X$ the Seshadri constant $\varepsilon(L,x)$ is computed by the two lines in the rulings on X passing through x.

Proposition 2.7 particular takes into account the Picard number of the underlying surface. $\square$

Turning now to the lower bounds on Seshadri constants, note first that if L is ample, then the inequality $\varepsilon(L) > 0$ follows from Seshadri's ampleness criterion. This bound cannot be improved in general, there are examples due to Miranda [24, Proposition 5.12] showing that $\varepsilon(L,x)$ can become arbitrarily small. However there is a uniform lower bound

$$\varepsilon(L,x) \geq \frac{1}{\dim X}$$

due to Ein, Küchle and Lazarsfeld [15] valid at a very general point of X. In fact it is conjectured that, intuitively speaking, ample line bundles at a very general point x are as positive as the very ample ones, i.e. $\varepsilon(L,x) \geq 1$ holds. This conjecture was proved for algebraic surfaces by Ein and Lazarsfeld [16].

Another intriguing question in the area of Seshadri constants is the problem of their rationality. Though tempting to state as a conjecture, for the lack of enough evidence, we restrict ourself merely to asking the following

Question 1.6. *Let X be a smooth projective surface and let L be an ample line bundle on X. Is then the global Seshadri constant $\varepsilon(L)$ a rational number?*

This question found an affirmative answer for abelian surfaces in [3] and for Enriques surfaces in [36]. The following lemma shows that in order to give an affirmative answer to the above question it suffices to find a single point $x \in X$ such that $\varepsilon(L,x)$ is submaximal. Note that the statement is not obvious because it could happen that there is a sequence of submaximal values of $\varepsilon(L, x_n)$ at points x_n converging to an irrational limit.

Lemma 1.7. *Let (X, L) be a polarized surface. If there exists a point $x \in X$ such that the Seshadri constant of L at x is submaximal $\varepsilon(L, x) < \sqrt{L^2}$ then $\varepsilon(L)$ is a rational number.*

P r o o f. The following argument was suggested by Thomas Bauer. The claim follows also from recent results of Oguiso [29, Corollary 2].

Suppose that $\alpha = \varepsilon(L)$ and there exists a sequence (C_n, x_n) of irreducible curves and points on X such that

$$\alpha_n = \frac{L.C}{\mathrm{mult}_{x_n} C_n} \longrightarrow \alpha.$$

Let β be a rational number in the interval $\alpha < \beta < \sqrt{L^2}$. Without loss of generality we can assume that $\alpha_n < \beta$ for all n. It follows from the Riemann-Roch theorem that there exists a positive integer q (not depending on n) and a sequence of divisors $D_n \in |qL|$ such that

$$\frac{L.D_n}{\mathrm{mult}_{x_n} D_n} < \sqrt{L^2} + \delta,$$

where δ satisfies $0 < \delta < \dfrac{L^2 - \sqrt{L^2}\beta}{\beta}$. Assuming that C_n is not a component of D_n we have

$$qL.C_n = D_n.C_n \geq \mathrm{mult}_{x_n} D_n \cdot \mathrm{mult}_{x_n} C_n > \frac{qL^2}{\sqrt{L^2} + \delta} \, \mathrm{mult}_{x_n} C_n,$$

which gives $\dfrac{L.C_n}{\text{mult}_{x_n} C_n} > \beta$, a contradiction. Hence every curve C_n is a component of D_n. This shows that the degree of C_n (with respect to L) is uniformly bounded. But then there are only finitely many possible multiplicities of curves C_n so that the sequence α_n being convergent must in fact stabilize. This shows that α is a rational number. $\square$

Corollary 1.8. *Let (X, L) be a polarized surface. If L^2 is a square then $\varepsilon(L)$ is a rational number.*

P r o o f. Either $\varepsilon(L)$ is maximal and then $\varepsilon(L) = \sqrt{L^2}$ or it is submaximal and Lemma 1.7 applies. $\square$

Now we pass to the notion of multiple point Seshadri constants which brings us closer to the Nagata Conjecture. Let X be a smooth projective variety, L a nef line bundle on X and $x_1, \ldots, x_r$ distinct points in X. Then the *r-tuple Seshadri constant of L at $x_1, \ldots, x_r$* is the number

$$(8) \qquad \varepsilon(L; x_1, \ldots, x_r) = \inf_{C \cap \{x_1, \ldots, x_r\} \neq \emptyset} \frac{L.C}{\sum \text{mult}_{x_i} C} \;,$$

where the infimum is taken over all (irreducible) curves passing through at least one of the points $x_1, \ldots, x_r$. Of course the obvious counterparts of definitions (4) and (6) provide the equivalent way to define the multiple point Seshadri constant.

If L is ample, then we have the following inequality relating, in particular, the multipoint Seshadri constant to the constants at the individual points

$$(9) \qquad \frac{1}{\sum_{i=1}^{r} \frac{1}{\varepsilon(L, x_i)}} \leq \varepsilon(L; x_1, \ldots, x_r) \leq \sqrt[\dim X]{\frac{L^{\dim X}}{r}}.$$

Indeed, given a curve C we may arrange the points so that $x_1, \ldots, x_s \in C$ and $x_{s+1}, \ldots, x_r \notin C$ with $s \geq 1$. By definition (2) we have then $\dfrac{L.C}{\text{mult}_{x_i} C} \geq \varepsilon(L, x_i)$ for $i = 1, \ldots, s$. Summing up the reciprocities we get

$$\frac{\sum_{i=1}^{s} \text{mult}_{x_i} C}{L.C} \leq \sum_{i=1}^{s} \frac{1}{\varepsilon(L, x_i)}.$$

Now we can replace s by r on both sides since this makes only the number on the right hand side possibly bigger. Taking again the reciprocities we obtain (9).

2. Around the Nagata Conjecture. The problem of the global generation of line bundles of the form (1) was already investigated by Ballico and

Coppens [1], however their cohomological conditions seem difficult to verify in general. In Theorem 2.10 we provide a new effective criterion for our Problem B.

Problems B and C can be viewed together in the general framework of understanding positivity of a line bundle as its ability to separate 0-dimensional subschemes of given length. We state a result on Problem B in Proposition 2.10.

Finally, in Theorem 2.9 we address the ampleness of L. In this direction Biran [6, Corollary 2.1.B] generalized the result of Küchle and Xu to the case $k_1 = \cdots = k_r = 2$ and proved that L is ample provided again $L^2 > 0$, $d \geq 6$, and the points are very general. The case of homogeneous multiplicity $k_1 = \cdots = k_r = 3$ was solved by Tutaj-Gasińska [38]. She shows that also in this case L is ample if it satisfies the necessary requirement $L^2 > 0$ and if d is sufficiently big, here $d \geq 10$.

Now we pass to the Nagata Conjecture which, as already remarked, can be viewed as a special case either of Problem A or D. The Conjecture gives a necessary lower bound on the degree d assuming that the linear system $|L|$ is non-empty.

Nagata Conjecture. *Let $P_1, \ldots, P_r$ be $r \geq 10$ very general points in $\mathbb{P}^2$ and let $k_1, \ldots, k_r$ be fixed non-negative integers. If $C \subset \mathbb{P}^2$ is a curve of degree d such that $\mathrm{mult}_{P_i} C \geq k_i$, then*

$$d > \frac{1}{\sqrt{r}} \sum_{i=1}^{r} k_i.$$

Amazingly the Conjecture still escapes any solution apart from the case settled by Nagata himself in which the number of blown up points r is a square. If r is not a square, then the strong inequality in the above conjecture (which was essential for the Hilbert's problem) is, of course, equivalent to the weak one. However, if r is a square, then it would be interesting to know whether one can relax the assumption that the points are very general at the price of allowing the weak inequality.

Nagata Conjecture, weak inequality. *Let $P_1, \ldots, P_r$ be $r \geq 9$ general points in $\mathbb{P}^2$ and let $k_1, \ldots, k_r$ be fixed non-negative integers. If $C \subset \mathbb{P}^2$ is a curve of degree d such that $\mathrm{mult}_{P_i} C \geq k_i$, then*

$$d \geq \frac{1}{\sqrt{r}} \sum_{i=1}^{r} k_i.$$

Apparently this is not known, even if r is a square (see also the recent work of Harbourne [18]).

For the purpose of this paper it is convenient to use the duality between the cone of effective curves and the cone of ample divisors and formulate the Nagata Conjecture in the language of Seshadri constants (see [41], [2]). For a polarized variety (X, L) and general points $P_1, \ldots, P_r \in X$ we use the short-hand notation $\varepsilon(L; r) := \varepsilon(L; P_1, \ldots, P_r)$.

Nagata Conjecture and Seshadri constants. *For $r \geq 9$ we have*

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); r) = \frac{1}{\sqrt{r}},$$

i.e. the Seshadri constant $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); r)$ attains its maximal possible value.

It might be worth to point out here that in the Seshadri constants formulation the case when the number of points r is a square $r = s^2$ and the points are very general can be easily proved as follows. Let C be a smooth curve of degree s and let $P_1, \ldots, P_{s^2}$ be arbitrary points on C. Since C is in irreducible we check that the Seshadri constant $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); P_1, \ldots, P_{s^2}) = \frac{1}{s}$. In fact, if it were smaller, say equal $\varepsilon < \frac{1}{s}$, then there would be a curve D passing through $P_1, \ldots, P_{s^2}$ such that its proper transform on the blowup would spoil the nefness of $H - \frac{1}{s} \sum_{i=1}^{s^2} E_i$. But on $\mathbb{P}^2$ this would imply that D has a negative intersection with C which is absurd. This shows that the Seshadri constant is maximal for the special choice of points $P_1, \ldots P_{s^2}$ and then by the semi-continuity result in the spirit of [29] we conclude that it must be maximal for s^2 very general points.

Looking at the Conjecture from this point of view it becomes apparent that there is no need to restrict attention to $\mathbb{P}^2$, not even to stick to dimension two. However in the case of algebraic surfaces recent results of Biran [7, Theorem 1] on symplectic packings suggest a challenging effective version of the conjecture. Note, that the connection between Seshadri constants and symplectic packings was observed already by McDuff and Polterovich [26] and exploited by Xu [39] and Lazarsfeld [25]. We refer to [6, Section 7] for a nice overview.

Conjecture 2.1 (Biran-Nagata)**.** *Let (X, L) be a polarized surface. Let k_0 be an integer such that the linear system $|k_0 L|$ contains a smooth non-rational*

curve and let $r_0 = k_0^2 L^2$. Then for $r \geq r_0$ the Seshadri constant

$$\varepsilon(L; r) = \sqrt{\frac{L^2}{r}}$$

is maximal.

The assumptions on the number r_0 cannot be weakened in general. This is easily seen for the hyperplane bundle on the projective plane (cf. Example 2.4). Another example of this kind is provided by (X, Θ) being a principally polarized abelian surface. Then there are no rational curves on X; hence $k_0 = 1$, and by the above formula $r_0 = \Theta^2 = 2$. The conjecture does not hold for $r = 1$ as observed by Steffens [33].

We want to present yet another formulation of the Nagata Conjecture which brings into the discussion the bigness of considered divisors. Let, as usual, $\pi : Y \longrightarrow X$ be the blow up of X at points $P_1, \ldots, P_r$ with exceptional divisors $E_1, \ldots E_r$. Let $H = \pi^* L$ and $E = \sum_{i=1}^{r} E_i$. We consider the ray $H - \lambda E$ in the Néron-Severi space $N_{\mathbb{R}}^1(Y)$. Then the Seshadri constant $\varepsilon(L; P_1, \ldots, P_r)$ is this positive value of λ for which $H - \lambda E$ is merely nef but not ample, i.e. for which the ray hits the boundary of the nef cone. One can also consider the following number

$$\tau = \tau(L; P_1, \ldots, P_r) := \sup\{\lambda > 0 : \quad H - \lambda E \text{ is big}\},$$

i.e. the value of λ for which the ray hits the boundary of the pseudo-effective cone of Y. If $\rho(X) = 1$ and $r = 1$ then Corollary 1.5 implies that $\varepsilon(L; P) \cdot \tau(L; P) = \sqrt{L^2}$, i.e. either the Seshadri constant of L at P is maximal and in this case $H - \varepsilon(L; P)E$ is nef but not big or it is submaximal, in which case $H - \varepsilon(L; P)E$ is not nef but big. This remains true in the multiple point framework. Keeping the notation we have

Biran-Nagata Conjecture and bigness. *For any $r \geq r_0$ the $\mathbb{R}$-divisor*

$$H - \sqrt{\frac{H^2}{r}} \cdot E$$

is not big.

No matter which way stated the Nagata Conjecture seems out of reach at the moment. However it seems reasonable to ask for lower bounds on Seshadri constants $\varepsilon(L; r)$. In this direction Xu [40, Lemma] proved the following result which we shall use repeatedly.

Lemma 2.2 (Xu)**.** *Let $P_1, \ldots, P_s$ be $s \geq 9$ general points in $\mathbb{P}^2$ and let C be a reduced and irreducible curve of degree p passing through the points P_i with the multiplicity $\mathrm{mult}_{P_i} C = m_i$, for $i = 1, \ldots, s$. Then*

$$p^2 \geq \sum_{i=1}^{s} m_i^2 - m_j$$

for arbitrary $j \in \{1, \ldots, s\}$ with $m_j > 0$.

Using the above lemma Xu gave in [39, Theorem 1(a)] the following bound:

$$(10) \qquad \varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); r) \geq \frac{\sqrt{r-1}}{r} = \frac{1}{\sqrt{r}}\sqrt{1 - \frac{1}{r}} \quad \text{for } r \geq 10.$$

It seems that this was the first general bound obtained in this direction. Combining Lemma 2.2 with Reider Theorem we improved the above bound in [37] and showed that

$$(11) \qquad \varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); r) \geq \frac{1}{\sqrt{r+1}} = \frac{1}{\sqrt{r}}\sqrt{1 - \frac{1}{r+1}} \quad \text{for } r \geq 10.$$

Using results on multiple point Seshadri constants proved in Proposition 2.7 and Corollary 2.8 we approximate further the value of $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); r)$ conjectured by Nagata and show the following improvement of (10) and (11).

Theorem 2.3. *For $r \geq 10$ we have*

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); r) \geq \frac{\sqrt{49r+8}}{7r+1} > \frac{1}{\sqrt{r}}\sqrt{1 - \frac{1}{8r}}.$$

This inequality is of interest even if r is a square as we are concerned here with general points.

In the opposite direction the range of r between 1 and 9 is discussed in the following example. As a consequence we show that the Harbourne-Hirschowitz Conjecture implies the Nagata Conjecture. This was already observed by Ciliberto [9, Remark 5.12].

Example 2.4. In the following table we summarize the values of r-tuple Seshadri constants on $\mathbb{P}^2$ and describe curves which compute it.

r	$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); r)$	curve	number of curves
1	1	line through P_1	a pencil
2	$\frac{1}{2}$	line through P_1, P_2	1
3	$\frac{1}{2}$	line through P_i, P_j, $i \neq j$	3
4	$\frac{1}{2}$	line through P_i, P_j, $i \neq j$	6
		or conic through P_1, P_2, P_3, P_4	a pencil
5	$\frac{2}{5}$	conic through $P_1, \ldots, P_5$	1
6	$\frac{2}{5}$	conic through any 5 points	6
7	$\frac{3}{8}$	cubics through $P_1, \ldots, P_7$ with node at P_i	7
8	$\frac{6}{17}$	sextic with double points at $P_1, \ldots, P_7$	8
		and a triple point at P_8	
9	$\frac{1}{3}$	cubic through $P_1, \ldots, P_9$	1

Remark 2.5. Note that the existence of the curves computing Seshadri constant in the above cases follows in a straightforward manner from the Riemann-Roch Theorem. It is easy to check that for $r \geq 10$ Riemann-Roch never produces submaximal curves. However Syzdek [34] shows that the Riemann-Roch kind of argument need not always to produce submaximal curves in the whole range $1 \leq r < N_0$ of Conjecture 2.1.

Remark 2.6. Now we explain how Harbourne-Hirschowitz implies the Nagata Conjecture. Suppose that the Conjecture fails for some $r \geq 10$, i.e. there exists an irreducible and reduced curve C of degree d with multiplicities $m_1, \ldots, m_r$ at general points $P_1, \ldots, P_r$ and such that $d < \dfrac{1}{\sqrt{r}} \cdot \sum\limits_{i=1}^{r} m_i$. By the monodromy argument there are curves $C_\sigma \cong C$ with multiplicities $m_{\sigma(1)}, \ldots m_{\sigma(r)}$ for any permutation $\sigma \in \Sigma_r$. The union of these curves is a divisor D with

homogeneous multiplicities, say m, which also spoils the Nagata Conjecture, i.e.

$$(12) \qquad\qquad\qquad\qquad D^2 < rm^2.$$

For the proper transform $\widetilde{C}$ of C the Harbourne-Hirschowitz Conjecture implies that $\widetilde{C}^2 \geq p_a(C) - 1$ and $\widetilde{C}^2 = -1$ if and only if $\widetilde{C}$ is rational. It $\widetilde{C}$ were a (-1)-curve, then D would be a homogeneous (-1)-configuration. But all such configurations consists only of curves listed in Example 2.4. Hence it must be $\widetilde{C}^2 \geq 0$ and consequently also $\widetilde{D}^2$ which contradicts (12).

Note also that the above example shows that Corollary 1.5 fails for multiple point Seshadri constants. However the last column of the table suggests that on surfaces the number of curves computing the multiple point Seshadri constant in the submaximal case could be bounded by the number of points r. The following proposition takes this and the Picard number of the surface into account. This proposition has its ancestors in [35] (Propositions 1.8 and 4.5) and in [34]. Examples discussed above show that the formulation is optimal. The proof is much simpler than in the cited papers.

Proposition 2.7. *Let (Y, L) be a polarized surface with Picard number ρ and let $P_1, \ldots, P_r$ be points in Y such that $\varepsilon := \varepsilon(L; P_1, \ldots, P_r)$ is submaximal. Then there are at most $\rho + r - 1$ irreducible and reduced curves computing ε.*

P r o o f. Let $\pi : X \longrightarrow Y$ be the blowing up of Y at $P_1, \ldots, P_r$ with exceptional divisors $E_1, \ldots, E_r$ and let $H := \pi^* L$. Suppose $C_1, \ldots, C_s$ are irreducible and reduced curves computing ε, $\widetilde{C}_1, \ldots, \widetilde{C}_r$ are their proper transforms. The $\mathbb{Q}$-divisor $M := H - \varepsilon \sum_{i=1}^{r} E_i$ is nef and big and we have $M. \left(\sum_{i=1}^{s} \lambda_i \widetilde{C}_i \right) = 0$ for arbitrary $\lambda_i \geq 0$. The Hodge Index Theorem implies that the intersection matrix of $\widetilde{C}_1, \ldots, \widetilde{C}_s$ is negative definite. Since $\rho(X) = \rho + r$ it implies the assertion $s \leq \rho + r - 1$. $\square$

It would be interesting to know whether for a surface Y with the Picard number ρ and $\varepsilon(L; P_1, \ldots, P_r) = \sqrt{\dfrac{L^2}{r}}$ maximal the existence of more than $r + \rho - 1$ curves computing the Seshadri constant implies that there are infinitely many such curves.

Proposition 2.7 has an interesting consequence in the case $P_1, \ldots, P_r$ are general points. We say that an r-tuple $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ is *almost-homogeneous* if all but at most one of the coordinates are equal. We say that a curve C is almost-homogeneous at $P_1, \ldots, P_r$ if the r-tuple $(\operatorname{mult}_{P_1} C, \ldots, \operatorname{mult}_{P_r} C)$ is almost-homogeneous.

Corollary 2.8. *Let (X, L) be a polarized surface with Picard number $\rho(X) = 1$ and let $P_1, \ldots, P_r$ be general points on X. Then any irreducible L-submaximal curve C is almost-homogeneous.*

P r o o f. Since the points are general the monodromy group acts as the full symmetric group S_r, i.e. if there is an irreducible curve C with multiplicities $(\text{mult}_{P_1} C, \ldots, \text{mult}_{P_r} C)$, then there exists an irreducible curve C_σ with multiplicities $\text{mult}_{P_i} C_\sigma = \text{mult}_{P_{\sigma(i)}} C$ for $i = 1, \ldots, r$ and $\sigma \in S_r$. The only possibility that there are at most r curves in the set $\{C_\sigma\}_{\sigma \in S_r}$ is that they are almost-homogeneous. $\square$

This implies in turn, that it suffices to rule out the existence of almost-homogeneous submaximal curves in order to prove the Nagata Conjecture. Before we show our result approximating the Nagata Conjecture we address first Problem D of the introduction and show the following

Proposition 2.9. *Let $\pi : X \longrightarrow \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ in r general points and let $k \geq 2$ and d be integers such that $d \geq 3k + 1$. If $r \leq \dfrac{d^2}{k^2} - 1$, then the line bundle $L = dH - k \sum\limits_{i=1}^{r} E_i$ is ample.*

P r o o f. We apply the Nakai-Moishezon criterion. First of all we have

$$L^2 = d^2 - rk^2 \geq k^2 > 0.$$

Next we claim that $L.C > 0$ for all irreducible and reduced curves $C \subset X$. This is obvious if C is one of the exceptional curves so we may assume that $C = \pi^* F - \sum\limits_{i=1}^{r} m_i E_i$, where F is a reduced and irreducible plane curve of degree p. Furthermore, without loss of generality we may assume that $m_1 \geq \cdots \geq m_s \geq 1$ and $m_{s+1} = \cdots = m_r = 0$. Computing $L.C$ we see that our claim is equivalent to

$$(13) \qquad\qquad dp > k \sum_{i=1}^{s} m_i.$$

Since $\dfrac{d}{k} \geq \sqrt{r+1} \geq \sqrt{s+1}$ and $\left(\sum\limits_{i=1}^{s} m_i \right)^2 \leq s \sum\limits_{i=1}^{s} m_i^2$ it suffices to show that

$$s \sum_{i=1}^{s} m_i^2 < p^2(s+1).$$

From Lemma 2.2 we have

(14)
$$\sum_{i=1}^{s} m_i^2 \le p^2 + m_s$$

so that we are done if

(15)
$$sm_s < p^2$$

holds. In fact (14) implies (15) unless $m_1 = \cdots = m_s = 1$. In the latter case in (13) we have to show

$$p\frac{d}{k} > s.$$

If $s \le 5$, then, using the assumption $d \ge 3k + 1$, it is enough to show $3p \ge s$, which follows by the assumption that the points $P_1, \ldots, P_r$ are general. So we assume that $s \ge 6$. Since $\frac{d}{k} \ge \sqrt{s+1}$, it is enough to prove that

$$p > \frac{s}{\sqrt{s+1}}.$$

From the generality assumption again we have $\binom{p+2}{2} - s \ge 1$ or equivalently $p \ge \frac{\sqrt{8s+9} - 3}{2}$. Now the claim follows from the simple observation that the real valued function

$$f(s) = \frac{\sqrt{8s+9} - 3}{2} - \frac{s}{\sqrt{s+1}}$$

is positive for $s \ge 6$. $\square$

Using the above result we pass to Problem B and prove a criterion for the global generation of a line bundle of the form $L = dH - k \sum_{i=1}^{r} E_i$. In the case of fat points, i.e. $k \ge 2$ this result seems to be new. Note that the bound (11) is its consequence as explained in [37].

Proposition 2.10. *Let $\pi : X \longrightarrow \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ in r general points and let $k \ge 2$ and d be integers such that $d \ge 3k + 1$. If $r \le \dfrac{(d+3)^2}{(k+1)^2} - 1$, then the line bundle $L = dH - k \sum_{i=1}^{r} E_i$ is globally generated.*

P r o o f. Let $N = L - K_X = (d+3)H - (k+1) \sum_{i=1}^{r} E_i$. First of all we have

$$N^2 = (d+3)^2 - r(k+1)^2 \ge (k+1)^2 > 5.$$

It follows directly from Proposition 2.9 that N is ample. Thus N satisfies assumptions of Reider Theorem [30]. If $L = K_X + N$ fails to be globally generated, then there exists a curve $D \subset X$ such that $N.D = 1$ and $D^2 = 0$. Let $p, m_1, \ldots, m_r$ be integers such that $D \equiv pH - \sum_{i=1}^{r} m_i E_i$. Computing $N.D$ we get $(d+3)p = (k+1) \sum_{i=1}^{r} m_i + 1$, which implies $\sum_{i=1}^{r} m_i > 0$, so that in fact it must be

$$(16) \qquad \sum_{i=1}^{r} m_i \geq 1.$$

Taking into account $\dfrac{d+3}{k+1} \geq \sqrt{r+1}$ we have

$$p\sqrt{r+1} \leq \sum_{i=1}^{r} m_i + \frac{1}{k+1}.$$

Since $D^2 = 0$ we have $p^2 = \sum_{i=1}^{r} m_i^2$, hence

$$(r+1) \sum_{i=1}^{r} m_i^2 \leq \left(\sum_{i=1}^{r} m_i + \frac{1}{k+1} \right)^2.$$

Now, the right hand side is bounded by $r \sum_{i=1}^{r} m_i^2 + \dfrac{2}{k+1} \sum_{i=1}^{r} m_i + \dfrac{1}{(k+1)^2}$ so that

$$(k+1)^2 \sum_{i=1}^{r} m_i^2 \leq 2(k+1) \sum_{i=1}^{r} + 1.$$

Since $\sum_{i=1}^{r} m_i^2 \geq \sum_{i=1}^{r} m_i$ we get $(k^2 - 1) \sum_{i=1}^{r} m_i \leq 1$ which in view of $k \geq 2$ and (16) gives a contradiction. $\square$

Finally we prove our bound on multiple point Seshadri constants on $\mathbb{P}^2$.

Proof of Theorem 2.3. Let $P_1, \ldots, P_r$ be $r \geq 10$ general points in $\mathbb{P}^2$. To begin with we assume that the Nagata Conjecture is false, i.e. there exists an $\mathcal{O}_{\mathbb{P}^2}(1)$-submaximal irreducible curve $C \subset \mathbb{P}^2$ of degree p with multiplicities $m_1, \ldots, m_r$ at $P_1, \ldots, P_r$ respectively with

$$(17) \qquad \frac{p}{\sum_{i=1}^{r} m_i} < \frac{1}{\sqrt{r}}$$

Corollary 2.8 implies in particular that C is almost homogeneous, i.e. there are integers m and n such that up to renumbering of the points, we have $m = m_1 = \cdots = m_{r-1}$ and $n = m_r$. Now, we claim that

$$
\frac{p}{(r-1)m+n} \geq
\begin{cases}
\dfrac{1}{\sqrt{r}}\sqrt{1 - \dfrac{1}{21r}} & \text{if } \quad m = n \\[3mm]
\dfrac{1}{\sqrt{r}}\sqrt{1 - \dfrac{5r+1}{(7r-1)^2}} & \text{if } \quad m > n \\[3mm]
\dfrac{1}{\sqrt{r}}\sqrt{1 - \dfrac{6r+1}{(7r+1)^2}} & \text{if } \quad m < n
\end{cases}
$$

Taking this for granted the assertion of the Theorem follows from the elementary observation that the last condition is the weakest for all $r \geq 10$.

Thus it remains to prove the above inequalities.

Homogeneous case $m = n$.
It is an easy corollary from [9, Remark 5.7] that in this case there are no submaximal curves for $m \leq 20$. So we can assume $m \geq 21$ which together with Lemma 2.2 gives

$$
\frac{p}{rm} \geq \sqrt{\frac{1}{r} - \frac{1}{r^2 m}} \geq \sqrt{\frac{1}{r} - \frac{1}{21 r^2}}.
$$

Inhomogeneous case a) $m \geq n + 1$.
In this case by Lemma 2.2 we have

$$
p^2 \geq (r-1)m^2 + n^2 - n,
$$

which is equivalent to

$$
\frac{p^2}{((r-1)m+n)^2} \geq \frac{(r-1)m^2 + n^2 - n}{((r-1)m+n)^2}.
$$

Now, it's easy to check that the function on the right is decreasing with n increasing in the range $0 \leq n \leq m - 1$. Hence, we can assume that $n = m - 1$, which gives

$$
\frac{p^2}{((r-1)m+n)^2} \geq \frac{rm^2 - 3m + 2}{(rm-1)^2}.
$$

The function on the right grows for $m \geq 4$ and since all special almost-homogeneous linear series for $m \leq 6$ are (-1)-special [23, Theorem A] repeating the

argument of Remark 2.6 we can assume that $m \geq 7$, so that

$$\frac{p}{(r-1)m+n} \geq \frac{\sqrt{49r-19}}{7r-1}.$$

Inhomogeneous case b) $m + 1 \leq n$.

Similarly as in the previous case we have

$$\frac{p^2}{((r-1)m+n)^2} \geq \frac{(r-1)m^2+n^2-m}{((r-1)m+n)^2}.$$

In this case the function on the right grows for $n \geq m+1$, hence we can assume $n = m+1$ which gives

$$\frac{p^2}{((r-1)m+n)^2} \geq \frac{rm^2+m+1}{(rm+1)^2}.$$

The function on the right grows for $m \geq 2$ so we can assume as in the case a) that $m \geq 7$ which gives

$$\frac{p}{(r-1)m+n} \geq \frac{\sqrt{4r+3}}{2r+1}. \qquad \qquad \square$$

Remark 2.11. Note that Lemma 2.2 actually holds for arbitrary polarized surfaces (X, L) so that one can easily obtain bounds along the lines of the above theorem valid for arbitrary surfaces. We restrict to the planar case for the sake of the simplicity both in the statement and in the proof. For the same reason we were not struggling for the optimal statement emphasizing rather methods than the particular bound.

Recently, for a very ample line bundle L, Harbourne provided bounds of the form $\sqrt{\dfrac{L^2}{r}} \cdot \sqrt{1 - \dfrac{1}{ar}}$, where a is a parameter depending on the degree of L and the number of points [18, Theorem 1.1]. Precise statement of his result is quite technical, so we postpone it here. Let us mention however that though his parameters a cannot be computed uniformly, they exhibit an optimal asymptotical behaviour.

Finally let us point out that there exists a whole series of bounds, specifically on $\mathbb{P}^2$, [31], [32], [19], which in many concrete cases are the best available at the present. Again, they involve a lot of technical assumptions which can never be checked in the general situation.

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