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DIRECT AND INVERSE SPECTRAL PROBLEMS FOR $(2N+1)$-DIAGONAL, COMPLEX, SYMMETRIC, NON-HERMITIAN MATRICES

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Abstract. We consider the following linear $(2N)$-th order difference equation: $\sum_{i=1}^{N}(\alpha_{k-i,i}y_{k-i} + \alpha_{k,i}y_{k+i}) + \alpha_{k,0}y_k = \lambda^N y_k$, $k = 0,1,2,\ldots$, where $\alpha_{i,j} \in \mathbb{C} : \alpha_{i,N} \neq 0$, λ is a complex parameter, $(y_0, y_1, \ldots, y_k, \ldots) = \vec{y}^T$ is a vector solution, N is a fixed integer. It can be written in the following matrix form: $J\vec{y} = \lambda^N \vec{y}$, where J is a $(2N+1)$-diagonal, symmetric matrix. We give an easy procedure for solving of the direct and the inverse spectral problems for the equation. Guseynov used a procedure of the Gelfand-Levitan type for the case $N = 1$. We use another procedure and this procedure is more easy and transparent.

1. Introduction. Let us consider the following linear difference equation of the second order:

$$b_0y_0 + a_0y_1 = \lambda y_0,$$

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$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n = 1, 2, \ldots, \tag{1}$$

where $a_n, b_n \in \mathbb{C}, a_n \neq 0$, λ is a complex parameter, $(y_0, y_1, \ldots)$ is a vector solution.

Such an equation was considered by Guseynov in [1]. He gave a procedure for construction of a spectral function. He gave necessary and sufficient conditions for a functional σ to be the spectral function. He showed how to reconstruct the difference equation using σ and a sequence of signs.

Our aim here is twofold. First, we shall give a little another definition of the spectral function and an easy procedure of reconstruction of difference equation (1). Second, we shall extend our procedure for the following linear difference equation of the $(2N)$-th order:

$$\sum_{i=1}^{N} (\alpha_{k-i,i} y_{k-i} + \alpha_{k,i} y_{k+i}) + \alpha_{k,0} y_k = \lambda^N y_k,\ k = 0, 1, 2, \ldots, \tag{2}$$

where $\alpha_{i,j} \in \mathbb{C} : \alpha_{i,N} \neq 0$, λ is a complex parameter, $(y_0, y_1, \ldots, y_k, \ldots) = \vec{y}^T$ is a vector solution, N is a fixed integer, and all $\alpha_{i,j}$ and p_k with negative indexes here are equal to zero.

Equation (2) can be written in the following matrix form:

$$J\vec{y}(\lambda) = \lambda^N \vec{y}(\lambda), \tag{3}$$

where J is a $(2N+1)$-diagonal, symmetric, non-Hermitian matrix:

$$J = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \ldots & \alpha_{0,N} & 0 & 0 & \ldots \\ \alpha_{0,1} & \alpha_{1,0} & \alpha_{1,1} & \ldots & \alpha_{1,N-1} & \alpha_{1,N} & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots \\ \alpha_{0,N} & \alpha_{1,N-1} & \alpha_{2,N-2} & \ldots & \cdot & \cdot & \cdot & \ldots \\ 0 & \alpha_{1,N} & \alpha_{2,N-1} & \ldots & \cdot & \cdot & \cdot & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4}$$

Notice that the case of equation (3) with Hermitian $(2N+1)$-diagonal matrix J is closely related to the orthogonal polynomials on radial rays in the complex plane. The measure of orthogonality plays a role of a spectral function. For the history of this type equations and recent results see [2] and references therein (for the case of five-diagonal matrices see [3, 4]). Symmetric, non-Hermitian matrices have a much more complicated structure (see [5]) and we can hardly expect for integral representations of their spectral functions.

In Section 3 we shall give necessary and sufficient conditions for a linear functional σ to be a spectral function of (2). If we know a sequence of signs, we can reconstruct difference equation (2) from its spectral function. The procedure of reconstruction is easy and transparent.

2. Spectral problems for three-diagonal matrices. Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ be a solution of (1) with the initial condition $p_0(\lambda) = 1$. It is not hard to see that $p_n(\lambda)$ is a polynomial of degree n. Let us define a linear with respect to the both arguments functional $\sigma(u, v)$, $u, v \in \mathbb{P}$, where $\mathbb{P}$ is the space of all polynomials. We define it in the following way:

$$\sigma(p_n(\lambda), p_m(\lambda)) = \delta_{n,m}, \ n, m = 0, 1, 2, \dots. \tag{5}$$

Using linearity we extend the definition of σ for arbitrary $u, v \in \mathbb{P}$. We notice that from linearity it follows that

$$\sigma(u, v) = \sigma(v, u), \ u, v \in \mathbb{P}. \tag{6}$$

Using recurrence relation (1) we write:

$$\sigma(\lambda p_n(\lambda), p_m(\lambda)) = \sigma(a_{n-1}p_{n-1} + b_n p_n + a_n p_{n+1}, p_m) =$$

$$= a_{n-1}\delta_{n-1,m} + b_n\delta_{n,m} + a_n\delta_{n+1,m}; \tag{7}$$

$$\sigma(p_n(\lambda), \lambda p_m(\lambda)) = \sigma(p_n, a_{m-1}p_{m-1} + b_m p_m + a_m p_{m+1}) =$$

$$= a_{m-1}\delta_{n,m-1} + b_m\delta_{n,m} + a_m\delta_{n,m+1}, \ n, m = 0, 1, 2, \dots. \tag{8}$$

Comparing (7) with (8) for $n = m - 1; m; m + 1$, we see that the expressions coincide. But for $n < m - 1$ or $n > m + 1$ we obtain zero in equations (7) and (8). So, we get

$$\sigma(\lambda p_n(\lambda), p_m(\lambda)) = \sigma(p_n(\lambda), \lambda p_m(\lambda)), \ n, m = 0, 1, 2, \dots. \tag{9}$$

Using linearity, from (9) we derive

$$\sigma(\lambda u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda v(\lambda)), \ u, v \in \mathbb{P}. \tag{10}$$

Conversely, let a sequence of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$, $\deg p_n = n$, $p_0 = 1$, be given. Also let a linear functional $\sigma(u, v)$ which satisfies (5), (10) be given. Then we can derive (1). Really, since $\deg p_n = n$, one can write:

$$\lambda p_n(\lambda) = \sum_{k=0}^{n+1} a_{n,k} p_k(\lambda), \ a_{n,k} \in \mathbb{C} : a_{n,n+1} \neq 0, n = 0, 1, 2, \dots. \tag{11}$$

Using relations (5), (6), (10) we get:

$$\sigma(\lambda p_n, p_j) = a_{n,j},\ j = 0, 1, 2, \ldots, n+1;$$

$$\sigma(\lambda p_n, p_j) = \sigma(p_n, \lambda p_j) = \sigma(\lambda p_j, p_n) = \left\{ \begin{array}{c} 0,\ j+1 < n \\ a_{j,n},\ j+1 \geq n \end{array} \right. ,$$

and

$$a_{n,j} = \left\{ \begin{array}{c} 0,\ j+1 < n \\ a_{j,n},\ j+1 \geq n \end{array} \right. ,\ j = 0, 1, 2, \ldots, n+1. \tag{12}$$

Relation (11) can be written in the following form:

$$\lambda p_n(\lambda) = a_{n,n-1}p_{n-1}(\lambda) + a_{n,n}p_n(\lambda) + a_{n,n+1}p_{n+1}(\lambda),\ n = 0, 1, 2, \ldots. \tag{13}$$

We put by definition: $a_n = a_{n,n+1},\ b_n = a_{n,n}$. Using relation (12), from relation (13) we get (1).

Definition. *A linear with respect to the both arguments functional* $\sigma(u, v)$, $u, v \in \mathbb{P}$, *which satisfies* (5), *we shall call* **the spectral function** *of difference equation* (1).

The following theorem is a reformulation of [1, Theorem 2.1, p. 242], but the method of proof is quite another:

Theorem 1. *A linear with respect to the both arguments functional* $\sigma(u, v)$, $u, v \in \mathbb{P}$, *is the spectral function of a difference equation of type* (1) *iff:*

1) $\sigma(\lambda u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda v(\lambda)),\ u, v \in \mathbb{P}$;

2) $\sigma(1, 1) = 1$;

3) *For arbitrary polynomial* $u_k(\lambda)$ *of degree* k, *there exists a polynomial* $\hat{u}_k(\lambda)$ *of degree* k *such that:*

$$\sigma(u_k(\lambda), \hat{u}_k(\lambda)) \neq 0.$$

P r o o f. Necessity. Let $\sigma(u, v)$ be a spectral function of a difference equation (1). Relation 1) can be derived in the same manner as relation (10) was derived. Relation 2) is obvious from (5) since $p_0 = 1$. For a polynomial $u_k(\lambda), \deg u_k = k$, we can write: $u_k(\lambda) = \sum_{j=0}^{k} \xi_j p_j(\lambda),\ \xi_j \in \mathbb{C} : \xi_k \neq 0$. Put $\hat{u}_k(\lambda) = \sum_{j=0}^{k} \overline{\xi_j} p_j(\lambda)$. Using orthonormality relation (5) we get

$$\sigma(u_k(\lambda), \hat{u}_k(\lambda)) = \sum_{j=0}^{k} |\xi_j|^2 > 0,$$

and relation 3) is verified.

Sufficiency. Let $\sigma(u,v)$ be a linear functional satisfying relations 1),2),3). Let us construct a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$. Put $p_0 = 1$. Let $R_1(\lambda)$ be arbitrary polynomial of degree 1. Then we can write: $R_1(\lambda) = \mu_1\lambda + \alpha_{1,0}p_0(\lambda), \mu_1, \alpha_{1,0} \in \mathbb{C}, \mu_1 \neq 0$. Then

$$\sigma(R_1(\lambda), p_0(\lambda)) = \sigma(\mu_1\lambda + \alpha_{1,0}p_0(\lambda), p_0(\lambda)) = \mu_1\sigma(\lambda, p_0) + \alpha_{1,0}.$$

If we take $\alpha_{1,0} = -\mu_1\sigma(\lambda, p_0)$ then

$$R_1(\lambda) = \mu_1(\lambda - \sigma(\lambda, p_0)p_0(\lambda)),$$

$$\sigma(R_1(\lambda), p_0(\lambda)) = 0, \tag{14}$$

From relation 3) it follows that there exists a polynomial $\hat{R}_1(\lambda), \deg \hat{R}_1 = 1$, such that

$$M_1 := \sigma(R_1(\lambda), \hat{R}_1(\lambda)) \neq 0.$$

For the polynomial $\hat{R}_1(\lambda)$ we can write: $\hat{R}_1(\lambda) = \nu_1 R_1(\lambda) + \beta_{1,0}p_0(\lambda), \nu_1, \beta_{1,0} \in \mathbb{C}, \nu_1 \neq 0$. Then

$$M_1 = \sigma(R_1(\lambda), \hat{R}_1(\lambda)) = \sigma(R_1(\lambda), \nu_1 R_1(\lambda) + \beta_{1,0}p_0(\lambda)) = \nu_1\sigma(R_1(\lambda), R_1(\lambda));$$

$$\frac{\nu_1}{M_1}\sigma(R_1(\lambda), R_1(\lambda)) = 1.$$

Put by definition

$$\tilde{\mu}_{1,1} = \sqrt{\frac{\nu_1}{M_1}},$$

where we take arbitrary branch of the square root. Put

$$p_1(\lambda) = \tilde{\mu}_{1,1}R_1(\lambda).$$

Then $\deg p_1 = 1$ and

$$\sigma(p_i, p_j) = \delta_{i,j},\ i,j = 0,1.$$

Suppose that we have constructed such polynomials $p_0(\lambda), p_1(\lambda), \ldots, p_{n-1}(\lambda)$; $\deg p_i = i,\ n \geq 2$, that

$$\sigma(p_i, p_j) = \delta_{i,j},\ i,j = 0,1,\ldots,n-1. \tag{15}$$

Consider arbitrary polynomial $R_n(\lambda)$ of degree n. We can write: $R_n(\lambda) = \mu_n\lambda^n + \sum_{i=0}^{n-1} \alpha_{n,i}p_i(\lambda)$, $\mu_n, \alpha_{n,i} \in \mathbb{C}, \mu_n \neq 0$. Then

$$\sigma(R_n(\lambda), p_j(\lambda)) = \sigma(\mu_n\lambda^n + \sum_{i=0}^{n-1} \alpha_{n,i}p_i(\lambda), p_j(\lambda)) = \mu_n\sigma(\lambda^n, p_j(\lambda)) + \alpha_{n,j},$$

$$j = 0, 1, \ldots, n-1.$$

Let us take $\alpha_{n,j} = -\mu_n\sigma(\lambda^n, p_j(\lambda)), j = 0, 1, 2, \ldots, n-1$. Then

(16) $$R_n(\lambda) = \mu_n(\lambda^n - \sum_{i=0}^{n-1} \sigma(t^n, p_i(t))p_i(\lambda)),$$

$$\sigma(R_n(\lambda), p_j(\lambda)) = 0,\ j = 0, 1, \ldots, n-1.$$

As it follows from relation 3), there exists a polynomial $\hat{R}_n(\lambda)$, $\deg \hat{R}_n = n$, such that

$$M_n := \sigma(R_n(\lambda), \hat{R}_n(\lambda)) \neq 0.$$

For the polynomial $\hat{R}_n(\lambda)$ we can write the following representation: $\hat{R}_n(\lambda) = \nu_n R_n(\lambda) + \sum_{i=0}^{n-1} \beta_{n,i}p_i(\lambda)$, $\nu_n, \beta_{n,i} \in \mathbb{C}$, $\nu_n \neq 0$. Then

$$M_n = \sigma(R_n(\lambda), \hat{R}_n(\lambda)) = \sigma(R_n(\lambda), \nu_n R_n(\lambda) + \sum_{i=0}^{n-1} \beta_{n,i}p_i(\lambda)) = \nu_n\sigma(R_n(\lambda), R_n(\lambda));$$

$$\frac{\nu_n}{M_n}\sigma(R_n(\lambda), R_n(\lambda)) = 1.$$

Put by definition

(17) $$\tilde{\mu}_{n,n} = \sqrt{\frac{\nu_n}{M_n}},$$

where we take arbitrary branch of the square root. We put

(18) $$p_n(\lambda) = \tilde{\mu}_{n,n}R_n(\lambda).$$

Then $\deg p_n = n$ and

$$\sigma(p_i, p_j) = \delta_{i,j},\ i, j = 0, 1, \ldots, n.$$

We continue this process and obtain a sequence of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$, $\deg p_n = n$, such that: $p_0(\lambda) = 1$ and $\sigma(p_i, p_j) = \delta_{i,j},\ i, j = 0, 1, 2, \ldots$. As it has been shown, if we have a linear functional $\sigma(u, v),\ u, v \in \mathbb{P}$, and a sequence of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$, $\deg p_n = n,\ p_0 = 1$, which satisfy (5), (10), we can derive (1). The functional σ will be a spectral function of this equation as it follows from the definition of the spectral function.

The proof is complete. $\square$

Let an equation of type (1) be given and using (5) we have constructed its spectral function $\sigma(u, v)$. How we can reconstruct our difference equation (1)? We must use the procedure of construction of $\{p_n(\lambda)\}_{n=0}^{\infty}$ in the proof of the last theorem. We put $p_0 = 1$ and then begin from relation (15) for $n = 1, 2, 3, \ldots$. During this procedure we must take $\mu_n = 1$ for $R_n(\lambda)$ in (16) and take $\tilde{\mu}_{n,n}$ in (17) such that $\operatorname{Arg} \tilde{\mu}_{n,n} \in [0, \pi)$ if $\operatorname{Arg} \mu_{n,n} = \operatorname{Arg} \dfrac{1}{a_n a_{n-1} \ldots a_0} \in [0, \pi)$, or $\operatorname{Arg} \tilde{\mu}_{n,n} \in [\pi, 2\pi)$ otherwise, where $\mu_{n,n}$ is the leading coefficient of p_n which is connected with (1). Really, let $\{P_n(\lambda)\}_{n=0}^{\infty}$ be a system of polynomials which is constructed following this procedure. Suppose that $P_i(\lambda) = p_i(\lambda),\ i = 0, 1, \ldots, n-1; n \geq 1$, i.e. the first $n-1$ constructed polynomials coincide with polynomials defined from (1). Then for $p_n(\lambda)$ we can write: $p_n(\lambda) = \mu_{n,n}\lambda^n + \sum\limits_{i=0}^{n-1} \tilde{\alpha}_{n,i} P_i(\lambda)$. Then

$$0 = \sigma(p_n(\lambda), p_i(\lambda)) = \sigma(p_n(\lambda), P_i(\lambda)) = \mu_{n,n}\sigma(\lambda^n, P_i(\lambda)) + \tilde{\alpha}_{n,i},\ i = 0, 1, \ldots, n-1.$$

We get $\tilde{\alpha}_{n,i} = -\mu_{n,n}\sigma(\lambda^n, P_i(\lambda))$ and

$$p_n(\lambda) = \mu_{n,n}(\lambda^n - \sum_{i=0}^{n-1} \sigma(\lambda^n, P_i(\lambda))P_i(\lambda)) = \mu_{n,n}R_n(\lambda).$$

Comparing this relation with (18) (keeping in mind that p_n in (18) we now denote by P_n and $\mu_n = 1$) we get

$$P_n(\lambda) = \frac{\tilde{\mu}_{n,n}}{\mu_{n,n}} p_n(\lambda).$$

Then

$$1 = \sigma(P_n(\lambda), P_n(\lambda)) = \left(\frac{\tilde{\mu}_{n,n}}{\mu_{n,n}}\right)^2 \sigma(p_n(\lambda), p_n(\lambda)) = \left(\frac{\tilde{\mu}_{n,n}}{\mu_{n,n}}\right)^2,$$

$$A_n := \tilde{\mu}_{n,n}^2 = \mu_{n,n}^2 \neq 0.$$

So, $\tilde{\mu}_{n,n}$ and $\mu_{n,n}$ are square roots of A_n and according to our choice of branches of these roots, they coincide. Hence $P_n = p_n$. Consequently, $P_i = p_i,\ i = 0, 1, 2, \ldots$. But the coefficients of equation (1) are uniquely defined by σ, $\{p_n\}_0^{\infty}$.

If we define a sequence of signs $\{\pm, \pm, \dots\}$, where "+" in the n-th place means that Arg $\frac{1}{a_n a_{n-1} \dots a_0} \in [0,\pi)$, and "-" means that Arg $\frac{1}{a_n a_{n-1} \dots a_0} \in [\pi, 2\pi)$, then the spectral function σ and this sequence will uniquely define difference equation (1).

3. Spectral problems for difference equations of the $(2N)$-th order. Let us consider a linear difference equation of the $(2N)$-th order (2). Let $\{p_n(\lambda)\}_{n=0}^\infty$ be a solution of (2) with initial conditions $p_i(\lambda) = \lambda^i$, $i = 0, 1, 2, \dots, N-1$. It is clear from recurrence relation (2) that $\deg p_n(\lambda) = n$. Define a linear with respect to the both arguments functional $\sigma(u,v), u,v \in \mathbb{P}$, from relation (5). Using linearity we extend it for arbitrary $u, v \in \mathbb{P}$. We notice that from linearity it follows that relation (6) is fulfilled. From recurrence relation (2) it follows that

$$\sigma(\lambda^N p_n(\lambda), p_m(\lambda)) = \sigma(\sum_{i=1}^N (\alpha_{n-i,i} p_{n-i} + \alpha_{n,i} p_{n+i}) + \alpha_{n,0} p_n, p_m) =$$

$$= \sum_{i=1}^N (\alpha_{n-i,i} \delta_{n-i,m} + \alpha_{n,i} \delta_{n+i,m}) + \alpha_{n,0} \delta_{n,m}; \tag{19}$$

$$\sigma(p_n(\lambda), \lambda^N p_m(\lambda)) = \sigma(p_n, \sum_{j=1}^N (\alpha_{m-j,j} p_{m-j} + \alpha_{m,j} p_{m+j}) + \alpha_{m,0} p_m) =$$

$$= \sum_{j=1}^N (\alpha_{m-j,j} \delta_{n,m-j} + \alpha_{m,j} \delta_{n,m+j}) + \alpha_{m,0} \delta_{n,m}, \ n, m = 0, 1, 2, \dots. \tag{20}$$

Comparing the right-hand sides of (19) and (20) for $n = m-j$, $j = 1, 2, \dots, N; n = m; n = m+j$, $j = 1, 2, \dots, N$, we see that they coincide. But if $n < m - N$ or $n > m + N$ the right-hand sides are equal to zero. So, we obtain

$$\sigma(\lambda^N p_n(\lambda), p_m(\lambda)) = \sigma(p_n(\lambda), \lambda^N p_m(\lambda)), \ n, m = 0, 1, 2, \dots.$$

Using linearity, from this relation we derive that

$$\sigma(\lambda^N u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda^N v(\lambda)), \ u, v \in \mathbb{P}. \tag{21}$$

Conversely, let $\{p_n(\lambda)\}_{n=0}^{\infty}$, $\deg p_n = n$, be a sequence of polynomials such that $p_i(\lambda) = \lambda^i$, $i = 0, 1, \ldots, N-1$, and $\sigma(u, v)$ be a linear functional which satisfies (5), (21). Then we can write:

$$\lambda^N p_k(\lambda) = \sum_{i=0}^{N+k} \xi_{k,i} p_i(\lambda),\ \xi_{k,i} \in \mathbb{C} : \xi_{k,N+k} \neq 0; k = 0, 1, 2, \ldots. \tag{22}$$

Using (5),(21),(6) we get:

$$\sigma(\lambda^N p_k(\lambda), p_j(\lambda)) = \xi_{k,j},\ j = 0, 1, 2, \ldots, k+N;$$

$$\sigma(\lambda^N p_k(\lambda), p_j(\lambda)) = \sigma(p_k(\lambda), \lambda^N p_j(\lambda)) = \sigma(\lambda^N p_j(\lambda), p_k(\lambda)) =$$
$$= \begin{cases} \xi_{j,k},\ j+N \geq k \\ 0,\ j+N < k \end{cases}.$$

We get

$$\xi_{k,j} = \xi_{j,k},\ j = k-N, k-N+1, \ldots, k+N;$$
$$\xi_{k,j} = 0,\ j = 0, 1, 2, \ldots, k-N-1; k = 0, 1, 2, \ldots.$$

Using these relations, from relation (22) we derive:

$$\lambda^N p_k(\lambda) = \sum_{i=k-N}^{k+N} \xi_{k,i} p_i(\lambda) = \sum_{j=1}^{N} \xi_{k,k+j} p_{k+j}(\lambda) + \sum_{j=1}^{N} \xi_{k,k-j} p_{k-j}(\lambda) +$$

$$+\xi_{k,k} p_k(\lambda) = \sum_{j=1}^{N} (\xi_{k-j,k} p_{k-j}(\lambda) + \xi_{k,k+j} p_{k+j}(\lambda)) + \xi_{k,k} p_k(\lambda).$$

If we put $\alpha_{k,j} = \xi_{k,k+j}$, $j = 1, 2, \ldots, N; k = 0, 1, 2, \ldots$; and $\alpha_{k,0} = \xi_{k,k}$, $k = 0, 1, 2, \ldots$; then we obtain (2).

Definition. *A linear with respect to the both arguments functional* $\sigma(u, v)$, $u, v \in \mathbb{P}$, *which satisfies* (5), *we shall call* **the spectral function** *of difference equation* (2).

Theorem 2. *A linear with respect to the both arguments functional* $\sigma(u, v), u, v \in \mathbb{P}$, *is the spectral function of a difference equation of type* (2) *iff:*

1) $\sigma(\lambda^N u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda^N v(\lambda))$, $u, v \in \mathbb{P}$;

2) $\sigma(\lambda^i, \lambda^j) = \delta_{i,j}$, $i, j = 0, 1, \ldots, N-1$;

3) *For arbitrary polynomial* $u_k(\lambda)$ *of degree* k *there exists a polynomial* $\hat{u}_k(\lambda)$ *of degree* k *such that:*

$$\sigma(u_k(\lambda), \hat{u}_k(\lambda)) \neq 0.$$

P r o o f. *Necessity.* Let $\sigma(u,v)$ be a spectral function of a difference equation (2). Relation 1) can be proved in the same way as we have proved relation (21). Relation 2) is obvious from (5) since $p_i(\lambda) = \lambda^i,\ i = 0,1,2,\dots,N-1$. For a polynomial $u_k(\lambda), \deg u_k = k$, we can write: $u_k(\lambda) = \sum_{j=0}^{k} \xi_j p_j(\lambda),\ \xi_j \in \mathbb{C} : \xi_k \neq 0$. Put $\hat{u}_k(\lambda) = \sum_{j=0}^{k} \overline{\xi_j} p_j(\lambda)$. Then, using (5) we get:

$$\sigma(u_k(\lambda), \hat{u}_k(\lambda)) = \sum_{j=0}^{k} |\xi_j|^2 > 0,$$

and relation 3) holds true.

Sufficiency. Let $\sigma(u,v)$ be a linear functional which satisfies conditions 1),2),3). Let us construct a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$. Put $p_i(\lambda) = \lambda^i,\ i = 0,1,2,\dots,N-1$. From relation 2) we get:

$$\sigma(p_i, p_j) = \delta_{i,j},\ i,j = 0,1,\dots,N-1.$$

Suppose that we have built such polynomials $p_0(\lambda), p_1(\lambda), \dots, p_{n-1}(\lambda)$; $\deg p_i = i,\ n \geq N$, that

$$\sigma(p_i, p_j) = \delta_{i,j},\ i,j = 0,1,\dots,n-1. \tag{23}$$

Consider arbitrary polynomial $R_n(\lambda)$ of degree n. We can write: $R_n(\lambda) = \mu_n \lambda^n + \sum_{i=0}^{n-1} \alpha_{n,i} p_i(\lambda),\ \mu_n, \alpha_{n,i} \in \mathbb{C}, \mu_n \neq 0$. Then

$$\sigma(R_n(\lambda), p_i(\lambda)) = \sigma(\mu_n \lambda^n + \sum_{j=0}^{n-1} \alpha_{n,j} p_j(\lambda), p_i(\lambda)) = \mu_n \sigma(\lambda^n, p_i(\lambda)) + \alpha_{n,i},$$

$$i = 0,1,\dots,n-1.$$

If we take $\alpha_{n,i} = -\mu_n \sigma(\lambda^n, p_j(\lambda)),\ j = 0,1,2,\dots,n-1$, then

$$R_n(\lambda) = \mu_n (\lambda^n - \sum_{i=0}^{n-1} \sigma(t^n, p_i(t)) p_i(\lambda)), \tag{24}$$

and

$$\sigma(R_n(\lambda), p_i(\lambda)) = 0,\ i = 0,1,\dots,n-1.$$

From relation 3) it follows that there exists a polynomial $\hat{R}_n(\lambda), \deg \hat{R}_n = n$, such that

$$M_n := \sigma(R_n(\lambda), \hat{R}_n(\lambda)) \neq 0.$$

For the polynomial $\hat{R}_n(\lambda)$ we can write: $\hat{R}_n(\lambda) = \nu_n R_n(\lambda) + \sum\limits_{i=0}^{n-1} \beta_{n,i} p_i(\lambda),\ \nu_n, \beta_{n,i} \in \mathbb{C}, \nu_n \neq 0$. Then

$$M_n = \sigma(R_n(\lambda), \hat{R}_n(\lambda)) = \sigma(R_n(\lambda), \nu_n R_n(\lambda) + \sum_{i=0}^{n-1} \beta_{n,i} p_i(\lambda)) = \nu_n \sigma(R_n(\lambda), R_n(\lambda));$$

$$\frac{\nu_n}{M_n} \sigma(R_n(\lambda), R_n(\lambda)) = 1.$$

Put

$$\tilde{\mu}_{n,n} = \sqrt{\frac{\nu_n}{M_n}}, \tag{25}$$

where we take arbitrary branch of the square root. We put

$$p_n(\lambda) = \tilde{\mu}_{n,n} R_n(\lambda). \tag{26}$$

Then $\deg p_n = n$ and

$$\sigma(p_n, p_n) = 1;$$

$$\sigma(p_i, p_j) = \delta_{i,j},\ i, j = 0, 1, \ldots, n.$$

Applying this procedure for $n = N, N+1, N+2, \ldots$, we build such a sequence of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty},\ \deg p_n = n$, that $p_i(\lambda) = \lambda^i,\ i = 0, 1, \ldots, N-1$, and

$$\sigma(p_i, p_j) = \delta_{i,j},\ i, j = 0, 1, 2, \ldots.$$

As it has been already shown, if we have a linear functional $\sigma(u, v),\ u, v \in \mathbb{P}$ and a sequence of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$, $\deg p_n = n,\ p_i(\lambda) = \lambda^i,\ i = 0, 1, \ldots, N-1$, which satisfy (5),(21), we can derive (2). From the definition of the spectral function it follows that σ will be a spectral function of this equation.

The proof is complete. □

Let an equation of type (2) be given and using (5) we have built its spectral function $\sigma(u, v)$. How we can reconstruct our difference equation (2)? We must use the procedure of construction of $\{p_n(\lambda)\}_{n=0}^{\infty}$ in the proof of the previous theorem. In this procedure we must take $\mu_n = 1$ for $R_n(\lambda)$ in (24) and take $\tilde{\mu}_{n,n}$ in (25) such that $\operatorname{Arg} \tilde{\mu}_{n,n} \in [0, \pi)$ if $\operatorname{Arg} \mu_{n,n} = \operatorname{Arg} \dfrac{1}{\alpha_{k,N} \alpha_{k-N,N} \alpha_{k-2N,N} \cdots \alpha_{k-[\frac{k}{N}]N,N}} \in$

$[0, \pi)$, or Arg $\tilde{\mu}_{n,n} \in [\pi, 2\pi)$ otherwise, where $\mu_{n,n}$ is the leading coefficient of p_n which is defined from (2).

We can define a sequence of signs $\{\pm, \pm, \ldots\}$ where "+" appears if Arg $\dfrac{1}{\alpha_{k,N}\alpha_{k-N,N}\alpha_{k-2N,N}\cdots\alpha_{k-[\frac{k}{N}]N,N}} \in [0, \pi)$, and "–" appears if Arg $\dfrac{1}{\alpha_{k,N}\alpha_{k-N,N}\alpha_{k-2N,N}\cdots\alpha_{k-[\frac{k}{N}]N,N}} \in [\pi, 2\pi)$. This sequence and the spectral function allow us uniquely reconstruct difference equation (2).

REFERENCES

[1] G. S. Guseynov. Definition of an infinite nonself-adjoint Jacobi matrix by its generalized spectral function. *Matem. zametki* **23**, *2* (1978), 237–246 (in Russian).

[2] S. M. Zagorodnyuk. On generalized Jacobi matrices and orthogonal polynomials. *New York J. Math.* **9**, (2003), 117–136.

[3] S. M. Zagorodnyuk. On five-diagonal Jacobi matrices and orthogonal polynomials on rays in the complex plane. *Serdica Math. J.* **24**, *3–4* (1998), 257–282.

[4] S. M. Zagorodnyuk. On a characteristic property of polynomials satisfying difference equation of the 4-th order. Visn. Khark. Univ., Ser. "Matem., prykl. matem. i mekh.", **582** (2003), 15–25 (in Russian).

[5] F. R. Gantmaher. Theory of matrices. Moscow, Nauka, 1967 (in Russian).

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