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TRIANGULAR MODELS AND ASYMPTOTICS OF CONTINUOUS CURVES WITH BOUNDED AND UNBOUNDED SEMIGROUP GENERATORS*

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Abstract. In this paper classes of K^r-operators are considered – the classes of bounded and unbounded operators A with equal domains of A and A^*, finite dimensional imaginary parts and presented as a coupling of a dissipative operator and an antidissipative one with real absolutely continuous spectra and the class of unbounded dissipative K^r-operators A with different domains of A and A^* and with real absolutely continuous spectra. Their triangular models are presented. The asymptotics of the corresponding continuous curves with generators from these classes are obtained in an explicit form. With the help of the obtained asymptotics the scattering theory for the couples (A^*, A) when A belongs to the introduced classes is constructed.

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1. Introduction. The study of nonselfadjoint operators is based on the methods of the characteristic functions and its development began with the works of M. S. Livšic [19] and his associates in 1950's [6, 5] and later that of I. Gohberg, M. Krein, B. Sz. Nagy, C. Foias, L. de Branges, J. Rownyak [8, 23, 3, 4], M. S. Livšic, V. Vinnikov [21, 29] at al. Later on M. S. Livšic and A. A. Yantsevich in their book [22] proposed an interesting idea for an investigation of continuous curves (in particular of random processes) with the help of the theory of the nonselfadjoint operators and their characteristic functions. This idea was expanded and developed in the work of K. Kirchev, V. Zolotarev [11, 14, 15] at al. The essence of the theory, created from M. S. Livšic, is the connection between the theory of the nonselfadjoint operators and the theory of the bounded analytic functions on the upper half-plane and it considers mostly operators in a Hilbert space with a finite dimensional or trace class imaginary part.

The matrix function

$$W(\lambda) = I_E - iP_E(A - \lambda I)^{-1}P_E L$$

is called a characteristic function of a bounded operator $A : H \longrightarrow H$ with $\dim(A - A^*)H < \infty$, where $E = (A - A^*)H$, $L = \frac{1}{i}(A - A^*)|_E$ and $\Phi = P_E$ is the orthogonal projector of the Hilbert space H onto E. $W(\lambda)$ is defined and analytic in the set of all regular points of A, analytic in a neighbourhood $|\lambda| > a$ of $\lambda = \infty$, $W(\infty) = I$ and $W(\lambda)$ possesses the metric properties

$$\begin{aligned} W^*(\lambda)LW(\lambda) &\geq L \quad (\operatorname{Im}\lambda > 0), \\ W^*(\lambda)LW(\lambda) &= L \quad (\operatorname{Im}\lambda = 0), \\ W^*(\lambda)LW(\lambda) &\leq L \quad (\operatorname{Im}\lambda < 0) \end{aligned} \tag{1.1}$$

for a regular point λ of the operator A. In other words to every bounded operator A in a Hilbert space with a finite dimensional imaginary part there corresponds a matrix valued function which characterizes this operator up to an unitary equivalence on the principal subspace of A.

More generally the characteristic function of $A : H \longrightarrow H$ can be introduced in the form

$$W(\lambda) = I - i\Phi(A - \lambda I)^{-1}\Phi^* L$$

by the so called operator colligation $X = (A; H, \Phi, E; L)$, where E is a Hilbert space, $\Phi : H \longrightarrow E$ and $L : E \longrightarrow E$ are bounded linear operators with $L^* = L$ and $(A - A^*)/i = \Phi^* L\Phi$.

As an arbitrary finite matrix can be presented in a triangular form by a corresponding unitary mapping analogous problem can be solved for classes of nonselfadjoint operators. The operators from these classes are presented in the so called triangular models using unitary mappings. The study of these models may give much information about the original operators.

The main point of this investigations is the relation between the invariant subspaces of the operator A and the factorizations of the characteristic function $W(\lambda)$ (given by Potapov's factorization theorem).

The methods and results of the operator colligation theory have important applications not only in the investigations of various classes of linear nonselfadjoint operators, but also in the scattering theory, in the theory of nonstationary random processes or more generally in the theory of the continuous curves $g(t)$ in a Hilbert space H:

$$g(t) = e^{itA} f \quad (f \in H).$$

The obtaining of the asymptotic behaviour of the nonstationary curves, generated by classes of nonselfadjont operators, allows us to construct a scattering theory for the couple (A^*, A), where A is an operator from a given class.

An analogous theory can be developed for unbounded operators.

The class of the unbounded operators possesses richer properties than the class of the bounded nonselfadjoint operators. But the problems, considered in [6, 22, 11, 14, 15, 29, 12] for the bounded case, concerning triangular models, characteristic functions, continuous curves, their asymptotics, their corresponding correlation functions in a connection with the spectral properties of the operators are too complicated in the unbounded case.

Triangular models, characteristic functions of unbounded operators are considered by A. V. Kuzhel [16, 17, 18], operator colligations and characteristic functions are considered by A. G. Rutkas [26], triangular models and nondissipative curves are considered by the authors in [13].

Note that models of various classes of nonselfadjoint linear operators were constructed by different methods. Sometimes it is possible to construct different models of the same class of operators adapted to the solution of various particular problems.

In our applications in the scattering theory we use the so called time-dependent method – in other words in a time-dependent approach one deals with the asymptotic behaviour of $T_t = e^{itA}$ more or less directly.

The class of characteristic functions, introduced for bounded operators A by M. S. Livšic and M. S. Brodskii [19, 6, 20], and for unbounded densely defined operators A by A. G. Rutkas [26], with the help of the channel representations of the imaginary part $\operatorname{Im} A$ of A or operator colligations is more general than the class of characteristic functions for the operator A from [17, 18] or [28]. But the functions from [17, 18] and [28] can be used for closed unbounded operators A with nonempty resolvent set $\rho(A)$.

In this paper we present wide classes of K^r-operators (i. e. closed operators A in a Hilbert space with an Hermitian part of A with deficiency index (r, r) $(0 < r < \infty)$ and a nonempty resolvent set).

We present the classes of nondissipative operators $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ (part 2, part 3) and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ (part 4, part 5), their triangular models, corresponding continuous curves, their asymptotics, correlation functions and an application in the scattering theory, considered by the authors in [1, 12, 2, 13]. The operators from $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ are classes of K^r- operators with r dimensional imaginary parts and equal domains of the operator and its adjoint. In other words following the denotations of A. Kuzhel ([18]) we have $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}, \widetilde{\mathbf{\Lambda}}_{\mathbb{R}} \subset K_I^r$.

We present also the continuation of these investigations for a class of unbounded dissipative K^r-operators A with domains $D_A \neq D_{A^*}$ concerning analogous problems as in the cases of $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ (part 6).

The class $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ describes the class of all bounded nondissipative operators A in a Hilbert space with a finite dimensional imaginary part and presented as a coupling of a dissipative operator and an antidissipative one with real absolutely continuous spectra. The class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ describes the class of all unbounded nondissipative operators A in a Hilbert space with a dense domain $D_A = D_A^*$, with a finite dimensional imaginary part and presented as a coupling of a dissipative operator and an antidissipative one with real absolutely continuous spectra. We introduce the next triangular models

$$(1.2)\qquad \begin{gathered} Af(x) = \alpha(x)f(x) - i\int_0^x f(\xi)\Pi(\xi)S^*\Pi^*(x)d\xi + \\ +i\int_x^l f(\xi)\Pi(\xi)S\Pi^*(x)d\xi + i\int_0^x f(\xi)\Pi(\xi)L\Pi^*(x)d\xi \end{gathered}$$

for $f \in \mathbf{L}^2(0, l; \mathbb{C}^n)$, $L = J_1 - J_2 + S + S^*$, $\alpha(x)$ is a bounded nondecreasing function in $[0, l]$ (see part 2) and

$$(1.3)\qquad \begin{gathered} Af(x) = \alpha(x)f(x) + i\int_{-\infty}^x f(\xi)\Pi(\xi)(J_1 - J_2)\Pi^*(x)d\xi + \\ +i\int_{-\infty}^{+\infty} f(\xi)\Pi(\xi)S\Pi^*(x)d\xi \end{gathered}$$

for $f \in D_A \subset \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$, $L = J_1 - J_2 + S + S^*$, $\alpha(x)$ is an unbounded nondecreasing function in $\mathbb{R}$ (see part 4), describing the classes $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ correspondingly up to an unitary equivalence on the principal subspaces. (The selfadjoint matrix $L : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ and $n \times m$ matrix function $\Pi(x)$ satisfy appropriate conditions – see part 2 and part 4 correspondingly.) We define in an appropriate way families of operators and prove their properties which present these families as semigroups of operators from the class (C_0). The families $T_t = e^{itA}$ $(t \leq 0, t \geq 0)$, where A has the form (1.2), are semigroups of operators from the class (C_0) with generators from $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ (see part 2, part 3).

The families

$$T_t f = -\frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} (e^{it(\xi - i\delta)}((A - (\xi - i\delta)I)^{-1} f - \\ - e^{it(\xi + i\delta)}(A - (\xi + i\delta)I)^{-1} f) d\xi \tag{1.4}$$

$(t > 0,\ t < 0)$, where the integral on the right hand side of (1.4) is in the sense of a principal value and A has the form (1.3), are semigroups of operators from the class (C_0) with generators from $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ (see part 4, part 5).

These semigroups determine the exponential function by the equalities $e^{itA} = T_t$ and generate continuous curves $e^{itA} f$ for the models (1.2) and (1.3).

The asymptotics of the curves $e^{itA} f$ with A from the form (1.2) are given by the next theorem (see part 2):

Theorem 1.1. *Let for the model $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$, defined by* (1.2), *the next conditions hold:* 1) *the function $\alpha : [0;l] \longrightarrow \mathbb{R}$ satisfies* **(i)**, **(ii)**, **(iii)** (*page* 112)*;*

2) *$Q^*(x)$ is a smooth matrix function on $[0;l]$;*

3) *$B(x) \in C_{\alpha_1}[0;l]$ $(0 < \alpha_1 \leq 1)$.*

Then the curve $e^{itA} f(x)$ for each $f \in H_0$ after the change of the variable $x = \sigma(u)$ satisfies the relation

$$\left\| e^{itA} f(\sigma(u)) - e^{itu} S_{\pm} f(\sigma(u)) \right\|_{L^2} \to 0 \tag{1.5}$$

as $t \to \pm\infty$.

H_0 is a suitable subspace of $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$ and in the relation (1.5) we have used the next denotations

$$S_{\pm} f(\sigma(u)) = \\ = (\widehat{S}^{\pm} f(\sigma(u))) T_{\pm} \Pi(\sigma(u)) (J_1 |t|^{i\widetilde{B}_1(u)} J_1 + J_2 |t|^{-i\widetilde{B}_2(u)} J_2) Q(\sigma(u)), \tag{1.6}$$

(1.7) $$\widetilde{S}_{\pm}f(\sigma(u)) = (\widehat{S}_{\pm}f(\sigma(u)))T_{\pm},$$

(1.8) $$\widehat{S}_{\pm}f = \widetilde{S}_{11}f + \widetilde{S}_{22}f + \widetilde{S}_{12}^{\pm}f,$$

(1.9) $$\widetilde{S}_{kk}f(\sigma(u)) = \int_a^u \widetilde{f}'(w) \overset{\curvearrowleft}{\int_a^w} e^{\frac{(-1)^{k+1}i\widetilde{B}_k(v)}{v-u}dv} dwJ_k, \quad k = 1, 2,$$

(1.10) $$\widetilde{S}_{12}^{\pm}f(\sigma(u)) = -\int_a^b \widetilde{f}'(w)\widetilde{F}_w^{\mp}(u, b)dwS,$$

(1.11) $$\begin{aligned} T_{\pm}h = h\Big(&J_1U_{2a}(u)(u-a)^{i\widetilde{B}_1(u)}e^{\mp\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I + i\widetilde{B}_1(u))J_1 + \\ &+ J_2\widetilde{U}_{2a}(u)(u-a)^{-i\widetilde{B}_2(u)}e^{\pm\frac{\pi}{2}\widetilde{B}_2(u)}\mathbf{\Gamma}^{-1}(I - i\widetilde{B}_2(u))J_2\Big)\Pi^*(\sigma(u))\end{aligned}$$

for each $f \in H_0$ and each $h \in \mathbb{C}^m$. (The operators $\widetilde{F}_w^{\pm}(u, b)$, $U_{2a}(u)$, $\widetilde{U}_{2a}(u)$ are denoted in part 2 with the help of multiplicative integrals.) The asymptotics of the curves $T_tf = e^{itA}f$ with A from the form (1.3) (when $\alpha(x) = x$) are presented by the next theorem (see part 4):

Theorem 1.2. *Let for the model A defined by (4.35) the next conditions hold:*

1) $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$, $\|Q^{*\prime}(x)\| \leq C$;
2) $B(x) \in C_{\alpha_1}(\mathbb{R})$ $(0 < \alpha_1 \leq 1)$.

Then the curve T_tf has the next asymptotics for each $f \in S(\mathbb{R}; \mathbb{C}^n)$

(1.12) $$\|T_tf(x) - e^{itx}S_{\pm}f(x)\|_{\mathbf{L}^2} \to 0$$

as $t \to \pm\infty$, where

(1.13) $$S_{\pm}f(x) = (\widehat{S}_{\pm}f(x))T_{\pm}\Pi(x)(J_1|t|^{iB_1(x)}J_1 + J_2|t|^{-iB_2(x)}J_2)Q(x)$$

(1.14) $$\widehat{S}_{\pm}f = \widetilde{S}_{11}f + \widetilde{S}_{22}f + \widetilde{S}_{12}^{\pm}f,$$

(1.15)
$$\widetilde{S}_{kk}f(x) = \int_{-\infty}^{x} \widetilde{f}'(w) \int_{-\infty}^{\overleftarrow{w}} e^{\frac{(-1)^{k-1} i B_k(v)}{v-x} dv} dw J_k, \quad k = 1, 2,$$
$$\widetilde{S}_{12}^{\pm} f(x) = - \int_{-\infty}^{+\infty} \widetilde{f}'(w) \widetilde{F}_w^{\mp}(x, +\infty) dw S,$$

for each $f \in S(\mathbb{R}; \mathbb{C}^n)$,

(1.16)
$$T_{\pm} h = h \Big(J_1 V_{-\infty}(x) e^{\mp \frac{\pi}{2} B_1(x)} \mathbf{\Gamma}^{-1} (I + iB_1(x)) J_1 + + J_2 \widetilde{V}_{-\infty}(x) e^{\pm \frac{\pi}{2} B_2(x)} \mathbf{\Gamma}^{-1} (I - iB_2(x)) J_2 \Big) \Pi^*(x) \quad (h \in \mathbb{C}^m).$$

and $\widetilde{f}(w)$ *is defined by* (4.45)*.*

Here the operators $\widetilde{F}_w^{\pm}(x, = \infty)$, $V_{-\infty}(x)$, $\widetilde{V}_{-\infty}(x)$ are defined in part 4 with the help of multiplicative integrals. We present the case, when $\alpha(x) = x$, to avoid the complications of the writing. The case of an arbitrary unbounded nondecreasing function $\alpha(x)$ can be considered analogously after a suitable change of the variable.

The obtaining of the asymptotics of the corresponding continuous curves $e^{itA}f$ allows to construct a scattering theory for the couples (A^*, A) and solve the basic problems from the scattering theory concerning the similarity of A and A^*, of A and the operator of a multiplying by an independent variable (see part 3, part 5).

The solutions of these problems is based on the obtaining of the form and the existence of the wave operators for the couples (A^*, A) as strong limits. The next two theorems describe the wave operators for the couple (A^*, A) in the case when $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ (see part 3).

Theorem 1.3. *Let for the model* $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$*, defined by* (1.2)*, the next conditions hold*

1) *the function* $\alpha : [0; l] \longrightarrow \mathbb{R}$ *satisfies* **(i)**, **(ii)**, **(iii)** (*page* 112)*;*
2) $Q^*(x)$ *is a smooth matrix function on* $[0; l]$*;*
3) $B(x) \in C_{\alpha_1}[0; l]$ $(0 < \alpha_1 \leq 1)$.

Then there exist the limits

(1.17)
$$\lim_{t \to \pm\infty} (e^{itA^*} e^{-itA} f, g) = (\widetilde{S}_{\mp}^* \widetilde{S}_{\mp} f, g)$$

for all f*,* $g \in \mathbf{L}^2(0, l; \mathbb{C}^n)$ *and the operator* A *satisfies the equalities*

(1.18)
$$\widetilde{S}_{\pm}^* \widetilde{S}_{\pm} A = \widetilde{S}_{\pm}^* \mathcal{Q} \widetilde{S}_{\pm}$$

onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$ *after the change of the variable* $x=\sigma(u)$ *where* $\mathcal{Q}f(\sigma(u)) = f(\sigma(u))u$.

Theorem 1.4. *Let for the model* $A\in\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$, *defined by* (1.2), *the next conditions hold*

1) *the function* $\alpha:[0;l]\longrightarrow\mathbb{R}$ *satisfies* **(i)**, **(ii)**, **(iii)** (*page* 112)*;*
2) $Q^*(x)$ *is a smooth matrix function on* $[0;l]$*;*
3) $B(x)\in C_{\alpha_1}[0;l]$ $(0<\alpha_1\leq 1)$.

Then there exist the limits

$$s-\lim_{t\to\pm\infty} e^{itA^*}e^{-itA}$$

on $\mathbf{L}^2(0,l;\mathbb{C}^n)$.

The analogous results concerning wave operators for the couple (A^*,A) when $A\in\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ are obtained in the theorems (see part 5):

Theorem 1.5. *Let for the model* $A\in\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$, *defined by* (4.35), *the next conditions hold:*

1) $\|Q^{*\prime}(x)\|\in\mathbf{L}^2(\mathbb{R})$, $\|Q^{*\prime}(x)\|\leq C$*;*
2) $B(x)\in C_{\alpha_1}(\mathbb{R})$ $(0<\alpha_1\leq 1)$.

Then there exist the limits

$$\lim_{t\to\pm\infty}(e^{itA^*}e^{-itA}f,g)=(\widetilde{S}^*_{\mp}\widetilde{S}_{\mp}f,g) \tag{1.19}$$

for all $f,g\in\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ *and the operator* A *satisfies the equalities*

$$\widetilde{S}^*_{\pm}\widetilde{S}_{\pm}A=\widetilde{S}^*_{\pm}\mathcal{Q}\widetilde{S}_{\pm} \tag{1.20}$$

onto $S(\mathbb{R};\mathbb{C}^n)$, *where* $\mathcal{Q}f(x)=xf(x)$ *and* $\widetilde{S}_{\pm}$ *are defined by* (4.55).

Theorem 1.6. *Let for the model* $A\in\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$, *defined by* (4.35), *the next conditions hold:*

1) $\|Q^{*\prime}(x)\|\in\mathbf{L}^2(\mathbb{R})$, $\|Q^{*\prime}(x)\|\leq C$*;*
2) $B(x)\in C_{\alpha_1}(\mathbb{R})$ $(0<\alpha_1\leq 1)$.

Then there exist the strong limits $s-\lim\limits_{t\to\pm\infty} e^{itA^*}e^{-itA}$ *onto* $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$.

It is worthwhile to mention that for the considered classes $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ the analytical solution of the problems, presented in [14, 15, 12], is obtained using the characteristic function of M. S. Livšic (see part 2, part 4). It is presented with the help of the multiplicative integrals from the form

$$\overset{l}{\underset{0}{\overset{\rightarrow}{\int}}} e^{i\frac{T(v)}{v-\lambda}dv} \quad \text{and} \quad \overset{+\infty}{\underset{-\infty}{\overset{\rightarrow}{\int}}} e^{i\frac{T(v)}{v-\lambda}dv}$$

($T(v)$ is a nonnegative (nonpositive) matrix function) and the multiplicative integrals from these forms take part in the representations of the operators (1.5) and (1.13), describing the asymptotics of the corresponding continuous curves (see part 2 and part 4). Moreover the results for the bounded operators from $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and the unbounded operators A from $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ with domains $D_A = D_{A^*}$ are close each other.

A natural continuation of these investigations is the consideration of the class of K^r-operators A with domains $D_A \neq D_{A^*}$. In part 6 of this paper we present the class of dissipative K^r-operators A with domains $D_A \neq D_{A^*}$ and solve analogous problems as in the cases of the classes $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$. For the triangular model of A. Kuzhel ([18])

$$(1.21)\qquad \begin{aligned} Af(x) &= \alpha(x)f(x)+ \\ &+i\int_{-\infty}^{x} f(\xi)(\alpha(\xi)+i)\Pi(\xi)\overset{x}{\underset{\xi}{\overset{\rightarrow}{\int}}} e^{i\alpha(v)B(v)dv}\,\Pi^*(x)(\alpha(x)-i)d\xi \end{aligned}$$

(where $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is an unbounded nondecreasing function, the matrix functions $\Pi(x)$, $B(x) = \Pi^*(x)\Pi(x)$ satisfy appropriate conditions) describing the class of all unbounded dissipative K^r-operators with domain $D_A \subset \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$, $D_A \neq D_{A^*}$ and with real spectrum, we define a family of operators

$$(1.22)\qquad T_t f(x) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{it(\xi-i\delta)}(A-(\xi-i\delta)I)^{-1}f(x)d\xi$$

($t > 0$) in a sense of a principal value in a suitable subset of $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ (see part 6). After proving of the properties of the family (1.22), showing that this family is a semigroup from the class (C_0), we obtain the asymptotics of the continuous curves, defined by the equality $e^{itA}f = T_t f$ with generators (1.21) (in the case when $\alpha(x) = x$). These asymptotics are given by the theorem:

Theorem 1.7. *Let for the model A, defined by* (6.33)*, the next conditions hold:*

1) $\|B(x)\| \leq C$, $\|xB(x)\| \leq C$ $\forall x \in \mathbb{R}$;

2) $B(x) \in C_{\alpha_1}(\mathbb{R})$, $xB(x) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_1 \leq 1,\ 0 < \alpha_2 \leq 1)$;

3) $\|B(x)\| \in \mathbf{L}(\mathbb{R})$, $\|xB(x)\| \in \mathbf{L}(\mathbb{R})$;

4) $Q^*(x)$ *is a smooth matrix function on* $\mathbb{R}$ *and* $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$.

Then the curve $T_t f(x)$ *for each* $f(x) = (A - \lambda_0 I)^{-1}(A - \mu_0 I)^{-1} h(x)$, $h \in D_1 \cap H_0 \cap S(\mathbb{R}, \mathbb{C}^n)$ *has the next asymptotics*

$$\|T_t f(x) - e^{itx} S_+ f(x)\|_{\mathbf{L}^2} \longrightarrow 0 \tag{1.23}$$

as $t \to +\infty$, *where* $S_+ f(x)$ *is defined by the equality*

$$S_+ f(x) = \int\limits_{-\infty}^{x} \widetilde{h}(w) \overset{\overset{w}{\leftarrow}}{\int\limits_{-\infty}} e^{i\frac{1+vx}{v-x}B(v)dv} dw V_{-\infty}(x) t^{i\widetilde{B}(x)} e^{-\frac{\pi}{2}\widetilde{B}(x)} \cdot \\ \cdot \mathbf{\Gamma}^{-1}(I + i\widetilde{B}(x))\Pi^*(x)\tfrac{x-i}{x-\lambda_0} \cdot \tfrac{1}{x-\mu_0}, \tag{1.24}$$

$V_{-\infty}(x)$ *and* $\widetilde{h}(x)$ *are defined by* (6.45) *and* (6.61) *correspondingly.*

In our considerations we have essentially used the characteristic function, introduced by A. Kuzhel ([17, 18]), which is different from the characteristic function of M. S. Livšic ([22]) and A. G. Rutkas ([26]), applied in the other two considered classes and the proved properties of the multiplicative integrals

$$\overset{\overset{b}{\rightarrow}}{\int\limits_{a}} e^{-i\frac{1+\lambda v}{v-\lambda}B(v)dv} \quad \text{or} \quad \overset{\overset{b}{\rightarrow}}{\int\limits_{a}} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda}B(v)dv}$$

(Im $\lambda \neq 0$, $-\infty \leq a < b \leq +\infty$) describing the characteristic function of A. Kuzhel (see part 6). As in the previous two classes of operators $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ the obtained explicit form of the asymptotics of the continuous curves $e^{itA}f$ with A from the form (1.21) allows to apply these results in the scattering theory for the couples (A^*, A). The next theorems (see part 6) give the form and the existence of the wave operator $W_-(A^*, A)$ as a strong limit:

Theorem 1.8. *Let for the model* A, *defined by* (6.33), *the next conditions hold:*

1) $\|B(x)\| \leq C$, $\|xB(x)\| \leq C$ $\forall x \in \mathbb{R}$;

2) $B(x) \in C_{\alpha_1}(\mathbb{R})$, $xB(x) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_1 \leq 1,\ 0 < \alpha_2 \leq 1)$;

3) $\|B(x)\| \in \mathbf{L}(\mathbb{R})$, $\|xB(x)\| \in \mathbf{L}(\mathbb{R})$, $\|e^{-2\pi\widetilde{B}(x)}\|_{\mathbf{L}^2} < 1$;

4) $Q^*(x)$ *is a smooth matrix function on* $\mathbb{R}$ *and* $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$.
Then there exists the limit

$$\lim_{t\to-\infty}(e^{itA^*}e^{-itA}f,g) = (\widetilde{S}_+^*\widetilde{S}_+f,g) \tag{1.25}$$

for all $f,g \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ *and the operator* A *satisfies the equality*

$$\widetilde{S}_+^*\widetilde{S}_+A = \widetilde{S}_+^*\mathcal{Q}\widetilde{S}_+ \tag{1.26}$$

onto the subspace $\widetilde{\mathcal{D}}_0$, *where* $\mathcal{Q}f(x) = xf(x)$ *and* $\widetilde{S}_+$ *is defined by* (6.82).

Theorem 1.9. *Let for the model* A, *defined by* (6.33), *the next conditions hold:*

1) $\|B(x)\| \le C$, $\|xB(x)\| \le C$ $\forall x \in \mathbb{R}$;

2) $B(x) \in C_{\alpha_1}(\mathbb{R})$, $B(x) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_1 \le 1,\ 0 < \alpha_2 \le 1)$;

3) $\|B(x)\| \in \mathbf{L}(\mathbb{R})$, $\|xB(x)\| \in \mathbf{L}(\mathbb{R})$, $\|e^{-2\pi\widetilde{B}(x)}\|_{\mathbf{L}^2} < 1$;

4) $Q^*(x)$ *is a smooth matrix function on* $\mathbb{R}$ *and* $\|Q^{*\prime}(x)\| \in \mathbf{L}(\mathbb{R})$.
Then there exists the strong limit

$$s-\lim_{t\to-\infty} e^{itA^*}e^{-itA} \quad \textit{onto } \mathbf{L}^2(\mathbb{R};\mathbb{C}^n).$$

All results in this paper are obtained in an explicit form by using of multiplicative integrals and the introduced analogue in $\mathbb{C}^m$ of the classical gamma-function. The results in parts 2, 3, 4, 5 are considered by the authors in [1, 12, 2, 13]. The results in part 6 are new and they have not been published till now in other papers.

It has to mention that analogously it can be considered the case of a coupling A of a dissipative K^r-operator and an antidissipative K^r-operator with different domains D_A and D_{A^*} essentially using the methods and the results, presented in the last part of this paper for the dissipative K^r-operators. The separate consideration of the dissipative case of K^r-operators A with different domains of A and its adjoint is important from the viewpoint of the necessity of introducing of other preliminary technical results and propositions, concerning the multiplicative integrals, which describe the characteristic function of A. Kuzhel.

2. Triangular model and asymptotics of nondissipative curves with bounded semigroup generators from the class $\widetilde{\Omega}_{\mathbb{R}}$. In this part we consider a new form of the triangular model of M. S. Livšic, introduced by the authors in [1, 12]. This new form allows to obtain results for the class of nondissipative operators (a coupling of a dissipative operator and an antidissipative one) which are similar to the results of L. A. Sakhnovich [27] for dissipative operators.

The results of this part and the next part of the paper are presented by the authors in [1, 12, 2].

Let H be a separable Hilbert space and let A be a bounded linear nonselfadjoint operator in H with a finite nonhermitian rank, i.e. with a finite dimensional imaginary part $\dim(A - A^*)H < +\infty$. (Analogously it can be considered the case when the imaginary part of A belongs to the trace class.)

Let $\alpha(x)$ be a bounded nondecreasing function on a finite interval $[0; l]$ which is continuous at 0 and continuous from the left on $(0; l]$, $\Pi(x)$ is a measurable $n \times m$ $(1 \leq n \leq m)$ matrix function on $[0; l]$, whose rows are linearly independent at each point of a set of positive measure, and satisfying the condition

$$\operatorname{tr} \Pi^*(x)\Pi(x) = 1. \tag{2.1}$$

Let the operator $\widetilde{\Phi} : \mathbf{L}^2(0, l; \mathbb{C}^n) \to \mathbb{C}^m$ have the form $\widetilde{\Phi} f(x) = \int\limits_0^l f(x)\Pi(x)dx$ and $L : \mathbb{C}^m \longrightarrow \mathbb{C}^m$, $L^* = L$, $\det L \neq 0$. For the selfadjoint operator $L : \mathbf{C}^m \longrightarrow \mathbf{C}^m$ with $\det L \neq 0$ we can assume without loss of generality that L has the representation

$$L = J_1 - J_2 + S + S^*, \tag{2.2}$$

where J_1, J_2, S, $S^* : \mathbb{C}^m \longrightarrow \mathbb{C}^m$,

$$J_1 = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix},\ J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r_1} \end{pmatrix},\ S = \begin{pmatrix} 0 & 0 \\ \widehat{S} & 0 \end{pmatrix}, \tag{2.3}$$

I_k is the identity matrix in $\mathbb{C}^k$ $(k = r_1, m - r_1)$, $\widehat{S}$ is a $(m - r_1) \times r_1$ matrix, r_1 is the number of the positive eigenvalues and $m - r_1$ is the number of the negative eigenvalues of the operator L. This is possible because we can always find an invertible operator $V : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ (see [1]), so that

$$L = V(J_1 - J_2 + S + S^*)V^*.$$

We consider the model (1.2) in the Hilbert space $\mathbf{L}^2(0,l;\mathbb{C}^n)$ which can be embedded in a colligation

$$X = (A; \mathbf{L}^2(0,l;\mathbb{C}^n), \Phi, \mathbb{C}^m; L), \tag{2.4}$$

where the operator $\Phi : \mathbf{L}^2(0,l;\mathbb{C}^n) \longrightarrow \mathbb{C}^m$ is defined by

$$\Phi f(x) = \int\limits_0^l f(x)\Pi(x)dx. \tag{2.5}$$

Then

$$(A - A^*)/i = \Phi^* L \Phi \tag{2.6}$$

where

$$\Phi^* h = h\Pi^*(x). \tag{2.7}$$

If the matrix function $B(x) = \Pi^*(x)\Pi(x)$ satisfies the condition

$$B(x)J_1 = J_1 B(x) \tag{2.8}$$

for almost all $x \in [0;l]$ then the model (1.2) describes the class of all linear bounded nonselfadjoint operators in a Hilbert space (up to an unitary equivalence) presented as a coupling of a dissipative operator and an antidissipative one (see [1]). In other words the model A from the form (1.2) has the representation

$$A = P_1 A P_1 + P_2 A P_2 + P_1 A P_2. \tag{2.9}$$

The operators P_1, $P_2 : \mathbf{L}^2(0,l;\mathbb{C}^n) \longrightarrow \mathbf{L}^2(0,l;\mathbb{C}^n)$, defined by the equalities

$$P_1 f(x) = f(x)\Pi(x)J_1 Q(x), \quad P_2 f(x) = f(x)\Pi(x)J_2 Q(x), \tag{2.10}$$

are orthogonal projectors in $\mathbf{L}^2(0,l;\mathbb{C}^n)$, where $Q(x)$ is a measurable $m \times n$ matrix function satisfying the condition

$$\Pi(x)Q(x) = I \tag{2.11}$$

for almost all $x \in [0;l]$ and I is the identity matrix in $\mathbb{C}^n$. Then

$$
\begin{aligned}
&P_1AP_1f(x) = \alpha(x)f(x)\Pi(x)J_1Q(x) + i\int_0^x f(\xi)\Pi(\xi)J_1\Pi^*(x)d\xi,\\
(2.12)\qquad &P_2AP_2f(x) = \alpha(x)f(x)\Pi(x)J_2Q(x) - i\int_0^x f(\xi)\Pi(\xi)J_2\Pi^*(x)d\xi,\\
&P_1AP_2f(x) = i\int_0^l f(\xi)\Pi(\xi)S\Pi^*(x)d\xi, \quad P_2AP_1f(x) = 0.
\end{aligned}
$$

The operator $A_1 = P_1A$ is a dissipative operator onto the subspace $H_1 = P_1\mathbf{L}^2(0,l;\mathbb{C}^n)$, $A_2 = P_2A$ is an antidissipative one onto the subspace $H_2 = P_2\mathbf{L}^2(0,l;\mathbb{C}^n)$ and the subspace H_1 is an invariant subspace of the operator A. In other words the operator A is a coupling of the operators A_1 and A_2: $A = A_1 \vee A_2$.

Conversely, an arbitrary linear bounded nonselfadjoint operator $C \in \Omega_{\mathbb{R}}$, presented as a coupling of a dissipative operator and an antidissipative one with real spectra determined by a bounded nondecreasing function $\alpha : [0;l] \longrightarrow \mathbb{R}$ is unitary equivalent to the model (1.2) (on the principal subspace) with appropriate matrix functions $\Pi(x)$, $B(x)$ and L, and $B(x)$ satisfies the condition (2.8).

Let us denote by $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ the set of all operators $A \in \mathbf{\Omega}_{\mathbb{R}}$ with the representation (1.2) (up to the unitary equivalence), satisfying the condition (2.8). This class describes nondissipative curves with basic operators from $\mathbf{\Omega}$ with real spectra having a limit of the corresponding correlation function as $t \to \pm\infty$. Then the complete characteristic function of the colligation (2.4) with $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ has the form (see [1])

$$
W(\lambda) = \overset{l}{\underset{0}{\overrightarrow{\int}}} e^{\frac{iJ_2B(\theta)J_2}{\lambda-\alpha(\theta)}Ld\theta} \overset{l}{\underset{0}{\overrightarrow{\int}}} e^{\frac{iJ_1B(\theta)J_1}{\lambda-\alpha(\theta)}Ld\theta}.
$$

We shall be considering only operators from the class $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ with an absolutely continuous spectrum. This means that the inverse function $\sigma(u)$ of the function $\alpha(x)$ generates a measure ν which is an absolutely continuous measure with respect to the Lebesgue measure (the singular part of ν is 0), i.e. the function $\sigma(u)$ is absolutely continuous.

We will present the asymptotics of the nondissipative curves $e^{itA}f(x)$, $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$, generated by the operators $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ with an absolutely continuous spectra ([12]). In the course of the obtaining of these asymptotics we have used the existence and the form of the limits

$$(2.13)\qquad s-\lim_{\delta\to 0}\overset{\overrightarrow{b}}{\underset{a}{\int}} e^{\frac{-iT(v)}{v-(x\pm i\delta)}dv} = s-\lim_{\delta\to 0}\overset{\overrightarrow{x-\delta}}{\underset{a}{\int}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(v)} \overset{\overrightarrow{b}}{\underset{x+\delta}{\int}} e^{\frac{-iT(v)}{v-x}dv}$$

(for almost all $x\in[a;b]$) for an integrable nonnegative matrix function $T(v)$ and some corollaries and results of this formula (2.13) that have been considered by L.A.Sakhnovich in [27] ($s-\lim$ denotes a strong limit). The formula (2.13) is an analogue for multiplicative integrals of the well-known Privalov's theorem [24] for the limit values for the integral $f(\lambda)=\int\limits_a^b \frac{p(t)}{t-\lambda}dt$ in the scalar case.

In this part we will denote by $\|\ \ \|$ the norm of a matrix function in $\mathbb{C}^n$ and by $\|\ \ \|_{L^2}$ – the norm in $\mathbf{L}^2(0,l;\mathbb{C}^n)$.

We recall that a matrix function $T(x)$ is said to be a matrix function from the class $C_\alpha[a;b]$ $(\alpha>0)$ if $T(x)$ is an integrable nonnegative or nonpositive matrix function on $[a;b]$ and satisfies the condition

$$\|T(x_1)-T(x_2)\|\le C|x_1-x_2|^\alpha$$

for some constant $C>0$ and for all $x_1,x_2\in[a;b]$.

For a nonnegative (nonpositive) matrix function $T(x)\in C_\alpha[a;b]$ $(\alpha>0)$ let us denote the next operators

$$(2.14)\qquad F_w^{\pm}(x,u)=s-\lim_{\delta\to 0}\overset{\overrightarrow{u}}{\underset{w}{\int}} e^{\frac{-iT(v)}{v-(x\pm i\delta)}dv},$$

$$(2.15)\qquad P_w(x,u)=F_w^{+}(x,u)-F_w^{-}(x,u)$$

for all w,u,x such that $a\le w<u\le b$, $a\le x\le b$ and

$$(2.16)\qquad F_w^{\pm}(x,u)=s-\lim_{\delta\to 0}\overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \overset{\overrightarrow{u}}{\underset{x+\delta}{\int}} e^{\frac{-iT(v)}{v-x}dv},$$

$$(2.17)\qquad \begin{aligned} R_w^{\pm 1}(x) &= (F_w^{\pm}(x,u)(F_w^{\pm}(x,u))^*)^{\frac{1}{2}} = \\ &= s-\lim_{\delta\to 0}\overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{\frac{-iT(v)}{v-x}dv} e^{\pm\pi T(x)} \left(\overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{\frac{-iT(v)}{v-x}dv}\right)^{-1}, \end{aligned}$$

$$U_w(x,u) = R_w^{\mp 1}(x) F_w^{\pm}(x,u) = s - \lim_{\delta \to 0} \overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{\frac{-iT(v)}{v-x} dv} \overset{\overrightarrow{u}}{\underset{x+\delta}{\int}} e^{\frac{-iT(v)}{v-x} dv}, \tag{2.18}$$

$$U_{1w}(x,u) = \lim_{\delta \to 0} \overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{\frac{-iT(v)}{v-x} dv} e^{-i \int\limits_{x+\delta}^{u} \frac{T(x)}{v-x} dv}, \tag{2.19}$$

$$U_{2w}(x) = \lim_{\delta \to 0} \overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{\frac{-iT(v)}{v-x} dv} e^{i \int\limits_{w}^{x-\delta} \frac{T(x)}{v-x} dv}, \tag{2.20}$$

$$P_{2w}(x,u) = (R_w(x) - R_w^{-1}(x)) U_{2w}(x) e^{-iT(x) \ln \frac{u-x}{x-w}}, \tag{2.21}$$

$$Q_w(x,u) = P_{2w}(x,u) e^{iT(x) \ln(u-x)} e^{-iT(u) \ln(u-x)}, \tag{2.22}$$

$$Q_w(x) = P_{1w}(x) e^{iT(x) \ln(x-w)} , \tag{2.23}$$

$$P_{1w}(x) = (R_w(x) - R_w^{-1}(x)) U_{2w}(x) \tag{2.24}$$

for all w, u, x such that $a \le w < x < u \le b$. (The existence of these limits follows from the formula (2.13) about the limit values for multiplicative integrals ([27]).)

Using these notations for $T(x) \in C_\alpha[a;b]$ $(0 < \alpha \le 1)$ we shall recall several inequalities obtained by L. A. Sakhnovich in [27] which we will use in our proofs:

$$\|U_w(x,u) - U_{1w}(x,u)\| \le \int\limits_x^u \frac{\|T(x) - T(v)\|}{|x-v|} dv \tag{2.25}$$

for all w, u, x such that $a \le w \le x \le u \le b$,

$$\|U_{2a}(x_1) - U_{2a}(x_2)\| \le C \left(\frac{x_2 - x_1}{x_1 - a} \right)^{\alpha'}, \tag{2.26}$$

$$\|R_a(x_1) - R_a(x_2)\| \le C\left(\frac{x_2 - x_1}{x_1 - a}\right)^{\alpha'}, \tag{2.27}$$

$$\|F_a^{\pm}(x_1, b) - F_a^{\pm}(x_2, b)\| \le C\left(\left(\frac{x_2 - x_1}{x_1 - a}\right)^{\alpha'} + \left(\frac{x_2 - x_1}{b - x_2}\right)^{\alpha'}\right), \tag{2.28}$$

for all $x_1, x_2 \ : \ a \le x_1 < x_2 \le b$ where $C > 0$ is a suitable constant and $\alpha' = \alpha/(1+\alpha)$.

From the limit formula (2.13) for multiplicative integrals follows the next representation

$$\overset{\overset{b}{\rightarrow}}{\int_a} e^{\frac{-iT(v)}{v-\lambda}dv} = I + \frac{1}{2\pi i}\int_a^b \frac{P_a(x,b)}{x-\lambda}dx \tag{2.29}$$

for a nonnegative (nonpositive) matrix function $T(x) \in C_\alpha[a;b]$ $(0 < \alpha \le 1)$ for each $\lambda \in \mathbb{C} \setminus [a;b]$. This representation we use in the course of obtaining of the asymptotics of the continuous curves, generated by the model (1.2).

We shall recall also the next inequalities, obtained by the authors in [12] (Lemma 1, Lemma 2, Lemma 3) for a nonnegative (nonpositive) matrix function $T(x) \in C_\alpha[a;b]$ $(0 < \alpha \le 1)$:

$$\left\| e^{iT(x)\ln(u-x)} - e^{iT(u)\ln(u-x)} \right\| \le C(u-x)^{\alpha'} \tag{2.30}$$

for u, x: $a \le x < u \le b$ and $\forall \alpha' : 0 < \alpha' < \alpha$,

$$\|P_w(x,u) - Q_w(x,u)\| \le C(u-x)^{\alpha'} \tag{2.31}$$

for all w, u, x : $a \le w < x < u \le b$ and $\forall \alpha' : 0 < \alpha' < \alpha$,

$$\|Q_w(x_1) - Q_w(x_2)\| \le C\left(\frac{x_2 - x_1}{x_1 - w}\right)^{\alpha'} \tag{2.32}$$

for all $x_1, x_2 \ : \ a \le w < x_1 < x_2 \le b$ and $0 < \alpha' < \alpha$.

For a selfadjoint matrix function $T(u)$ in $\mathbb{C}^m$ we introduce the analogue in $\mathbb{C}^m$ of the classical gamma-function

$$\mathbf{\Gamma}(\varepsilon I - iT(u)) = \int_0^{+\infty} e^{-x} e^{((\varepsilon-1)I - iT(u))\ln x} dx \tag{2.33}$$

for $\varepsilon > 0$ and we present some properties which we have used for the asymptotic behaviour of the considered nondissipative curves in this part and the other parts of the paper (see [12], Lemma 4, Lemma 5, Lemma 6, Lemma 7).

Lemma 2.1. *If $T(u)$ is a selfadjoint operator in $\mathbb{C}^m$ then there hold the next equalities*

$$\int_0^{\infty e^{i\varphi}} e^{-z} e^{((\varepsilon-1)I - iT(u))\ln z} dz = \mathbf{\Gamma}(\varepsilon I - iT(u)) \tag{2.34}$$

for all φ: $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ and for all $\varepsilon\ :\ 0 < \varepsilon < 1$.

The equalities (2.34) have been obtained by the Cauchy's theorem for the operator function $G(z) = e^{-z} e^{((\varepsilon-1)I - iT(u))\ln z}$ and a suitable domain in $\mathbb{C}$.

Lemma 2.2. *If T is an arbitrary selfadjoint operator in $\mathbb{C}^m$ then the next equalities hold:*

$$\mathbf{\Gamma}(\varepsilon I - iT)\mathbf{\Gamma}(I - \varepsilon I + iT)\sin(\pi(\varepsilon I - iT)) = \pi I \tag{2.35}$$

for each $\varepsilon : 0 < \varepsilon < 1$ and

$$\lim_{\varepsilon \to 0} \mathbf{\Gamma}(\varepsilon I - iT) = \pi i \mathbf{\Gamma}^{-1}(I + iT)(\sinh(\pi T))^{-1}. \tag{2.36}$$

For future reference we will call a matrix function $T(x)$ on $[0; l]$ a smooth matrix function if $T(x)$ is differentiable and $T'(x)$ is continuous on $[0; l]$ (by norm in $\mathbb{C}^n$).

In this part we will assume that the matrix function $Q^*(x)$ is smooth on $[0; l]$, $B(x) \in C_{\alpha_1}[0; l]$ $(0 < \alpha_1 \leq 1)$ and the function $\alpha : [0; l] \longrightarrow \mathbb{R}$ satisfies the conditions:

(i) $\alpha(x)$ is continuous strictly increasing on $[0; l]$;

(ii) the inverse function $\sigma(u)$ of $\alpha(x)$ is absolutely continuous on $[a; b]$ (where $a = \alpha(0),\ b = \alpha(l)$);

(iii) $\sigma'(u)$ is continuous and satisfies the condition

$$|\sigma'(u_1) - \sigma'(u_2)| \leq C|u_1 - u_2|^{\alpha_2} \ \ (0 < \alpha_2 \leq 1) \tag{2.37}$$

for all $u_1,\ u_2 \in [a; b]$ and for some constant $C > 0$.

For our further applications we shall denote the matrix functions defined by (2.14) (or (2.16)), (2.15), (2.17), (2.18), (2.19), (2.20), (2.21), (2.22), (2.23), (2.24) with $F_w^{\pm}(x, u)$, $P_w(x, u)$, $R_w^{\pm 1}(x)$, $U_w(x, u)$, $U_{1w}(x, u)$, $U_{2w}(x)$, $P_{2w}(x, u)$, $Q_w(x, u)$, $Q_w(x)$, $P_{1w}(x)$ respectively for the nonnegative matrix function $T(x) =$

$J_1B(\sigma(x))J_1\sigma'(x)$ on $[a;b]$ and with $\widetilde{F}_w^{\pm}(x,u)$, $\widetilde{P}_w(x,u)$, $\widetilde{R}_w^{\pm 1}(x)$, $\widetilde{U}_w(x,u)$, $\widetilde{U}_{1w}(x,u)$, $\widetilde{U}_{2w}(x)$, $\widetilde{P}_{2w}(x,u)$, $\widetilde{Q}_w(x,u)$, $\widetilde{Q}_w(x)$, $\widetilde{P}_{1w}(x)$ respectively for the non-positive matrix function $T(x) = -J_2B(\sigma(x))J_2\sigma'(x)$ on $[a;b]$.

For the simplification of the writing suppose that the initial function

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x)) \in \mathbf{L}^2(0,l;\mathbb{C}^n)$$

is chosen from the dense set H_0 in $\mathbf{L}^2(0,l;\mathbb{C}^n)$ such that there exist $f_k'(x) \in \mathbf{L}^2(0,l)$, $k = 1,2,\dots,n$, and $f(0) = (0,0,\dots,0)$.

The next theorem gives the asymptotics of the projections of the curve $e^{itA}f$ as $t \to \pm\infty$ on the dense set in $\mathbf{L}^2(0,l;\mathbb{C}^n)$ (see [12]).

Theorem 2.3. *Let for the model $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$, defined by (1.2), the next conditions hold:*

1) *the function $\alpha : [0;l] \longrightarrow \mathbb{R}$ satisfies* **(i)**, **(ii)**, **(iii)**;
2) *$Q^*(x)$ is a smooth matrix function on $[0;l]$;*
3) *$B(x) \in C_{\alpha_1}[0;l]$ $(0 < \alpha_1 \le 1)$.*

Then the curves $P_1e^{itA}P_1f(x)$, $P_2e^{itA}P_2f(x)$, $P_1e^{itA}P_2f(x)$ for each $f \in H_0$ after the change of the variable $x = \sigma(u)$ have the next asymptotics

$$\|P_1e^{itA}P_1f(\sigma(u)) - e^{itu}\int_a^u \widetilde{f}'(w)U_{2w}(u)(u-w)^{i\widetilde{B}_1(u)}dw\cdot \\ \cdot|t|^{i\widetilde{B}_1(u)}e^{\mp\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))J_1\Pi^*(\sigma(u))\|_{L_2} \to 0, \tag{2.38}$$

$$\|P_2e^{itA}P_2f(\sigma(u)) - e^{itu}\int_a^u \widetilde{f}'(w)\widetilde{U}_{2w}(u)(u-w)^{-i\widetilde{B}_2(u)}dw\cdot \\ \cdot|t|^{-i\widetilde{B}_2(u)}e^{\pm\frac{\pi}{2}\widetilde{B}_2(u)}\mathbf{\Gamma}^{-1}(I-i\widetilde{B}_2(u))J_2\Pi^*(\sigma(u))\|_{L_2} \to 0, \tag{2.39}$$

$$\Big\|P_1e^{itA}P_2f(\sigma(u)) + \\ + e^{itu}\int_a^b \widetilde{f}'(w)(\widetilde{F}_w^{\mp}(u,b)-I)dwSU_{2a}(u)(u-a)^{i\widetilde{B}_1(u)}\cdot \\ \cdot |t|^{i\widetilde{B}_1(u)}e^{\mp\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))J_1\Pi^*(\sigma(u))\Big\|_{L_2} \to 0 \tag{2.40}$$

as $t \to +\infty$, where

$$\widetilde{f}(w) = f(\sigma(w))Q^*(\sigma(w)), \tag{2.41}$$

$$\widetilde{B}_k(u) = J_k B(\sigma(u)) J_k \sigma'(u), \quad k = 1, 2. \tag{2.42}$$

For the obtaining of the asymptotics (2.38), (2.39) and (2.40) we have used suitable representation of the projections of the curve $e^{itA}f$ after the change of the variable $x = \sigma(u)$ (see [12]):

$$P_1 e^{itA} P_1 f(\sigma(u)) = = -\frac{1}{2\pi i}\int\limits_{\gamma} \frac{e^{it\lambda}}{u-\lambda}\left(\int\limits_a^u \widetilde{f}'(w) \int\limits_w^{\overset{u}{\rightarrow}} e^{\frac{i}{\lambda - v}\widetilde{B}_1(v)dv} J_1 \Pi^*(\sigma(u)) dw\right) d\lambda, \tag{2.43}$$

$$P_2 e^{itA} P_2 f(\sigma(u)) = = -\frac{1}{2\pi i}\int\limits_{\gamma} \frac{e^{it\lambda}}{u-\lambda}\left(\int\limits_a^u \widetilde{f}'(w) \int\limits_w^{\overset{u}{\rightarrow}} e^{\frac{i}{v - \lambda}\widetilde{B}_2(v)dv} J_2 \Pi^*(\sigma(u)) dw\right) d\lambda, \tag{2.44}$$

$$\begin{aligned} P_1 e^{itA} P_2 f(\sigma(u)) = \\ = -\frac{1}{2\pi i}\int\limits_a^b \widetilde{f}'(w) \lim_{\varepsilon\to 0}\left(\int\limits_w^b \frac{e^{itx}}{(u-x)^{1-\varepsilon}} \widetilde{P}_w(x,b) S F_a^+(x,u) dx + \right. \\ \left. + \int\limits_a^u \frac{e^{itx}}{(u-x)^{1-\varepsilon}} (\widetilde{F}_w^-(x,b) - I) S P_a(x,u) dx\right) dw J_1 \Pi^*(\sigma(u)). \end{aligned} \tag{2.45}$$

for $f \in \widetilde{H}_0$, where γ is an arbitrary closed contour containing $[\alpha(0); \alpha(l)]$ ([1]). We have also used the properties of the multiplicative integrals and the gamma-function, the limits from the form:

$$\lim_{\varepsilon\to 0} \varepsilon \int\limits_w^u \left(\int\limits_x^u (\eta - x)^{\varepsilon - 1} P_w(x,\eta) d\eta\right) dx = 0, \tag{2.46}$$

$$\lim_{\varepsilon\to 0} \int\limits_w^u (u-x)^{\varepsilon-1} P_w(x,u) dx J_1 \Pi^*(\sigma(u)) = 2\pi i J_1 \Pi^*(\sigma(u)), \tag{2.47}$$

obtained in [12], the inequalities (2.30), (2.31), (2.32), the relations from the form

$$\lim_{\varepsilon\to 0}\int\limits_{w}^{u}\frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-i\widetilde{B}_1(u)\ln(u-x)}dx \sim \tag{2.48}$$
$$\sim \pi i e^{itu}t^{i\widetilde{B}_1(u)}e^{-\frac{\pi}{2}\widetilde{B}_1(u)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}_1(u))(\sinh(\pi\widetilde{B}_1(u)))^{-1}$$

as $t\to+\infty$ and other similar inequalities and relations.

It has to mention that the relations (2.38) and (2.39) present the asymptotics of the curve $e^{itT}f$ as $t\to\pm\infty$, where T is a dissipative and an antidissipative operator correspondingly

$$Tf(x)=\alpha(x)f(x)\pm i\int\limits_0^x f(\xi)\Pi(\xi)\Pi^*(x)d\xi.$$

In the dissipative case under somewhat different assumptions and using different ways a result similar to (2.38) has been obtained by L. A. Sakhnovich in [27].

Let us consider now the dense set $\widetilde{H}_0$ in $\mathbf{L}^2(0,l;\mathbb{C}^n)$ such that $f_k'(x)\in\mathbf{L}^2(0,l)$ $(k=1,2,\dots,n)$, $f(0)=(0,0,\dots,0)$, $f(l)=(0,0,\dots,0)$.

For the convenience of the writing let us use the denotations (1.9), (1.10), (1.8), (1.11) for each $f\in H_0$ and each $h\in\mathbb{C}^m$. Then using the properties of the multiplicative integrals we present the operators $S_\pm$ and $\widetilde{S}_\pm$ in $\widetilde{H}_0$ in the form (1.6) and (1.7)

Generalizing the results in Theorem 2.3 we are in a position to give the asymptotics of the nondissipative curve $e^{itA}f$ for the considered model A using the introduced denotations. These asymptotics are given by Theorem 1.1.

The obtained asymptotics (1.5) of the curve $e^{itA}f$ allow to consider the behaviour of the corresponding correlation function. The next theorem follows from the form of matrix functions $\widetilde{B}_1(u)$, $\widetilde{B}_2(u)$, the asymptotics (1.5) and straightforward calculations.

Theorem 2.4. *Let for the model $A\in\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$, defined by* (1.2), *the next conditions hold*

1) *the function $\alpha:[0;l]\longrightarrow\mathbb{R}$ satisfies* **(i)**, **(ii)**, **(iii)**;

2) *$Q^*(x)$ is a smooth matrix function on $[0;l]$;*

3) *$B(x)\in C_{\alpha_1}[0;l]$ $(0<\alpha_1\le 1)$.*

Then there exists the limit $\lim\limits_{\tau\to\pm\infty}V(t+\tau,s+\tau)$ for the nondissipative curve $e^{itA}f$

in the subspace H_0 *and after the change of the variable* $x = \sigma(u)$

$$\lim_{\tau\to\pm\infty} V(t+\tau, s+\tau) =$$

$$= \int_a^b e^{i(t-s)}\widetilde{S}_\pm(f(\sigma(u)))(\widetilde{S}_\pm(f(\sigma(u))))^*\sigma'(u)du = \tag{2.49}$$

$$= (e^{itu}\widetilde{S}_\pm f(\sigma(u)), e^{isu}\widetilde{S}_\pm f(\sigma(u)))$$

for all $t, s \in \mathbb{R}$ *where* $V(t,s) = (e^{itA}f, e^{isA}f)$ *is the correlation function of the curve* $e^{itA}f$ *and the operators* $\widetilde{S}_\pm$ *are defined for all* $f \in H_0$ *by* (1.7)*.*

The next theorem allows to extend the relations (1.5) and (2.49) in the whole space $\mathbf{L}^2(0,l;\mathbb{C}^n)$.

Theorem 2.5. *The operators* $S_\pm$ *and* $\widetilde{S}_\pm$*, defined by* (1.6) *and* (1.7)*, are bounded linear operators in the subspace* $\widetilde{H}_0$*.*

The boundedness of the operators $S_\pm$ and $\widetilde{S}_\pm$ follows from the boundedness of $\widetilde{S}_{11}$, $\widetilde{S}_{22}$ and $\widehat{S}_{12}^\pm$ which follows from the dissipative operator P_1AP_1 and the antidissipative operator P_2AP_2, based on the obtained asymptotics (2.38) and (2.39) correspondingly. For the obtaining of the boundedness of $\widehat{S}_{12}^\pm$ we have also used the boundedness of the operator

$$G(f(\sigma(u)) = \int_u^b \widetilde{f}'(w) \int_w^{\overset{b}{\rightarrow}} e^{\frac{i\widetilde{B}_2(v)}{v-w}dv} dwJ_2 \tag{2.50}$$

and the suitable representation of $\widehat{S}_{12}^\pm$ in the form

$$\widehat{S}_{12}^\pm f(\sigma(u)) = -\left(\int_a^u \widetilde{f}'(w) \int_a^{\overset{w}{\leftarrow}} e^{\frac{-i\widetilde{B}_2(v)}{v-u}dv} dw\widetilde{F}_a^\mp(u,b) + \int_u^b \widetilde{f}'(w) \int_w^{\overset{b}{\rightarrow}} e^{\frac{i\widetilde{B}_2(v)}{v-u}dv} dw \right) S.$$

The next theorem shows the uniformly boundedness of the semigroups $\{e^{itA}\}_{t\geq 0}$ and $\{e^{itA}\}_{t\leq 0}$ on $\widetilde{H}_0$ which implies that that $\{e^{itA}\}$ are semigroups from the class (C_0).

Theorem 2.6. *For the model* $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$*, defined by* (1.2)*, the family of operators* $\{e^{itA}\}$ *is uniformly bounded.*

The proof of this theorem is based on the representation

$$(2.51) \qquad \Phi e^{itA} f = \int_a^b (e^{itA} f(\sigma(u)))\Pi(\sigma(u))\sigma'(u)du = \frac{1}{2\pi}\int_a^b e^{itu}\widehat{G}f(\sigma(u))du$$

for each $f \in \widetilde{H}_0$ where

$$\widehat{G}f(\sigma(u)) = (\widetilde{S}_{11}f(\sigma(u)))P_a(u,b) - (\widetilde{S}_{22}f(\sigma(u)))\widetilde{P}_a(u,b) -$$
$$- \int_a^b \widetilde{f}'(w)(\widetilde{F}_w^+(u,b)S(F_a^+(u,b) - I) - \widetilde{F}_w^-(u,b)S(F_a^-(u,b) - I))dw,$$

the inequality

$$(2.52) \qquad \|e^{itA}f\|^2_{L^2} \le \|f\|^2_{L^2} + \frac{1}{2\pi}\|L\|.\|\widehat{G}f\|^2_{L^2}$$

for each $f \in \widetilde{H}_0$ and the boundedness of $\widehat{G}$ in $\widetilde{H}_0$.

Now from Theorem 2.5 it follows that the operators $S_\pm$ and $\widetilde{S}_\pm$ can be extended by continuity in $\mathbf{L}^2(0,l;\mathbb{C}^n)$. Then using (2.52) for all $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$ the next relations hold

$$(2.53) \qquad \left\|e^{itA}f(\sigma(u)) - e^{itu}S_\pm f(\sigma(u))\right\|_{L^2} \to 0 \quad as\ t \to \pm\infty\ ,$$

$$(2.54) \qquad \lim_{\tau\to\pm\infty} V(t+\tau, s+\tau) = (e^{itu}\widetilde{S}_\pm f(\sigma(u)), e^{isu}\widetilde{S}_\pm f(\sigma(u))).$$

It is important to mention the next fact: it can be shown that the operators $\widetilde{S}_+$ and $\widetilde{S}_-$ do not depend on the choice of the matrix function $Q(x)$ satisfying the condition (2.11) on $[0;l]$. (We have used this matrix function $Q(x)$ to define the orthogonal projectors (2.10) which present the considered model A as a suitable coupling of a dissipative operator and an antidissipative one.)

3. Wave operators and a scattering operator for the couple $(\mathbf{A}^*, \mathbf{A})$ with $\mathbf{A} \in \widetilde{\boldsymbol{\Omega}}_{\mathbb{R}}$. The obtained asymptotics (2.53) for the nondissipative curve generated by the operator A from the class $\widetilde{\boldsymbol{\Omega}}_{\mathbb{R}}$ allow us to consider the basic terms from the scattering theory: wave operators and a scattering operator for the couple (A^*, A) as in the selfadjoint case [25, 9, 10] and in the dissipative case [27]. Using the introduced wave operators of A we will show that the operator

A is similar to the operator of a multiplying by the independent variable in a subspace of $\mathbf{L}^2(0,l;\mathbb{C}^n)$ after a change of the variable.

Let $A \in \widetilde{\mathbf{\Omega}}_{\mathbb{R}}$, L, $\Pi(x)$, $Q(x)$, $B(x)$, $\alpha(x)$, the colligation X are stated as in part 2.

The form and the existence of the wave operators as storng limits for the couple (A^*, A) are presented by Theorem 1.3 and Theorem 1.4 with the help of the explicit form of the asymptotics (2.53) for the nondissipative curve $e^{itA}f$.

The equality (1.17) follows immediately from (2.54) and the equality (1.18) follows from the proved equality

$$(e^{itu}\widetilde{S}_{\pm}Af, e^{isu}\widetilde{S}_{\pm}f) = (ue^{itu}\widetilde{S}_{\pm}f, e^{isu}\widetilde{S}_{\pm}f)$$

for all $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$ and all $t, s \in \mathbb{R}$.

If we denote $(W_{\pm}(A^*, A)f, g) = \lim\limits_{t\to\pm\infty}(e^{itA^*}e^{-itA}f, g)$ then the operators

$$W_{\pm}(A^*, A) = \widetilde{S}^*_{\mp}\widetilde{S}_{\mp}$$

are the wave operators as weak limits. Now from (1.18) it follows the similarity of A and A^*, given by the wave operators:

$$A^*W_{\pm}(A^*, A) = W_{\pm}(A^*, A)A.$$

The existence of the wave operators as strong limits follows from the colligation condition (2.6), the relations

$$\|W(t_2)f - W(t_1)f\|^2_{\mathbf{L}^2} = \left\| \int_{t_1}^{t_2} e^{i\tau A^*}\frac{A-A^*}{i}e^{-i\tau A}f\,d\tau \right\|^2_{\mathbf{L}^2} =$$
$$= \left\| \int_{t_1}^{t_2} \sum_{\alpha,\beta=1}^{m} (e^{-i\tau A}f, g_\alpha)(Le_\alpha, e_\beta)e^{i\tau A^*}g_\beta(x)\,d\tau \right\|^2_{\mathbf{L}^2} \leq$$
$$\leq M_1 \sum_{\alpha,\beta=1}^{m} |(Le_\alpha, e_\beta)|^2 \int_{t_1}^{t_2} |(\Phi e^{-i\tau A}f, e_\alpha)|^2 d\tau \int_{t_1}^{t_2} \|e^{i\tau A^*}\Phi^* e_\beta\|^2 d\tau$$

(M_1 is a suitable constant, $t_1, t_2 \in \mathbb{R}$, $f \in \mathbf{L}^2(0,l;\mathbb{C}^n)$) for the operator function $W(t) = e^{itA^*}e^{-itA}$, where $\{e_\alpha\}_1^m$ is an orthonormal basis in $\mathbb{C}^n$, $g_\alpha(x) = \Phi^* e_\alpha = e_\alpha \Pi^*(x)$ $(x \in [0;l])$, $\alpha = 1, 2, \ldots, m$, are channel elements of A. Then the integrability of $\|e^{itA^*}\Phi^* e_\beta\|^2_{\mathbf{L}^2}$ and $\|\Phi e^{-itA}f\|^2$ $(\forall f \in \mathbf{L}^2(0,l;\mathbb{C}^n))$ shows that

$$\|W(t_2)f - W(t_1)f\|_{\mathbf{L}^2} \to 0 \quad \textit{as} \;\; t_1, t_2 \to \pm\infty$$

which together with the uniformly bounded set of operators $W(t)$, $t \in \mathbb{R}$, implies that there exist the strong limits

$$s - \lim_{t\to\pm\infty} W(t) = s - \lim_{t\to\pm\infty} e^{itA^*}e^{-itA}$$

onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$.

The next theorem shows that the operator A from the class $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ is similar to the operator $\mathcal{Q}$ of a multiplying by the independent variable in a suitable subspace of $\mathbf{L}^2(0,l;\mathbb{C}^m)$ (after the change of the variable $x = \sigma(u)$). This allows us to define a scattering operator analogously as in the selfadjoint case and in the dissipative case.

Before continuing with the next theorem it should be noted that the inverse operator $T_\pm^{-1}$ of $T_\pm$ is defined by

$$\begin{aligned} T_\pm^{-1}h = h\Pi(\sigma(u))\sigma'(u)\cdot \\ \cdot\Big(J_1 i\lim_{\varepsilon\to 0}\mathbf{\Gamma}(\varepsilon I + i\widetilde{B}_1(u))e^{\pm\frac{\pi}{2}\widetilde{B}_1(u)}(u-a)^{-i\widetilde{B}_1(u)}U_{2a}^*(u)J_1 + \\ + J_2(-i)\lim_{\varepsilon\to 0}\mathbf{\Gamma}(\varepsilon I - i\widetilde{B}_2(u))e^{\mp\frac{\pi}{2}\widetilde{B}_2(u)}(u-a)^{i\widetilde{B}_2(u)}\widetilde{U}_{2a}^*(u)J_2\Big) \end{aligned} \tag{3.1}$$

for $h \in \mathbb{C}^n$.

Let us denote the next range of the operator $\widetilde{S}$

$$Y_1 = R(\widetilde{S}) = \Big\{g \in \mathbf{L}^2(0,l;\mathbb{C}^m) : \widetilde{S}f = g \quad \text{for some } f \in \mathbf{L}^2(0,l;\mathbb{C}^n)\Big\},$$

and $Y = \overline{Y_1}$ is the closure of Y_1.

Theorem 3.1. *The operators $\widehat{S}_\pm$ in $\mathbf{L}^2(0,l;\mathbb{C}^n)$ have bounded inverse operators $\widehat{S}_\pm^{-1}$ defined in Y and $A = \widehat{S}_\pm^{-1}\mathcal{Q}\widehat{S}_\pm$ onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$.*

Straightforward calculations and the obtained asymptotics (2.38), (2.39) show that the operators $G_\pm = G_{11} + G_{22} + G_{12}^\pm$ where

$$G_{11}g(\sigma(u)) = \frac{1}{2\pi}\frac{d}{du}\int_a^u g(\sigma(\tau))P_a(\tau,u)d\tau J_1 Q(\sigma(u))(\sigma'(u))^{-1},$$

$$G_{22}g(\sigma(u)) = -\frac{1}{2\pi}\frac{d}{du}\int_a^u g(\sigma(\tau))\widetilde{P}_a(\tau,u)d\tau J_2 Q(\sigma(u))(\sigma'(u))^{-1},$$

$$G_{12}^\pm g(\sigma(u)) = -G_{11}\widetilde{S}_{12}^\pm G_{22}g(\sigma(u))$$

in the subspace of all functions $g \in \mathbf{L}^2(0,l;\mathbb{C}^m)$ such that $|g'(x)| \leq C$ in $[0;l]$ (where $C > 0$ is a constant) are bounded inverse operators of $\widehat{S}_\pm$ and

$$A = \widehat{S}_\pm^{-1} \mathcal{Q} \widehat{S}_\pm \tag{3.2}$$

onto $\mathbf{L}^2(0,l;\mathbb{C}^n)$.

The proved equality (3.2) implies that the operator A which we consider after the change of the variable $x = \sigma(u)$ is similar to the operator $\mathcal{Q}$ of a multiplying by the independent variable u in the space Y.

The representation (3.2) allows us to introduce a scattering operator for the couple (A^*, A) using (1.7)

$$W_-^{-1}(A^*,A)W_+(A^*,A) = (\widetilde{S}_+^*\widetilde{S}_+)^{-1}\widetilde{S}_-^*\widetilde{S}_- = \widehat{S}_+^{-1}(T_+^*T_+)^{-1}T_-^*T_-\widehat{S}_-.$$

The obtained asymptotics of the curve $e^{itA}f$ allow to give an explicit form of the corresponding correlation function.

Theorem 3.2. *Let for the model $A \in \widetilde{\mathbf{\Omega}}_\mathbb{R}$, defined by* (1.2)*, the next conditions hold:*

1) *the function $\alpha : [0;l] \longrightarrow \mathbb{R}$ satisfies* **(i)**, **(ii)**, **(iii)**;
2) *$Q^*(x)$ is a smooth matrix function on $[0;l]$;*
3) *$B(x) \in C_{\alpha_1}[0;l]$ $(0 < \alpha_1 \leq 1)$.*

Then the correlation function $V(t,s)$ of the curve $e^{itA}f$ $(f \in \mathbf{L}^2(0,l;\mathbb{C}^n))$ has the representation

$$V(t,s) = (e^{i(t-s)u}\widetilde{S}_\pm f(\sigma(u)), \widetilde{S}_\pm f(\sigma(u)) +$$
$$+ \int_0^{\pm\infty} \sum_{\alpha,\beta=1}^m \Psi_\alpha(t+\tau)(Le_\alpha, e_\beta)\overline{\Psi_\beta(s+\tau)}d\tau,$$

where $\Psi_\alpha(t) = (e^{itu}\widehat{S}_\pm f, \widehat{S}_\pm^{-1^*} g_\alpha)$, $\alpha = 1,2,\ldots,m$, $\{e_\alpha\}_1^m$ is an orthonormal basis in $\mathbb{C}^m$ and $g_\alpha(x) = \Phi^* e_\alpha = e_\alpha \Pi^*(x)$ $(x \in [0,l]$, $\alpha = 1,2,\ldots,m)$ are the channel elements of the colligation X.

We have only considered the nonselfadjoint operator A with a finite dimensional imaginary part. Analogously one can find the asymptotics of the curve $e^{itA}f$ in the case when the imaginary part $(A - A^*)/i$ of A belongs to the trace class.

In order to conclude this part it should be remarked that one can find the asymptotics of nondissipative curves $e^{itA}f$ $(A \in \widetilde{\mathbf{\Omega}}_\mathbb{R})$ and apply in the scattering theory in the case of weaker assumptions for $B(x)$ and $\alpha(x)$. But this will give some technical complications of the consideration.

4. Triangular model and asymptotics of nondissipative curves with unbounded semigroup generators from the class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$. In this part we present nondissipative curves generated by a class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ of unbounded nondissipative operators and having asymptotics. The generators from $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ are unbounded operators A with a domain $D_A = D_{A^*}$, a finite dimensional imaginary part and presented as a coupling of a dissipative operator and an antidissipative one with real absolutely continuous spectra. We present the triangular model and basic terms and problems from the single operator colligation theory for the operators $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$.

In an appropriate way we introduce two families of operators from the class (C_0). These semigroups determine the exponential function e^{itA} for A. The obtaining of the asymptotics of the continuous nondissipative curves $e^{itA}f$ as $t \to \pm\infty$ and the limit of the corresponding correlation function in an explicit form allows to construct the scattering theory for the couple (A^*, A) (i.e. to consider a perturbation of a closed operator A with an operator $i\widetilde{A}$, where $\widetilde{A}$ is a finite dimensional selfadjoint operator).

The results of this part and the next part are presented by the authors in [13].

Let H be a separable Hilbert space. We consider the class $\mathbf{\Lambda}$ of all operators $A : D_A \longrightarrow H$ with a domain D_A, $\overline{D}_A = H$, $D_{A^*} = D_A$ and finite dimensional imaginary part (i.e. $\dim(A - A^*)D_A < \infty$). (Analogously it can be considered the case with a trace class imaginary part of A.) In this case the operator $A \in \mathbf{\Lambda}$ has the representation

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i}. \tag{4.1}$$

and A is a closed operator.

As in the bounded case in the study of an unbounded operator $A \in \mathbf{\Lambda}$ the set

$$X = (A; H, \Phi, E; L), \tag{4.2}$$

where H, E are Hilbert spaces, $\Phi : H \longrightarrow E$ and $L : E \longrightarrow E$ are linear bounded operators and $L^* = L$, are said to be the operator colligation if A satisfies the condition

$$(A - A^*)/i \subset \Phi^* L \Phi. \tag{4.3}$$

Note that every operator $A \in \mathbf{\Lambda}$ can be embedded in a colligation from the form (4.2) by setting (for example)

$$E = (A - A^*)H, \quad \Phi = \left| \frac{A - A^*}{i}\bigg|_E \right|^{1/2}, \quad L = \operatorname{sign} \frac{A - A^*}{i}\bigg|_E,$$

where the absolute value and the sign function are understood in the sense of the usual functional calculus for selfadjoint operators, $A - A^*$ is extended in H, $\dim E \geq \dim R((A - A^*)|_H)$ where $R((A - A^*)|_H)$ is the range of the extension of $(A - A^*)/i$ in H ([26]).

The set of non-real points of the spectrum $\sigma(A)$ of $A \in \mathbf{\Lambda}$ is finite or countable and the accumulation points of this set are real (see, for example, [8] or this can be obtained analogously as in the bounded case in [6]).

The characteristic operator function of the colligation X from the form (4.2)

$$W(\lambda) = I - iL\Phi(A - \lambda I)^{-1}\Phi^*, \tag{4.4}$$

defined onto E for $\lambda \notin \sigma(A)$, is analytic on the set $\rho(A)$ of the regular points of A and $W(\lambda)$ possesses metric properties as in the bounded case onto the linear subspace $G_\lambda \subset D_{W(\lambda)} = E$, determined by the condition $\Phi^* G_\lambda \subset (A - \lambda I)D_A$ for $\lambda \notin \sigma(A)$. Then the L-metric properties as in (1.1) hold onto G_λ (a such linear subspace G_λ always can be chosen ([26])).

The colligations $X_1 = (A_1; H_1, \Phi_1, E; L)$, $X_2 = (A_2; H_2, \Phi_2, E; L)$ with operators A_1, $A_2 \in \mathbf{\Lambda}$ are *unitary equivalent* if there exists an isometric operator $U : H_1 \longrightarrow H_2$, which satisfies the conditions $UA_1 = A_2U$, $\Phi_2^* = U\Phi_1^*$. From the unitary equivalence of the operators A_1 and A_2 it follows that $\sigma(A_1) = \sigma(A_2)$. When $\lambda \notin \sigma(A_1)$ then

$$(A_2 - \lambda I)^{-1} = U(A_1 - \lambda I)^{-1}U^*.$$

This equality implies that the characteristic operator functions $W(\lambda_1)$ and $W(\lambda_2)$ of the unitary equivalent colligations coincide for all $\lambda \notin \sigma(A_k)$, $k = 1, 2$.

Let $A \in \mathbf{\Lambda}$ and $A : D_A \longrightarrow H$. The subspace $H_1 \subset H$ is said to be the *invariant subspace* of A if $\overline{D_A \cap H_1} = H_1$ and $Af \in H_1$ when $f \in D_A \cap H_1$.

Now we will define a coupling of operators from $\mathbf{\Lambda}$. Let $A_1, A_2 \in \mathbf{\Lambda}$ and $A_k : D_{A_k} \longrightarrow H_k$, P_k be the orthogonal projectors of $H = H_1 \oplus H_2$ onto H_k, $k = 1, 2$. Let Γ be a linear operator in H with the properties

$$\Gamma H_1 = 0, \quad P_2 D_\Gamma = D_{A_2}, \quad R(\Gamma) \subset H_1$$

where D_Γ is the domain of Γ and $R(\Gamma)$ is the range of Γ. The operator

$$A = A_1 P_1 + A_2 P_2 + \Gamma$$

defined in $D_A = D_{A_1} \oplus D_{A_2}$ is said to be the *coupling* of A_1 and A_2, i.e. $A = A_1 \vee A_2$. Then $\overline{D}_A = H$, $\overline{D_A \cap H_1} = H_1$ and H_1 is an invariant subspace of A.

Conversely, if a subspace $H_1 \subset H$ is an invariant subspace of the operator $A \in \mathbf{\Lambda}$, defined in D_A, the orthogonal projector P_1 of H onto H_1 satisfies the relation $P_1(H_1 \cap D_A) \subset D_A$, then

$$A = P_1AP_1 + P_2AP_2 + P_1AP_2 \qquad (P_2 = I - P_1), \tag{4.5}$$

i.e. A is presented as a coupling of the operators $A_1 = AP_1 = P_1AP_1$ in H_1 and $A_2 = P_2AP_2$ in H_2 ($H_2 = H \ominus H_1$) and $D_{A_K} = P_kD_A$, $\overline{D}_{A_k} = H_k$, $k = 1, 2$.

If $\lambda \notin \sigma(A_1)$, $\lambda \notin \sigma(A_2)$ and $P_1AP_2D_A \subset R(A_1 - \lambda I)$ then $\lambda \notin \sigma(A)$ and the resolvent $(A - \lambda I)^{-1}$ of the operator $A = A_1 \vee A_2$ has the representation

$$(A - \lambda I)^{-1} = P_1(A_1 - \lambda I)^{-1}P_1 + P_2(A_2 - \lambda I)^{-1}P_2- \\ -P_1(A_1 - \lambda I)^{-1}P_1AP_2(A_2 - \lambda I)^{-1}P_2. \tag{4.6}$$

Let $A \in \mathbf{\Lambda}$, $A : D_A \longrightarrow H$. The biggest invariant subspace H_A of the operator A is said to be the *additional subspace* of A, if

1) for each $f \in H_A \cap D_A$ holds $(Af, g) = (f, Ag)$ $\forall g \in D_A$, i.e. $H_A \cap D_A$ is a subset of the Hermitian domain of A;

2) H_A and $H \ominus H_A$ are invariant subspaces of A. Then the additional subspace H_A and the so called *principal subspace* $H_1 = H \ominus H_A$ of A are invariant subspaces of A, A^*, $R_\lambda = (A - \lambda I)^{-1}$, $\widetilde{R}_z = (A^* - zI)^{-1}$ for $\lambda \notin \sigma(A)$, $z \notin \sigma(A^*)$ and

$$H_1 = \bigvee_{n=0,1,2,\dots} \left\{ R_{\lambda_0}^{n+1} \frac{A - A^*}{i} R_{\lambda_0}^* f, \quad f \in H \right\}$$

(for an arbitrary fixed λ_0: $\lambda_0 \notin \sigma(A)$, $\lambda_0 \notin \sigma(A^*)$).

Let $A \in \mathbf{\Lambda}$ with a domain D_A be embedded in a colligation X from the form (4.2). If $\{e_\alpha\}_1^m$ is an orthonormal basis in E we denote by $g_\alpha = \Phi^* e_\alpha$ the so called *channel elements* of A.

The next two theorems solve the basic problems of the single operator colligation theory for an unbounded operator $A \in \mathbf{\Lambda}$ concerning the coincidence of the characteristic operator functions of two operators and their unitary equivalence, multiplicative properties of the characteristic function of the coupling of two operators.

Theorem 4.1. *Let the operators $A_k : D_{A_k} \longrightarrow H_k$, $k = 1, 2$, belong to the class $\mathbf{\Lambda}$. Let the characteristic matrix functions of A_k ($k = 1, 2$)*

$$W_k(\lambda) = I - i\|((A_1 - \lambda I)^{-1} g_\alpha^k, g_\beta^k)\|L$$

(where $g_\alpha^k = \Phi_k^* e_\alpha$, $\alpha = 1, 2, \dots, m$ are the channel elements of A_k, $k = 1, 2$, $L^* = L$, $\det L \neq 0$) be equal. Then the operators $A_1|_{E_1}$ and $A_2|_{E_2}$ are unitarily equivalent on the principal subspaces H_{E_1} and H_{E_2} where

$$H_{E_k} = \bigvee_{n=0,1,2,\dots} \left\{ R_{k\lambda_0}^{n+1} g_\alpha^k : \ \alpha = 1, 2, \dots, m \right\}, \quad k = 1, 2,$$

$R_{k\lambda_0} = (A_k - \lambda_0 I)^{-1}$ ($k = 1, 2$), E_1 and E_2 are linear spans of $\{g_\alpha^1\}_1^m$ and $\{g_\alpha^2\}_1^m$ correspondingly.

Let the operator A with a domain D_A belong to the class $\mathbf{\Lambda}$ and H_0 be a subspace of H such that $P_0 D_A \subset H_0$, where P_0 is the orthogonal projector of H onto H_0. Let us define an operator A_0 onto $P_0 D_A$ by the equality

$$A_0 f = P_0 A f \quad \forall f \in P_0 D_A,$$

i.e. $A_0 f = A_0 P_0 g = P_0 A P_0 g$ for all $g \in D_A$. Let A be embedded in a colligation $X = (A; H, \Phi, E; L)$. Then the characteristic matrix function

$$W_0(\lambda) = I - i\Phi P_0 (A - \lambda I)^{-1} P_0 \Phi^* L = I - i\|((A - \lambda I)^{-1} g_\alpha^0, g_\beta^0)\| L \tag{4.7}$$

(where $g_\alpha^0 = P_0 g_\alpha = P_0 \Phi^* e_\alpha$, $\{e_\alpha\}_1^m$ is the orthonormal basis in E) is said to be the *projection* of the characteristic matrix function $W(\lambda)$ of A onto the subspace H_0.

Theorem 4.2. *Let the operator $A \in \mathbf{\Lambda}$ with a domain D_A be a coupling of the operators $A_k \in \mathbf{\Lambda}$, $A : D_{A_k} \longrightarrow H_k$, $k = 1, 2$, and $H_1 \oplus H_2 = H$. Then the characteristic operator function $W(\lambda)$ of A is a product of the projections $W(\lambda)$ onto H_1 and H_2:*

$$W(\lambda) = W_1(\lambda) W_2(\lambda), \tag{4.8}$$

where $W_k(\lambda) = I - i\Phi(P_k A P_k - \lambda)^{-1} P_k \Phi^* L$ are the projections of $W(\lambda)$ onto H_k, $k = 1, 2$.

Next we will present a model describing a class of unbounded operators from $\mathbf{\Lambda}$ with purely real absolutely continuous spectrum presented as a coupling of a dissipative operator and an antidissipative one.

Let A belong to the class $\mathbf{\Lambda}$ and let us suppose that the spectrum of A is real. In the case of a dissipative operator A the multiplicative representation of the characteristic operator function $W(\lambda)$ has the form

$$W(\lambda) = \int_{-\infty}^{\overset{+\infty}{\rightarrow}} e^{\frac{-i dE(\theta)}{\alpha(\theta) - \lambda}}, \tag{4.9}$$

where $\int_{-\infty}^{+\infty} \|dE(\theta)\| < \infty$ (see [27]). The dissipative operator $A \in \mathbf{\Lambda}$ with a real spectrum is an operator with an absolutely continuous spectrum if the characteristic operator function has the representation

$$W(\lambda) = \int_{-\infty}^{\overset{+\infty}{\rightarrow}} e^{\frac{-i\Pi^*(\theta)\Pi(\theta)}{\alpha(\theta) - \lambda} L d\theta}, \tag{4.10}$$

where $\int\limits_{-\infty}^{+\infty} \operatorname{tr}\Pi^*(\theta)\Pi(\theta)d\theta < \infty$ (see [27, 7]).

From (4.10) it follows ([27]) that the operator $A \in \mathbf{\Lambda}$ with a real absolutely continuous spectrum is unitary equivalent to the triangular model

$$Af(x) = \alpha(x)f(x) + i\int\limits_{-\infty}^{x} f(\xi)\Pi(\xi)Ld\xi\Pi^*(x), \quad (f \in \mathbf{L}^2(\mathbb{R}))$$

on the principal subspace, where $\|\Pi(x)\|^2$ is an integrable function on $\mathbb{R}$.

In this part we will denote by $\| \ \|$ the norm of a matrix function in $\mathbb{C}^n$ and by $\| \ \|_{\mathbf{L}^2}$ the norm in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$.

Let $\alpha(x)$ be an unbounded nondecreasing function on $\mathbb{R}$ which is continuous from the left, let $\Pi(x)$ be a measurable $n \times m$ $(1 \le n \le m)$ matrix function on $\mathbb{R}$ whose rows are linearly independent at each point of a set of a positive measure and satisfying the conditions $\int\limits_{-\infty}^{+\infty} \operatorname{tr}\Pi^*(\theta)\Pi(\theta)d\theta < \infty$ and

$$\int\limits_{-\infty}^{+\infty} \|\Pi(x)\|^2 dx < \infty \tag{4.11}$$

Let $L : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ be a selfadjoint matrix with $\det L \neq 0$. We can assume without loss of generality that L has the representation

$$L = J_1 - J_2 + S + S^*, \tag{4.12}$$

where $J_1,\ J_2,\ S : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ have the form (2.3) as in the bounded case

Our object is the model (1.3) with a domain

$$D_A = \{f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n) : \alpha(x)f(x) \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)\},$$

where $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n) = \{f(x) = (f_1(x), \ldots, f_n(x)) : \mathbb{R} \longrightarrow \mathbb{C}^n : f_k(x) \in \mathbf{L}^2(\mathbb{R}), k = 1, 2, \ldots, n\}$ is the Hilbert space with a scalar product

$$(f(x), g(x)) = \int\limits_{-\infty}^{+\infty} f(x)g^*(x)dx \quad (f, g \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)).$$

The domain D_A is dense in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$.

The condition (4.11) implies that $A : D_A \longrightarrow \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$. Then the domain D_{A^*} of A^* coincides with D_A, i.e. $D_{A^*} = D_A$. From the form of A^* it follows that A is a closed operator with a dense domain D_A in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$. Then the operator A can be presented in the form

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i},$$

$\dfrac{A - A^*}{2i} f(x) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{+\infty} f(\xi)\Pi(\xi)L\Pi^*(x)d\xi$, $\dim \frac{A-A^*}{2i} D_A < \infty$ and A belongs to the class $\mathbf{\Lambda}$. Then we can embed the operator A in a colligation

$$X = (A; \mathbf{L}^2(\mathbb{R};\mathbb{C}^n), \Phi, \mathbb{C}^m; L), \tag{4.13}$$

where the bounded operator $\Phi : \mathbf{L}^2(\mathbb{R};\mathbb{C}^n) \longrightarrow \mathbb{C}^m$ is defined by

$$\Phi f(x) = \int_{-\infty}^{+\infty} f(x)\Pi(x)dx. \tag{4.14}$$

The imaginary part of A takes the form

$$\left.\frac{A - A^*}{i}\right|_{D_A} = \Phi^* L \Phi|_{D_A}, \tag{4.15}$$

where

$$\Phi^* h = h\Pi^*(x), \quad h \in \mathbb{C}^m. \tag{4.16}$$

Let the matrix function $B(x) = \Pi^*(x)\Pi(x)$ satisfy the condition (as in the bounded case)

$$B(x)J_1 = J_1 B(x) \tag{4.17}$$

for almost all $x \in \mathbb{R}$. Let $Q(x)$ be a measurable $m \times n$ matrix function satisfying the condition

$$\Pi(x)Q(x) = I \tag{4.18}$$

for almost all $x \in \mathbb{R}$, where I is the identity matrix in $\mathbb{C}^n$. Then the operators P_1, $P_2 : \mathbf{L}^2(\mathbb{R};\mathbb{C}^n) \longrightarrow \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ defined by the equalities

$$P_1 f(x) = f(x)\Pi(x)J_1 Q(x), \quad P_2 f(x) = f(x)\Pi(x)J_2 Q(x). \tag{4.19}$$

are orthogonal projectors in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ Then from (1.3) and (4.19) it follows that

$$P_1AP_1f(x) = \alpha(x)f(x)\Pi(x)J_1Q(x) + i\int_{-\infty}^{x} f(\xi)\Pi(\xi)J_1\Pi^*(x)d\xi,$$

$$(4.20)\qquad P_2AP_2f(x) = \alpha(x)f(x)\Pi(x)J_2Q(x) - i\int_{-\infty}^{x} f(\xi)\Pi(\xi)J_2\Pi^*(x)d\xi,$$

$$P_1AP_2f(x) = i\int_{-\infty}^{+\infty} f(\xi)\Pi(\xi)S\Pi^*(x)d\xi, \qquad P_2AP_1f(x) = 0.$$

The form of A^* and the equalities (4.20) show that the operator $A_1 = P_1A$ is a dissipative operator onto the subspace $H_1 = P_1D_A$, $A_2 = P_2A$ is an antidissipative operator onto the subspace $H_2 = P_2D_A$ and $P_1\mathbf{L}^2(\mathbb{R};\mathbb{C}^n) \cap D_A$ is an invariant subspace of the operator A. This implies that A has the representation

$$(4.21)\qquad A = P_1AP_1 + P_2AP_2 + P_1AP_2$$

and A is a coupling of the dissipative operator A_1 and an antidissipative operator A_2: $A = A_1 \vee A_2$.

Conversely as in the bounded case ([1, 12]) if C is a linear operator from the class $\mathbf{\Lambda}$ with a real spectrum, presented as a coupling of a dissipative operator and an antidissipative one with real absolutely continuous spectra determined by an unbounded function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$, then C is unitary equivalent to the operator from the form (1.3) onto the principal subspace and the matrix function $B(x) = \Pi^*(x)\Pi(x)$ satisfies the condition (4.17).

Let us denote by $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ the set of all operators $A \in \mathbf{\Lambda}$ with the representation (1.3) (up to an unitary equivalence to the principal subspace) with purely real absolutely continuous spectrum and satisfying the condition (4.17). This class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ describes nondissipative unbounded operators A with a dense domain D_A in a Hilbert space, $D_A = D_{A^*}$ and presented as a coupling of a dissipative operator and an antidissipative one with real absolutely continuous spectra. The model (1.3) we call the **triangular model** of the operators from the class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$.

We will show that the operators from $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ generate the so called *nondissipative curves* having asymptotics and limits of the corresponding correlation function as $t \to \pm\infty$.

The next theorem gives the form of the characteristic operator function of the operators from the class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ using the multiplicative properties of the characteristic matrix function (Theorem 4.2), straightforward calculations and the properties of the multiplicative integrals.

Let $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ and let A have the representation (1.3). Let the matrix function $B(x)$ satisfies the condition (4.17), let P_1 and P_2 are defined by (4.19).

Theorem 4.3. *The characteristic matrix function of the model* (1.3) *has a representation*

$$W(\lambda) = \int_{-\infty}^{\overset{+\infty}{\rightarrow}} e^{\frac{iJ_2 B(\theta) J_2}{\lambda - \alpha(\theta)} L d\theta} \int_{-\infty}^{\overset{+\infty}{\rightarrow}} e^{\frac{iJ_1 B(\theta) J_1}{\lambda - \alpha(\theta)} L d\theta}, \tag{4.22}$$

where the matrix functions $\int_{-\infty}^{\overset{+\infty}{\rightarrow}} e^{\frac{iJ_1 B(\theta) J_1}{\lambda - \alpha(\theta)} L d\theta}$, $\int_{-\infty}^{\overset{+\infty}{\rightarrow}} e^{\frac{iJ_2 B(\theta) J_2}{\lambda - \alpha(\theta)} L d\theta}$ *are the projections of* $W(\lambda)$ *onto the subspaces* $P_1\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$, $P_2\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ *and the product in* (4.22) *is as matrices.*

Next we introduce semigroups with generators from $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ and obtain the asymptotic behaviour of the corresponding curves. That is way we need a suitable representation of the resolvent of the operator $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$. This representation is given by the next lemma.

At first for the considered model A from the form (1.3) in this part we assume that the matrix functions $\Pi(x)$ satisfies the conditions: $\|\Pi(x)\| \leq C$, $\|\Pi(x)\| \in \mathbf{L}^2(\mathbb{R})$ (for some constant $C > 0$). We also suppose that the function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the conditions:

(i) $\alpha(x)$ is continuous unbounded strictly increasing on $\mathbb{R}$;

(ii) the inverse function $\sigma(u)$ of $\alpha(x)$ is absolutely continuous on $\mathbb{R}$;

(iii) $\sigma'(u)$ is a bounded function on $\mathbb{R}$.

For the simplification of writing suppose that the initial function $f(x) = (f_1(x), \ldots, f_n(x))$ is chosen from the set

$$\begin{aligned} \widetilde{H}_0 = \{f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n) : f' \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n), \|f(x)\| \leq M_f, \\ \|f'(x)\| \leq M_f, \lim_{x\to\pm\infty} f(x)Q^*(x) = 0\} \end{aligned} \tag{4.23}$$

($M_f > 0$ is a constant) which is dense in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$.

Lemma 4.4. *Let A be the model* (1.3)*, let $Q(x)$ be a measurable matrix function on $\mathbb{R}$, the function $\|Q^{*\prime}(x)\|$ be bounded, $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$. Then for each $f \in \widetilde{H}_0$ and for each $\lambda \notin \mathbb{R}$ the resolvent $(A - \lambda I)^{-1} f$ has the representation*

$$(A - \lambda I)^{-1} f(x) = 1/(\alpha(x) - \lambda) Z(\lambda, x) f(x) \Pi^*(x), \tag{4.24}$$

$(\operatorname{Im}\lambda \neq 0)$ *where*

$$Z(\lambda,x)f(x) = \int\limits_{-\infty}^{+\infty} ((f(\xi)Q^*(\xi))'\Bigg(\overset{\overrightarrow{x}}{\int\limits_{\xi}} e^{\frac{-iJ_1B(\theta)J_1}{\alpha(\theta)-\lambda}d\theta} J_1+$$

$$+\overset{\overrightarrow{x}}{\int\limits_{\xi}} e^{\frac{iJ_2B(\theta)J_2}{\alpha(\theta)-\lambda}d\theta} J_2)\chi_{(-\infty;x]}(\xi)- \tag{4.25}$$

$$-\overset{\overrightarrow{+\infty}}{\int\limits_{\xi}} e^{\frac{J_2B(\theta)J_2}{\alpha(\theta)-\lambda}d\theta} S \overset{\overrightarrow{x}}{\int\limits_{-\infty}} e^{\frac{-iJ_1B(\theta)J_1}{\alpha(\theta)-\lambda}d\theta} J_1\Bigg)d\xi.$$

For further applications of the resolvent of the model A it has to mention the next representation of $(A-\lambda I)^{-1}f$ for each $f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$

$$(A-\lambda I)^{-1}f(x) = \frac{f(x)}{\alpha(x)-\lambda}-$$

$$-\frac{1}{\alpha(x)-\lambda}\int\limits_{-\infty}^{+\infty}\frac{1}{\alpha(\eta)-\lambda}X(\lambda,\eta)f(\eta)d\eta\Pi^*(x) = Y(\lambda,x)/(\alpha(x)-\lambda) \tag{4.26}$$

with $Y(\lambda,x) = f(x) - \int\limits_{-\infty}^{+\infty}(1/(\alpha(\eta)-\lambda))X(\lambda,\eta)f(\eta)d\eta\Pi^*(x)$ $(\forall\lambda \notin \mathbb{R})$. Here $X(\lambda,x)f(x) \in \mathbf{L}(\mathbb{R};\mathbb{C}^m)$ and $\|X(\lambda,\eta)f(\eta)\| \leq \widetilde{M}_f\|\widehat{f}(\eta)\|$, $\|\widehat{f}(\eta)\| \in \mathbf{L}(\mathbb{R})$, $\widetilde{M}_f$ is a suitable positive constant depending on the function f and $1/|\operatorname{Im}\lambda|$. From (4.26) it follows the inequality

$$\|(A-\lambda I)^{-1}f(x)\| \leq \frac{1}{|\alpha(x)-\lambda|}(\|f(x)\| + \widetilde{C}_f\|\Pi^*(x)\|) \tag{4.27}$$

where $\widetilde{C}_f > 0$ is a constant depending on f and $1/|\operatorname{Im}\lambda|$.

On the other side if $f \in \widetilde{H}_0$ from (4.24), the properties of the multiplicative integrals and the condition $\|B(x)\| \leq C$ we obtain the inequality

$$\|(A-\lambda I)^{-1}f(x)\| \leq \frac{1}{|\alpha(x)-\lambda|}C_f\|\Pi^*(x)\|, \tag{4.28}$$

where $C_f > 0$ is a constant depending only on the function $f \in \widetilde{H}_0$ (but not on $1/|\operatorname{Im}\lambda|$).

Now we will introduce families of operators $\{T_t\}_{t\geq 0}$, $\{T_t\}_{t\leq 0}$ which are semigroups of operators, generating the exponential function in the case of unbounded operator $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$. Then we will present the asymptotics of the curves $T_t f$, $f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ as $t \to \pm\infty$.

For the operator A from the class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ we may assume without loss of generality that it has the form (1.3).

Let $\|\Pi^{*\prime}(x)\|$ be bounded on $\mathbb{R}$, $\|\Pi^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$, $\lim\limits_{x\to\pm\infty} \Pi^*(x)Q^*(x) = 0$. Let the function $\alpha(x)$ satisfy the condition (**i**), (**ii**), (**iii**). Let $\|Q^{*\prime}(x)\|$ be bounded on $\mathbb{R}$, $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$.

It has to mention that in the particular case $L = I$ the operator A is dissipative and this case is noticed in [27]. In the case when $L = J_1 - J_2$ (i.e. $S = 0$) the operator A has the form $A = P_1AP_1 + P_2AP_2$ and the both spaces $H_1 = P_1\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ and $H_2 = P_2\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ are invariant subspaces of A. In this part we consider the case when $S \neq 0$.

For an arbitrary sufficiently small $\delta > 0$ and for every $f \in D_A$ we define the next families of operators $T_t f$ by the equality (1.4) in the case of $t > 0$ and in the case of $t < 0$, where the integral on the right hand side of (1.4) is in the sense of a principal value.

The existence of the integral in (1.4) for all $f \in D_A$ follows from the representation

$$\lim_{R\to\infty} \int_{-R}^{R} (e^{it(\xi-i\delta)}(A-(\xi-i\delta)I)^{-1}f - e^{it(\xi+i\delta)}(A-(\xi+i\delta)I)^{-1}f)d\xi =$$
$$= \lim_{R\to\infty} \int_{-R}^{R} \left(\frac{e^{it(\xi-i\delta)}}{\xi-i\delta-\lambda_0}(A-(\xi-i\delta)I)^{-1}g - \frac{e^{it(\xi+i\delta)}}{\xi+i\delta-\lambda_0}(A-(\xi+i\delta)I)^{-1}g \right) d\xi,$$

(where $g = (A-\lambda_0 I)f$, $\operatorname{Im}\lambda_0 > \delta > 0$ when $t > 0$ and $\operatorname{Im}\lambda_0 < -\delta < 0$ when $t < 0$), using the resolvent equation, the Lebesgue convergence theorem and the Residue theorem for the function $e^{itz}/(z-\lambda_0)$ and domains with suitable contours.

On the other hand $T_t f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ for each $f \in D_A$ which follows from (4.26) and straightforward calculations.

It has to mention also the independence of the definition (1.4) of $T_t f$ for $f \in D_A \cap \widetilde{H}_0$ on the choice of $\delta > 0$. In other words the introduced families $\{T_t\}_{t>0}$ and $\{T_t\}_{t<0}$ are well defined by (1.4) in $D_A \cap \widetilde{H}_0$.

Next two theorems describe the properties of the families of operators $\{T_t\}_{t>0}$ and $\{T_t\}_{t<0}$, which present these families as a semigroups of operators from the class (C_0) with a generator iA and solve the question of a differentiability of the considered semigroups. Only the boundedness of T_t will be presented further after the obtaining of the asymptotics of $T_t f$ as $t \to \pm\infty$.

Theorem 4.5. *The families of operators* $\{T_t\}_{t>0}$ *and* $\{T_t\}_{t<0}$*, defined by* (1.4)*, satisfy the conditions*

$$T_s T_t f = T_{s+t} f \quad \forall f \in D_A \;\; (\forall t, s > 0 \;\; \textit{and} \;\; \forall t, s < 0), \tag{4.29}$$

$$\lim_{t \to 0} T_t f = f \quad \forall f \in D_A \;\; (t \to 0, \;\; t > 0 \;\; \textit{and} \;\; t \to 0, \;\; t < 0). \tag{4.30}$$

Theorem 4.6. *The families of operators* $\{T_t\}_{t\geq 0}$ *and* $\{T_t\}_{t\leq 0}$*, defined by* (1.4)*, satisfy the conditions*

$$\frac{d}{dt} T_t f(x) = iAT_t f(x) \quad \forall f \in D_A, \tag{4.31}$$

$$\lim_{t \to 0} \frac{T_t f - f}{t} = iAf \quad \forall f \in D_A. \tag{4.32}$$

The relation (4.30) allows to define

$$T_0 f = f \quad \forall f \in D_A \tag{4.33}$$

and we can consider the families $\{T_t\}_{t\geq 0}$ and $\{T_t\}_{t\leq 0}$.

The proofs of Theorem 4.5 and Theorem 4.6 (see [13]) are based on the use of the representation $f = (A - \lambda_0 I)^{-1} g$ (for $f \in D_A$), the representation (4.26), the inequality (4.27), other appropriate inequalities, the Residue theorem, the Lebesgue convergence theorem.

The properties of the families of operators T_t allow us to define the *nondissipative curves* generated by the unbounded operators $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ using the families $\{T_t\}_{t\leq 0}$ and $\{T_t\}_{t\geq 0}$ for each $f \in \widetilde{H}_0$ by $T_t f$ when $t \leq 0$ and $t \geq 0$.

We will present in an explicit form of the asymptotics of the curves $T_t f$, generated by the unbounded operators A from $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ when f belongs to the suitable dense set in $\mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$.

Let $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ be the model from the form (1.3), let L, $\Pi(x)$, $Q(x)$, $B(x)$, P_1, P_2, $\alpha(x)$, $\sigma(u)$ be stated as above.

Before continuing with the asymptotics of the curves, generated from A, it has to mention that the model (1.3) after the change of the variable $x = \sigma(u)$ can be written in the form

$$Ag(u) = ug(u) + i\int_{-\infty}^{u} g(\eta)\widehat{\Pi}(\eta)(J_1 - J_2)\sigma'(\eta)d\eta\widehat{\Pi}^*(u) + \\ + i\int_{-\infty}^{+\infty} g(\eta)\widehat{\Pi}(\eta)S\sigma'(\eta)d\eta\widehat{\Pi}^*(u) \tag{4.34}$$

(where $g \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n; \sigma(u))$, $\|\widehat{\Pi}(u)\| \in \mathbf{L}^2(\mathbb{R}; \sigma(u)))$ when the function $f(\sigma(u)) \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n; \sigma(u))$, $\|\Pi(\sigma(u))\| \in \mathbf{L}^2(\mathbb{R})$ and the function $\alpha(x)$ satisfies the conditions (**i**), (**ii**), (**iii**).

For the simplification of writing we can consider the model

$$Af(x) = xf(x) + i\int_{-\infty}^{x} f(\xi)\Pi(\xi)(J_1 - J_2)d\xi\Pi^*(x) + \\ + i\int_{-\infty}^{+\infty} f(\xi)\Pi(\xi)Sd\xi\Pi^*(x) \tag{4.35}$$

(i.e. $\alpha(x) = x$) with a domain

$$D_A = \{f \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n) : xf(x) \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)\}. \tag{4.36}$$

It has to mention that the asymptotics of the curves generated by the model (4.34) can be obtained analogously to the asymptotics of the curves generated by (4.35) if we suppose the additional condition for $\sigma(u)$: $\sigma'(u) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_2 \leq 1)$ (i.e. $|\sigma'(u_1) - \sigma'(u_2)| \leq C|u_1 - u_2|^{\alpha_2}$ for all u_1, $u_2 \in \mathbb{R}$ and some constant $C > 0$).

Let the model $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ be defined by (4.35) with a domain D_A, defined by (4.36), let L, $\Pi(x)$, $Q(x)$, $B(x)$ be stated as above.

Before continuing with the next theorems we need some denotations. Let $B(x)$ belongs to the class $C_{\alpha_1}(\mathbb{R})$ $(0 < \alpha_1 \leq 1)$, i.e. $\|B(x_1) - B(x_2)\| \leq C|x_1 - x_2|^{\alpha_1}$ for all x_1, $x_2 \in \mathbb{R}$ and some constant $C > 0$. Let us denote the next operators analogously as in the bounded case in part 2

$$B_1(x) = J_1B(x)J_1, \quad B_2(x) = J_2B(x)J_2, \tag{4.37}$$

$$F_w^{\pm}(x,u) = s - \lim_{\delta\to 0} \int\limits_w^{\overrightarrow{u}} e^{\frac{-iB_1(v)}{v-(x\pm i\delta)}dv}, \quad \widetilde{F}_w^{\pm}(x,u) = s - \lim_{\delta\to 0} \int\limits_w^{\overrightarrow{u}} e^{\frac{iB_2(v)}{v-(x\pm i\delta)}dv}, \tag{4.38}$$

$$P_w(x,u) = F_w^{+}(x,u) - F_w^{-}(x,u), \qquad \widetilde{P}_w(x,u) = \widetilde{F}_w^{+}(x,u) - \widetilde{F}_w^{-}(x,u) \tag{4.39}$$

for all $w, u, x \in \mathbb{R}$ such that $-\infty \le w < u \le +\infty$ and

$$F_w^{\pm}(x,u) = s - \lim_{\delta\to 0} \int\limits_w^{\overrightarrow{x-\delta}} e^{\frac{-iB_1(v)}{v-x}dv} e^{\pm\pi B_1(x)} \int\limits_{x+\delta}^{\overrightarrow{u}} e^{\frac{-iB_1(v)}{v-x}dv}, \tag{4.40}$$

$$\widetilde{F}_w^{\pm}(x,u) = s - \lim_{\delta\to 0} \int\limits_w^{\overrightarrow{x-\delta}} e^{\frac{iB_2(v)}{v-x}dv} e^{\mp\pi B_2(x)} \int\limits_{x+\delta}^{\overrightarrow{u}} e^{\frac{iB_2(v)}{v-x}dv}, \tag{4.41}$$

$$V_{-\infty}(x) = s - \lim_{\delta\to 0} \int\limits_{-\infty}^{\overrightarrow{x-\delta}} e^{\frac{-iB_1(v)}{v-x}dv} e^{iB_1(x)\ln\delta}, \tag{4.42}$$

$$\widetilde{V}_{-\infty}(x) = s - \lim_{\delta\to 0} \int\limits_{-\infty}^{\overrightarrow{x-\delta}} e^{\frac{iB_2(v)}{v-x}dv} e^{-iB_2(x)\ln\delta} \tag{4.43}$$

for all w, u, x such that $-\infty \le w < x < u \le +\infty$. Now using (4.38) from (4.24), (4.25) it follows the existence of the limits $Z^{\pm}(\xi,x)f(x) = \lim\limits_{\delta\to 0} Z(\xi \pm i\delta, x)$ and these limits have the form

$$\begin{aligned} Z^{\pm}(\xi,x)f(x) = \int\limits_{-\infty}^{+\infty} \widetilde{f}'(w)((F_w^{\pm}(\xi,x)J_1 + \widetilde{F}_w^{\pm}(\xi,x)J_2)\chi_{(-\infty;x]}(w) - \\ -\widetilde{F}_w^{\pm}(\xi,\infty)S\widetilde{F}_{-\infty}^{\pm}(\xi,x)J_1)dw, \end{aligned} \tag{4.44}$$

where

$$\widetilde{f}(w) = f(w)Q^*(w) \quad for \;\; f \in \widetilde{H}_0. \tag{4.45}$$

Let us denote by $S(\mathbb{R};\mathbb{C}^n)$ the set of all smooth fast decreasing functions $f(x)=(f_1(x),\dots,f_n(x))$.

Now we are in a position to give the asymptotics of the nondissipative curves T_tf as $t\to\pm\infty$ for $f\in(\mathbb{R};\mathbb{C}^n)$. These asymptotics are presented by Theorem 1.2.

The main point of the obtaining of the asimptotics (1.12) is a suitable representation of the operators T_t, defined by (1.4) for the model A with the form (4.35) (see [13], Lemma 5.1):

$$\begin{aligned} T_tf(x) = &-\frac{1}{2\pi i}\int\limits_{\mathbb{R}\setminus\Delta}\frac{e^{it\xi}}{\xi-\lambda_0}\cdot\frac{1}{x-\xi}Z^-(\xi,x)g(x)\Pi^*(x)d\xi-\\ &-\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\Delta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}Z^-(\xi,x)f(x)\Pi^*(x)d\xi-\\ &-\frac{1}{2\pi i}\int\limits_{\Delta}\frac{e^{it\xi}}{\xi-\lambda_0}d\xi f(x)+e^{it\lambda_0}f(x) \end{aligned} \tag{4.46}$$

when $t>0$ and $\operatorname{Im}\lambda_0>0$,

$$\begin{aligned} T_tf(x) = &-\frac{1}{2\pi i}\int\limits_{\mathbb{R}\setminus\Delta}\frac{e^{it\xi}}{\xi-\lambda_0}\cdot\frac{1}{x-\xi}Z^+(\xi,x)g(x)\Pi^*(x)d\xi-\\ &-\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\Delta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}Z^+(\xi,x)f(x)\Pi^*(x)d\xi-\\ &-\frac{1}{2\pi i}\int\limits_{\Delta}\frac{e^{it\xi}}{\xi-\lambda_0}d\xi f(x)+e^{it\lambda_0}f(x) \end{aligned} \tag{4.47}$$

when $t<0$ and $\operatorname{Im}\lambda_0<0$ correspondingly for all $f\in\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$, where $\Delta=[x-\beta;x+\beta]$, $\beta>0$ is a suitable fixed number, $g(x)=(A-\lambda_0I)f(x)$, $Z^{\pm}(\xi,x)f(x)$ are defined by (4.44) and $\widetilde{f}(w)$ is defined by (4.45).

From the representations (4.46) and (4.47) it follows that the asymptotic behaviour of T_tf as $t\to\pm\infty$ depends only on the asymptotic behaviour of

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\Delta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}Z^{\mp}(\xi,x)f(x)\Pi^*(x)d\xi,$$

correspondingly. The other addends in (4.46) and (4.47) tend to 0 as $t\to+\infty$ and $t\to-\infty$ correspondingly.

Next we use the methods and the ideas as in [12, 2] for the bounded case, the properties of the multiplicative integrals in (4.44), appropriate inequalities concerning multiplicative integrals for the matrix functions $B_1(x)$, $B_2(x) \in C_{\alpha_1}(\mathbb{R})$ $(0 < \alpha_1 \leq 1)$ (as (2.26), (2.27), (2.28), (2.30), (2.25), (2.32) and other similar inequalities), the asymptotic behaviour

$$\lim_{\varepsilon\to 0}\int\limits_{w}^{u}\frac{e^{itx}}{(u-x)^{1-\varepsilon}}e^{-iC(u)\ln(u-x)}dx \sim$$
$$\sim \pi i e^{itu}t^{iC(u)}e^{-\frac{\pi}{2}C(u)}\mathbf{\Gamma}^{-1}(I+iC(u))(\sinh(\pi C(u)))^{-1}$$

as $t \to +\infty$ for a nonnegative matrix function $C(u)$, obtained in the bounded case. Then using these ideas, the Lebesgue convergence theorem and the Lebesgue lemma for the Fourier transform we obtain consecutively the asymptotics of the projections $P_1e^{itA}P_1f$, $P_1e^{itA}P_2f$, $P_1e^{itA}P_2f$, i.e.

$$(4.48)\qquad \begin{gathered}-\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int\limits_{x-\beta}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\left(\int\limits_{-\infty}^{x}\widetilde{f}'(w)F_w^-(\xi,x)dw\right)d\xi J_1\Pi^*(x)\sim\\ \sim e^{itx}\int\limits_{-\infty}^{x}\widetilde{f}'(w)U_{2w}(x)(x-w)^{iB_1(x)}dwt^{iB_1(x)}e^{frac\pi 2B_1(x)}\cdot\\ \cdot\mathbf{\Gamma}^{-1}(I+iB_1(x))J_1\Pi^*(x),\end{gathered}$$

$$(4.49)\qquad \begin{gathered}-\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int\limits_{x-\beta}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\left(\int\limits_{-\infty}^{x}\widetilde{f}'(w)\widetilde{F}_w^-(\xi,x)dw\right)d\xi J_2\Pi^*(x)\sim\\ \sim e^{itx}\int\limits_{-\infty}^{x}\widetilde{f}'(w)\widetilde{U}_{2w}(x)(x-w)^{-iB_2(x)}dwt^{-iB_2(x)}e^{\frac{\pi}{2}B_2(x)}.\\ .\mathbf{\Gamma}^{-1}(I-iB_2(x))J_2\Pi^*(x),\end{gathered}$$

$$\begin{gathered}\lim_{\varepsilon\to 0}\int\limits_{x-\beta}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\left(\int\limits_{-\infty}^{+\infty}\widetilde{f}'(w)\widetilde{F}_w^-(\xi,\infty)SF_{-\infty}^-(\xi,x)dw\right)d\xi J_1\Pi^*(x)\sim\\ \sim 2\pi i e^{itx}\int\limits_{-\infty}^{+\infty}\widetilde{f}'(w)\widetilde{F}_w^-(x,\infty)dwSV_{-\infty}(u)t^{iB_1(x)}e^{-\frac{\pi}{2}B_1(x)}.\\ .\mathbf{\Gamma}^{-1}(I+iB_1(x))J_1\Pi^*(x)\end{gathered}$$

as $t \to +\infty$, where

$$U_{2w}(x) = \lim_{\delta\to 0} \overset{\xrightarrow{x-\delta}}{\int_w} e^{\frac{-iB_1(v)}{v-x}dv} e^{i\int_w^{x-\delta}\frac{B_1(x)}{v-x}dv},$$

$$\widetilde{U}_{2w}(x) = \lim_{\delta\to 0} \overset{\xrightarrow{x-\delta}}{\int_w} e^{\frac{iB_2(v)}{v-x}dv} e^{i\int_w^{x-\delta}\frac{-B_2(x)}{v-x}dv}.$$

This proves the asymptotics (1.12) in the case when $t \to +\infty$. Analogously we obtain the asymptotics (1.12) in the case when $t \to -\infty$.

As in the bounded case in part 3 we obtain the boundedness of the operators $S_\pm$, defined by (1.13), using the dissipativness and the antidissipativness of the operators P_1AP_1 and P_2AP_2 correspondingly onto D_A, the decreasing function $\psi_1(t) = \|P_1T_tP_1f\|^2_{\mathbf{L}^2}$ and the increasing function $\psi_2(t) = \|P_2T_tP_2f\|^2_{\mathbf{L}^2}$ in $\mathbb{R}$ for each fixed $f \in S(\mathbb{R}, \mathbb{C}^n)$. We also use the obtained asymptotics (1.12), the boundedness of the operator

$$\int_x^{+\infty} \widetilde{f}'(w) \overset{\xrightarrow{+\infty}}{\int_w} e^{\frac{iB_2(v)}{v-x}dv} dw J_2 \tag{4.50}$$

in $S(\mathbb{R};\mathbb{C}^n)$ and the representation of $\widetilde{S}^{\pm}_{12}f$ in the form

$$\widetilde{S}^{\pm}_{12}f(x) =$$

$$= -\left(\int_{-\infty}^{x} \widetilde{f}'(w) \overset{\xleftarrow{w}}{\int_{-\infty}} e^{\frac{-iB_2(v)}{v-x}dv} dw \widetilde{F}^{\mp}_{-\infty}(x,\infty) + \int_x^{+\infty} \widetilde{f}'(w) \overset{\xrightarrow{+\infty}}{\int_w} e^{\frac{iB_2(v)}{v-x}dv} dw \right) S.$$

So we obtain the next theorem.

Theorem 4.7. *The operators $S_\pm$ and $\widehat{S}_\pm$, defined by* (1.13) *and* (1.15), *are bounded linear operators in the subspace $S(\mathbb{R};\mathbb{C}^n)$ of the smooth fast decreasing functions.*

The next theorem finishes the description of the families of operators $\{T_t\}_{t\ge 0}$, $\{T_t\}_{t\le 0}$ and together with the properties of these families, obtained above (Theorem 4.5, Theorem 4.6), shows that $\{T_t\}_{t\ge 0}$ and $\{T_t\}_{t\le 0}$ are semigroups of operators from the class (C_0).

Theorem 4.8. *The families of operators* $\{T_t\}_{t\geq 0}$, $\{T_t\}_{t\leq 0}$, *defined by* (1.4) *for the model A from the form* (4.35), *are uniformly bounded families of operators in* $S(\mathbb{R};\mathbb{C}^n)$.

The proof of Theorem 4.8 is analogous to the proof of Theorem 4.6 (see [13], Theorem 5.4). The uniformly boundedness of the families of operators $\{T_t\}_{t\geq 0}$, $\{T_t\}_{t\leq 0}$ follows from the colligation condition (4.15) and the inequalities

$$\|T_t f\|^2_{\mathbf{L}^2} \leq \|f\|^2_{\mathbf{L}^2} + \|L\| \int_0^{+\infty} |\Phi T_\tau f|^2 d\tau \leq \tag{4.51}$$

$$\leq \|f\|^2_{\mathbf{L}^2} + \frac{1}{2\pi}\|L\| \int_{-\infty}^{+\infty} |\widehat{G}f(x)|^2 dx = \|f\|^2_{\mathbf{L}^2} + \frac{1}{2\pi}\|L\|.\|\widehat{G}f\|^2_{\mathbf{L}^2}$$

for each $f \in S(\mathbb{R};\mathbb{C}^n)$. In (4.51) we have used the representation of

$$\Phi T_t f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\xi}\widehat{G}f(\xi)d\xi, \tag{4.52}$$

which follows after direct calculations from the properties of the multiplicative integrals and the limits, similar to the limits as in (2.46). We have also used the form of $\widehat{G}f(\xi)$

(4.53)

$$\widehat{G}f(\xi) = \int_{-\infty}^{+\infty} \widetilde{f}'(w)(P_w(\xi,+\infty)J_1 - \widetilde{P}_w(\xi,+\infty)J_2 -$$

$$-(\widetilde{F}_w^+(\xi,\infty)S(F_{-\infty}^+(\xi,\infty)-I) - \widetilde{F}_w^-(\xi,\infty)S(F_{-\infty}^-(\xi,\infty)-I))J_1)dw =$$

$$= \widetilde{S}_{11}f(\xi)P_{-\infty}(\xi,+\infty)J_1 - \widetilde{S}_{22}f(\xi)\widetilde{P}_{-\infty}(\xi,+\infty)J_2 -$$

$$-\int_{-\infty}^{+\infty} \widetilde{f}'(w)(\widetilde{F}_w^+(\xi,\infty)S(F_{-\infty}^+(\xi,\infty)-I) -$$

$$-\widetilde{F}_w^-(\xi,\infty)S(F_{-\infty}^-(\xi,\infty)-I))J_1 dw$$

using the introduced denotations (1.15), the boundedness of operator $\widehat{G}$ in $S(\mathbb{R};\mathbb{C}^n)$ given by the boundedness of $\widetilde{S}_{11}$, $\widetilde{S}_{22}$ and the operator (4.50) (see Theorem 4.7).

The proved boundedness of the operators from the semigroups $\{T_t\}_{t\leq 0}$ and $\{T_t\}_{t\geq 0}$ in the subspace $S(\mathbb{R};\mathbb{C}^n)$ (dense in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$) allows us to extend T_t by continuity in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ and to define the **exponential function** for the unbounded operators $A \in \widetilde{\mathbf{\Lambda}}_{\mathbf{R}}$ by

$$e^{itA} = T_t \quad (t \in \mathbb{R})$$

and to consider the *nondissipative curves* $e^{itA}f = T_t f$ for $f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$. Then Theorem 1.2 and Theorem 4.7 imply that for all $f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ the next relation holds

$$\|e^{itA}f(x) - e^{itx}S_{\pm}f(x)\|_{\mathbf{L}_2} \to 0 \quad as \;\; t \to \pm\infty.$$

The obtained asymptotics (1.12) allow to determine the behaviour of the correlation function $V(t+\tau, s+\tau) = (e^{i(t+\tau)A}f, e^{i(s+\tau)A}f)$ of the nondissipative curves $e^{itA}f$ as $\tau \to \pm\infty$.

Theorem 4.9. *Let for the model $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$, defined by* (4.35), *next conditions hold:*

1) $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$, $\|Q^{*\prime}(x)\| \leq C$;
2) $B(x) \in C_{\alpha_1}(\mathbb{R})$ $(0 < \alpha_1 \leq 1)$.

Then there exist the limits of the correlation function $\lim\limits_{\tau\to\pm\infty} V(t+\tau, s+\tau)$ *of the nondissipative curves $e^{itA}f$ for each $f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ and*

$$\lim_{\tau\to\pm\infty} V(t+\tau, s+\tau) = \int_{-\infty}^{+\infty} e^{i(t-s)x}\widetilde{S}_{\pm}f(x)(\widetilde{S}_{\pm}f(x))^* dx \tag{4.54}$$

for all $f \in S(\mathbb{R};\mathbb{C}^n)$, $t, s \in \mathbb{R}$, where $V(t,s) = (e^{itA}f, e^{isA}f)$ is the correlation function of the curve $e^{itA}f$ and the operators $\widetilde{S}_{\pm}$ are defined by

$$\widetilde{S}_{\pm}f(x) = T_{\pm}\widehat{S}_{\pm}f(x), \tag{4.55}$$

$\widehat{S}_{\pm}$, $T_{\pm}$ are defined by (1.15) *and* (1.16).

5. Wave operators and a scattering operator for the couple $(\mathbf{A}^*, \mathbf{A})$ with $\mathbf{A} \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ and applications. The obtained asymptotics (1.12) for the nondissipative curves $e^{itA}f$ generated by the unbounded operators A from the class $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ with domains $D_A = D_{A^*}$ allow us to apply these results for a constructing of the scattering theory for the couple (A^*, A) as in the selfadjoint case [25, 9, 10], as in the bounded dissipative case [27] and in the bounded nondissipative case [12]. In other words we consider a finite dimensional perturbation

$A+i\widetilde{A}$ of the closed operator A presented as a coupling of a dissipative operator and an antidissipative one with a dense domain D_A, where $\widetilde{A}$ is a finite dimensional selfadjoint operator. We obtain explicit forms of the wave operators of A, the scattering operator and the similarity of A to the operator of a multiplying by the independent variable in a subspace of $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$. We obtain also the form of the correlation function of the nondissipative curves $e^{itA}f$.

Let the model $A\in\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ be defined by (4.35), let L, $\Pi(x)$, $Q(x)$, $B(x)$, the colligation X be stated as in part 4.

Theorem 1.5 gives the form of the wave operators of the couple (A^*,A) as a weak limit.

The equality (1.19) implies the existence of the *wave operators* of the couple (A^*,A) defined by

$$(W_\pm(A^*,A)f,g)=\lim_{t\to\pm\infty}(e^{itA^*}e^{-itA}f,g)=(\widetilde{S}^*_\mp\widetilde{S}_\mp f,g)$$

and $W_\pm(A^*,A)=\widetilde{S}^*_\mp\widetilde{S}_\mp$ as weak limits. Theorem 1.6 proves the existence of the wave operators as a strong limit.

The proof of this theorem (see [13], Theorem 7.2) is analogous to the proof of Theorem 1.4. But in this case we have used the form of the operator $\widehat{G}$, defined by (4.53) and its boundedness, obtained in the course of the proving of Theorem 4.9.

Now from the equality (1.20) it follows that

$$A^*W_\pm(A^*,A)=W_\pm(A^*,A)A$$

which express the similarity of A and A^*, given by the wave operators $W_\pm(A^*,A)$.

The next theorem deals with the similarity of the model $A\in\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$, defined by (4.35), and the operator $\mathcal{Q}$ of a multiplying by an independent variable in a suitable subspace of $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ and presents this similarity in an explicit form. This allows us to define a scattering operator analogously as in the bounded selfadjoint case, the bounded dissipative case and the bounded nondissipative case.

Let us denote the range of the operator $\widehat{S}_\pm$ by

$$R(\widehat{S}_\pm)=\{g\in\mathbf{L}^2(\mathbb{R};\mathbb{C}^n):\widehat{S}_\pm f=g\ \ for\ \ some\ \ f\in\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)\}$$

and $Y_\pm=\overline{R(\widehat{S}_\pm)}$ be the closure of $R(\widehat{S}_\pm)$.

Theorem 5.1. *The operators $\widehat{S}_\pm$, defined by* (1.15) *in $S(\mathbb{R};\mathbb{C}^n)$, have inverse bounded operators $\widehat{S}_\pm^{-1}$ defined in $Y_\pm$ and*

$$A=\widehat{S}_\pm^{-1}\mathcal{Q}\widehat{S}_\pm \tag{5.1}$$

onto $S(\mathbb{R};\mathbb{C}^n)$.

The proof of this theorem (see [13], Theorem 7.3) follows the ideas of the proof of Theorem 3.1. For the operators

$$G_\pm = G_{11} + G_{22} + G_{12}^\pm,$$

where

$$G_{kk}g(x) = \frac{1}{2\pi}\frac{d}{dx}\int\limits_{-\infty}^{x} g(\tau)P_{-\infty}^{(k)}(\tau,x)d\tau J_k Q(x), \quad k=1,2,$$
$$G_{12}^\pm = -G_{11}\widetilde{S}_{12}^\pm G_{22}g(x)$$

(where $P_{-\infty}^{(1)}(\tau,x) = P_{-\infty}(\tau,x)$ and $P_{-\infty}^{(2)}(\tau,x) = \widetilde{P}_{-\infty}(\tau,x)$) in the subspace of all functions $g \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^m)$ such that $g' \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^m)$, $\|g'(x)\| \le C$ straightforward calculations show that $G_\pm$ are bounded inverse operators of $\widehat{S}_\pm$. Then from the representation (4.55) of $\widetilde{S}_\pm$, the existence and the form of $T_\pm^{-1}$

$$T_\pm^{-1}h = h\Pi(x)(J_1 i\lim_{\varepsilon\to 0}\mathbf{\Gamma}(\varepsilon I + iB_1(x))e^{\pm\frac{\pi}{2}B_1(x)}V_{-\infty}^*(x)J_1 +$$
$$+J_2(-i)\lim_{\varepsilon\to 0}\mathbf{\Gamma}(\varepsilon I - iB_2(x))e^{\mp\frac{\pi}{2}B_2(x)}\widetilde{V}_{-\infty}^*(x)J_2)$$

($h \in \mathbb{C}^n$) and the equality (1.20) we obtain (5.1).

Now Theorem 5.1 allows us to introduce a *scattering operator* for the couple (A^*,A) using the representation (4.55), Theorem 1.5 and Theorem 1.6

$$W_-^{-1}(A^*,A)W_+(A^*,A) = (\widetilde{S}_+^*\widetilde{S}_+)^{-1}\widetilde{S}_-^*\widetilde{S}_-.$$

The proved similarity (5.1) of the model A and the operator $\mathcal{Q}$ of a multiplying by an independent variable and the obtained limits (4.54) of the correlation function $V(t+\tau,s+\tau)$ as $\tau\to\pm\infty$ of the nondissipative curves $e^{itA}f$ allow us to obtain the form of the correlation function $V(t,s)$. We consider the function

$$W(t,s) = -\frac{\partial}{\partial\tau}V(t+\tau,s+\tau)|_{\tau=0} \tag{5.2}$$

called an *infinitesimal correlation function* of the curve $e^{itA}f$ ($f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$), introduced by M. S. Livšic and A. A. Yantsevich in [22] for a bounded operator A.

From the representation (5.2), the limits (4.54), the colligation condition (4.15) and the similarity (5.1) it follows the form of the correlation function. This form is given by the next theorem.

Theorem 5.2. *Let for the model $A \in \widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$, defined by* (4.35), *next conditions hold:*

1) $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$, $\|Q^{*\prime}(x)\| \leq C$;

2) $B(x) \in C_{\alpha_1}(\mathbb{R})$ $(0 < \alpha_1 \leq 1)$.

Then the correlation function $V(t,s)$ of the curve $e^{itA}f$ $(f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n))$ has the representation

$$V(t,s) = (e^{i(t-s)x}\widetilde{S}_{\pm}f(x), \widetilde{S}_{\pm}f(x)) + \int\limits_0^{\pm\infty} \sum_{\alpha,\beta=1}^{m} \Psi^{\pm}_{\alpha}(t+\tau)(Le_{\alpha}, e_{\beta})\overline{\Psi^{\pm}_{\alpha}(s+\tau)}d\tau,$$

where $\Psi^{\pm}_{\alpha}(t) = (e^{itx}\widehat{S}_{\pm}f(x), \widehat{S}_{\pm}^{-1^*}g_{\alpha}(x))$, $\alpha = 1,2,\ldots,m$, $\{e_{\alpha}\}_1^m$ is an orthonormal basis in $\mathbb{C}^m$ and $g_{\alpha}(x) = \Phi^* e_{\alpha} = e_{\alpha}\Pi^*(x)$ $(x \in \mathbb{R}$, $\alpha = 1,2,\ldots,m)$ are the channel elements of the colligation X defined by (4.13).

6. Triangular model and asymptotics of dissipative curves with unbounded semigroup generators iA with different domains of A and A*. The presented classes of operators $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ in the previous parts of this paper are K^r-operators A with domains $D_A = D_{A^*}$ and a finite dimensional imaginary parts (following the denotations in these cases $r = m$). We recall that a closed operator A in a Hilbert space H is called a quasi-Hermitian operator of a rank r (K^r-operator) if the restriction of the operator A onto the Hermitian domain of A is an Hermitian operator with finite and equal defect numbers (r,r) and nonempty resolvent set $\rho(A)$.

In this part we will continue our considerations for a class of dissipative unbounded K^r-operators A with domain $D_A \neq D_{A^*}$ and a real spectrum.

The results presented in this part are new and they have not been published till now in other papers.

In [18] A. Kuzhel has considered the triangular model of all K^r-operators A with a real spectrum and characteristic functions from the form

$$W(\lambda) = \int\limits_0^{\overset{l}{\rightarrow}} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda}}dE(v)J$$

where $\alpha(v)$ is a nondecreasing real function in $(0;l)$ (or $(-\infty;+\infty)$), $E(v)J$ is a monotonically increasing family of Hermitian matrices.

We consider the triangular model (1.21) where $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ is an unbounded nondecreasing function, $\Pi(x)$ is a measurable $n \times m$ $(1 \leq n \leq m$,

$r \leq m$) matrix function whose rows are linearly independent on each point of a set of positive measure and satisfying the conditions:

$$\int_{-\infty}^{+\infty} \|\Pi(x)\|^2 dx < +\infty, \tag{6.1}$$

$$\int_{-\infty}^{+\infty} \operatorname{tr} B(x) dx < +\infty, \tag{6.2}$$

where

$$B(x) = \Pi^*(x)\Pi(x)$$

and $f \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$.

The model (1.21) describes the class of all unbounded dissipative K^r-operators with a domain $D_A \subset \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$ with a real spectrum and with a characteristic matrix function

$$W(\lambda) = \overset{\overset{+\infty}{\rightarrow}}{\underset{-\infty}{\int}} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda}B(v)dv}. \tag{6.3}$$

The model (1.21) satisfies the condition $\operatorname{Im}(Af, f) \geq 0$, $\forall f \in D_A$, and consequently A is a dissipative operator.

On the other hand the operator A is densely defined (see Corollary 3.3, [18]).

Direct calculations show that the resolvent of A has the representation

$$(A - \lambda I)^{-1} f(x) = \frac{f(x)}{\alpha(x) - \lambda} -$$

$$-i \int_{-\infty}^{x} \frac{\alpha(\xi) + i}{\alpha(\xi) - \lambda} f(\xi)\Pi(\xi) \overset{\overset{x}{\rightarrow}}{\underset{\xi}{\int}} e^{-i\frac{1+\lambda\alpha(v)}{\alpha(v)-\lambda}B(v)dv} d\xi \Pi^*(x) \frac{\alpha(x) - i}{\alpha(x) - \lambda} \tag{6.4}$$

for each $\lambda \notin \mathbb{R}$ and $f \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$. Following [18] we obtain that for all λ: $\operatorname{Im} \lambda \neq 0$ the resolvent $(A - \lambda I)^{-1}$ is defined and bounded on the space $\mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$ and the spectrum of A lies on the real axis.

Let us consider the auxiliary selfadjoint operator

$$B_i = iR_i - iR_i^* + 2R_i^* R_i, \tag{6.5}$$

where $R_i = (A - iI)^{-1}$. Then following the ideas from [18] we obtain the representation

$$(6.6) \qquad B_i f(x) = \int_{-\infty}^{+\infty} \frac{\alpha(\xi)+i}{\alpha(\xi)-i} f(\xi)\Pi(\xi) \int_{\xi}^{\overrightarrow{+\infty}} e^{B(v)dv} d\xi \int_{x}^{\overleftarrow{+\infty}} e^{B(v)dv} \Pi^*(x) \frac{\alpha(x)-i}{\alpha(x)+i}.$$

Denoting the operators

$$(6.7) \qquad \Phi f(x) = \int_{-\infty}^{+\infty} \frac{\alpha(x)+i}{\alpha(x)-i} f(x)\Pi(x) \int_{x}^{\overrightarrow{+\infty}} e^{B(v)dv} dx, \quad \forall f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n),$$

$$(6.8) \qquad \Phi^* h = h \int_{x}^{\overleftarrow{+\infty}} e^{B(v)dv} \Pi^*(x) \frac{\alpha(x)-i}{\alpha(x)+i}, \quad h \in \mathbb{C}^m$$

it follows that

$$(6.9) \qquad B_i f = \Phi^* \Phi f.$$

Following the ideas of A.Kuzhel ([18]) the condition under which $D_A = D_{A^*}$ for the model A from the form (1.21) takes the form

$$D_A = D_{A^*} \iff \operatorname{tr} \int_{-\infty}^{+\infty} (\alpha^2(x)+1)B(x)dx < \infty,$$

(i.e. $D_A = D_{A^*}$ if and only if $(\alpha^2(x)+1)B(x)$ is an integrable matrix function on $\mathbb{R}$).

Remarks. It has to mention that the case when $(\alpha^2(x)+1)B(x)$ is an integrable matrix function on $\mathbb{R}$ the model A of A. Kuzhel, defined by (1.21), coincides with the model of M. S. Livšic for the dissipative operator

$$Af(x) = \alpha(x)f(x) + i \int_{-\infty}^{+\infty} f(\xi)\widetilde{\Pi}(\xi)d\xi\widetilde{\Pi}^*(x)$$

when $D_A = D_{A^*}$, where

$$\widetilde{\Pi}^*(x) = \int_{-\infty}^{\overrightarrow{x}} e^{i\alpha(v)B(v)dv} \Pi^*(x)(\alpha(x)-i)$$

and $\operatorname{tr}(\alpha^2(x)+1)B(x) = \operatorname{tr}\widetilde{\Pi}^*(x)\widetilde{\Pi}(x) = \operatorname{tr}\widetilde{B}(x)$.

Let us consider now the model A with the representation (1.21), where $(\alpha^2(x)+1)B(x)$ is not integrable function on $\mathbb{R}$. This implies that A is a dissipative closed densely defined operator with $D_A \neq D_{A^*}$.

Let us denote

$$D_1 = \{f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n) : \alpha(x)f(x) \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)\} \tag{6.10}$$

We define the family of operators $\{T_t\}_{t>0}$ by the equality (1.22) in the sense of a principal value, where $f = (A-\lambda_0 I)^{-1}g$, $\forall g \in D_1$, λ_0 is an arbitrary fixed number with $\operatorname{Im}\lambda_0 > 0$, δ is an arbitrary number with $0 < \delta < \operatorname{Im}\lambda_0$ and $t > 0$.

Theorem 6.1. *The operator T_t $(t > 0)$, defined by* (1.22)*, satisfies the conditions:*

1) *the integral in* (1.22) *exists in the sense of a principal value for each $f = (A-\lambda_0 I)^{-1}g$, where $g \in D_1$;*

2) *$T_t f(x) \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ for all $f = (A-\lambda_0 I)^{-1}g$, $g \in D_1$;*

3) *$T_t f(x)$ does not depend on the choice of the sufficiently small number $\delta > 0$ for $f = (A-\lambda_0 I)^{-1}g$, $g \in D_1$.*

P r o o f. Let $g \in D_1$ and $f = (A-\lambda_0 I)^{-1}g$, let λ_0 be a fixed number with $\operatorname{Im}\lambda_0 > 0$ and $\delta : 0 < \delta < \operatorname{Im}\lambda_0$. Then after calculations and using the Residue theorem the operator (1.22) takes the form

$$\begin{aligned} T_t f(x) &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{it(\xi-i\delta)}(A-(\xi-i\delta)I)^{-1}f(x)d\xi = \\ &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{it(\xi-i\delta)}(A-(\xi-i\delta)I)^{-1}(A-\lambda_0 I)^{-1}g(x)d\xi = \\ &= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{it(\xi-i\delta)}}{\xi-i\delta-\lambda_0}(A-(\xi-i\delta)I)^{-1}d(x)d\xi + e^{it\lambda_0}f(x). \end{aligned} \tag{6.11}$$

On the other hand from the definition of the multiplicative integral we have

$$\begin{aligned} &\left\| \overset{\rightarrow}{\int_{\xi}^{x}} e^{-i\frac{1+(\xi-i\delta)\alpha(v)}{\alpha(v)-(\xi-i\delta)}B(v)dv} \right\| = \\ &= \left\| \overset{\rightarrow}{\int_{\xi}^{x}} e^{\left(-i\frac{(1+\xi\alpha(v))(\alpha(v)-\xi)-\delta^2\alpha(v)}{(\alpha(v)-\xi)^2+\delta^2)} - \frac{\delta(\alpha^2(v)+1)}{(\alpha(v)-\xi)^2+\delta^2)}\right)B(v)dv} \right\| \leq 1, \end{aligned} \tag{6.12}$$

$\forall \xi, x \in \mathbb{R}$, $\xi \leq x$.

Now from the form (6.4) of the resolvent $(A - (\xi - i\delta)I)^{-1}g$ and the inequality (6.12) it follows the existence of the integral on the right hand side in the last equality in (6.11), i.e. $T_t f$ is defined by (1.22) for all $f = (A - \lambda_0 I)^{-1} g$, $g \in D_1$.

The condition 2) follows from the relations (6.11) and the representation of the resolvent $(A - (\xi - i\delta)I)^{-1}g$ for $g \in D_1$.

Finally for the obtaining of the independence of the definition of $T_t f$ by (1.22) on the choice of $\delta > 0$ $(0 < \delta < \operatorname{Im} \lambda_0)$ we apply the Residue theorem for the function $e^{itz}(A - zI)^{-1} f$ and for a suitable domain in the lower half-plane. Then using the form (6.4) of the resolvent and the inequalities as in (6.12), concerning the multiplicative integrals, we obtain

$$\int_{-\infty}^{+\infty} e^{it(\xi - i\tau)}(A - (\xi - i\tau)I)^{-1} f(x) d\xi = \int_{-\infty}^{+\infty} e^{it(\xi - i\delta)}(A - (\xi - i\delta)I)^{-1} f(x) d\xi$$

for arbitrary $\tau, \delta : 0 < \delta < \tau < \operatorname{Im} \lambda_0$, $f = (A - \lambda_0 I)^{-1} g$ for all $g \in D_1$. The proof is complete. $\square$

The condition 3) in Theorem 6.1 implies that the operators from the family $\{T_t\}_{t>0}$, defined by (1.22), onto the set

$$D_0 = \{f \in D_A : f = (A - \lambda_0 I)^{-1} g \ \ \forall g \in D_1\}$$

are well-defined operators.

Next we will show that the family $\{T_t\}_{t>0}$, defined by (1.22), possesses the properties of the semigroup of operators from the class (C_0) with a differentiability and a generator iA.

Theorem 6.2. *The family of operators* $\{T_t\}_{t>0}$ *defined by* (1.22) *satisfies the conditions*

$$T_s T_t f = T_{s+t} f \quad \forall f \in D_0 \quad (\forall t, s > 0), \tag{6.13}$$

$$\lim_{t \to 0} T_t f = f \quad \forall f \in D_0 \quad (t > 0). \tag{6.14}$$

P r o o f. The proof of (6.13) is analogous to the proof of (4.29) in the case when $t > 0$ (see [13], Theorem 4.1).

Let $f \in D_0$ and then $f = (A - \lambda_0 I)^{-1} g$, where $g \in D_1$. Let $\delta > 0$ be a fixed number such that $0 < \delta < \operatorname{Im} \lambda_0$. From the representation (6.4) of the resolvent, the definition (1.22) of T_t and the Resolvent equation we obtain

$$T_t f(x) = T_t (A - \lambda_0 I)^{-1} g(x) =$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{it(\xi - i\delta)} (A - (\xi - i\delta)I)^{-1} (A - \lambda_0 I)^{-1} g(x) d\xi =$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} ((A - (\xi - i\delta)I)^{-1} g(x) - (A - \lambda_0 I)^{-1} g(x)) d\xi =$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{it(\eta - i\delta)}}{\xi - i\delta - \lambda_0} \frac{1}{\alpha(x) - \eta + i\delta} g(x) d\eta +$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\alpha(\eta) + i) g(\eta) \Pi(\eta) \left(\int_{-\infty}^{+\infty} \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} \cdot \frac{\alpha(x) - i}{\alpha(x) - \xi + i\delta} \cdot \frac{1}{\alpha(\eta) - \xi + i\delta} \cdot \right.$$

$$\left. \cdot \int_{\eta}^{\overrightarrow{x}} e^{-i \frac{1 + (\xi - i\delta)\alpha(v)}{\alpha(v) - \xi + i\delta} B(v) dv} d\xi \right) d\eta \Pi^*(x) + e^{it\lambda_0} f(x).$$

These relations and the Lebesgue convergence theorem show that

$$\lim_{t \to 0, t > 0} T_t f(x) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} (A - (\xi - i\delta)I)^{-1} (A - \lambda_0)^{-1} g(x) d\xi =$$

$$(6.15) \qquad = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{1}{\xi - i\delta - \lambda_0} (A - (\xi - i\delta)I)^{-1} g(x) d\xi + f(x)$$

according to the norm $\| \quad \|_{\mathbf{L}^2}$.

Now we calculate the integral on the right hand side of (6.15) applying the Residue theorem for the function $\varphi(z) = 1/(z - \lambda_0)(A - zI)^{-1} g$ and the domain G with a contour $\Gamma_R = [AB] \cup L_R$, where

$$L_R = \{z = \sqrt{R^2 + \delta^2} e^{i\varphi}; -\pi + \psi_R \le \varphi \le -\psi_R\},$$

$$[AB] = \{z = x - i\delta; -R \le x \le R\},$$

($\psi_R = \arctan \dfrac{\delta}{R}$ for an arbitrary sufficiently large $R > 0$) and then letting $R \to \infty$. These calculations show that

$$\int\limits_{-\infty}^{+\infty} \frac{1}{\xi - i\delta - \lambda_0}(A - (\xi - i\delta)I)^{-1} g(x) d\xi = 0. \tag{6.16}$$

The equalities (6.16) and (6.15) imply that (6.14) holds and the theorem is proved. □

The relation (6.14) allows to define the operator T_t in the case when $t = 0$ by the equality

$$T_0 f(x) = \lim_{t \to 0, t > 0} T_t f(x) = f(x) \qquad \forall f \in D_0. \tag{6.17}$$

The next theorem solves the question of a differentiability of the family $\{T_t\}_{t \geq 0}$, defined by (1.22) and (6.17), and determines the generator of the considered family.

Theorem 6.3. *The family of operators $\{T_t\}_{t\geq 0}$, defined by* (1.22) *and* (6.17), *satisfies the conditions*

$$\frac{d}{dt} T_t f(x) = iAT_t f(x) \qquad \forall f \in \widehat{D}_0, \tag{6.18}$$

$$\lim_{t \to 0, t > 0} \frac{T_t f - f}{t} = iAf \qquad \forall f \in \widehat{D}_0, \tag{6.19}$$

where

$$\widehat{D}_0 = \{f \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n) : f = (A - \lambda_0 I)^{-1}(A - \mu_0 I)^{-1} h, \forall h \in D_1\}$$

($\mu_0 \neq \lambda_0$, $\operatorname{Im} \mu_0 > 0$).

P r o o f. The model A is a closed operator. Then from the representation (6.4) of the resolvent $(A - (\xi - i\delta)I)^{-1}$ direct calculations show that

$$AT_t f(x) = T_t Af(x) \qquad \forall f \in \widehat{D}_0. \tag{6.20}$$

Then if $f \in \widehat{D}_0$, i.e. $f = (A - \lambda_0 I)^{-1} g = (A - \lambda_0 I)^{-1}(A - \mu_0 I)^{-1} h$ $(h \in D_1)$, we obtain

$$iAT_t f(x) = iT_t A(A - \lambda_0 I)^{-1}(A - \mu_0 I)^{-1} h(x) =$$

$$(6.21)\quad = -\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} i(\xi - i\delta)(A - (\xi - i\delta)I)^{-1} g(x) d\xi + \\ + e^{it\lambda_0} i\lambda_0 (A - \lambda_0 I)^{-1} g(x) =$$

$$= -\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{d}{dt}\frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0}(A - (\xi - i\delta)I)^{-1} g(x) d\xi + \frac{d}{dt} e^{it\lambda_0} f(x).$$

Now we will show that the right side of (6.21) is equal to the derivative $\frac{d}{dt} T_t f(x)$ for each $f \in \widehat{D}_0$.

Indeed we have

$$(6.22)\quad \frac{2\pi i}{\tau}(T_{t+\tau} f - T_t f) + \\ + \int\limits_{-\infty}^{+\infty} \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} i(\xi - i\delta)(A - (\xi - i\delta)I)^{-1} g(x) d\xi - i\lambda_0 e^{it\lambda_0} f(x) = \\ = -\int\limits_{-\infty}^{+\infty} \left(\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)\right) \frac{e^{it(\xi - i\delta)}(A - (\xi - i\delta)I)^{-1} g(x)}{\xi - i\delta - \lambda_0} d\xi + \\ + 2\pi i\left(\frac{1}{\tau}(e^{i(t+\tau)\lambda_0} - e^{it\lambda_0}) f(x) - i\lambda_0 e^{it\lambda_0} f(x)\right).$$

These results show that it remains to prove that the integral on the right hand side of (6.22) tends to 0 as $\tau \to 0$. But

$$(6.23)\quad \lim_{\tau \to 0}\left(\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)\right) = 0$$

and direct calculations show that

$$(6.24)\quad \left|\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)\right| \le \begin{cases} \widetilde{C}_1 |\xi| & \text{when} \quad |\tau\xi| \ge \widetilde{\delta}, \\ \widetilde{C}_2 |\xi - i\delta| & \text{when} \quad |\tau\xi| \le \widetilde{\delta}, \end{cases}$$

where $\widetilde{\delta} > 0$ is an arbitrary fixed number, $\widetilde{C}_1$, $\widetilde{C}_2 > 0$ are suitable constant.

From the form of $g(x) = (A - \mu_0 I)^{-1}h(x)$ $(h \in D_1)$ and the form (6.4) of the resolvent of the model A we present the integral on the right hand side of the equality (6.22) in the form

(6.25)

$$\int\limits_{-\infty}^{+\infty} (\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)) \frac{e^{it(\xi - i\delta)}(A - (\xi - i\delta)I)^{-1}g(x)}{\xi - i\delta - \lambda_0} d\xi =$$
$$= \int\limits_{-\infty}^{+\infty} (\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)) \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} \frac{(A - (\xi - i\delta)I)^{-1}h(x)}{\xi - i\delta - \mu_0} d\xi -$$
$$- \int\limits_{-\infty}^{+\infty} (\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)) \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} \frac{1}{\xi - i\delta - \mu_0} d\xi g(x).$$

But direct calculations show that

(6.26)

$$\frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} (\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)) \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} \frac{1}{\xi - i\delta - \mu_0} d\xi g(x) =$$
$$= \left(\left(\frac{1}{\tau}(e^{i\tau\lambda_0} - 1) - i\lambda_0 \right) e^{it\lambda_0} - \left(\frac{1}{\tau}(e^{i\tau\mu_0} - 1) - i\mu_0 \right) e^{it\mu_0} \right) \frac{g(x)}{\lambda_0 - \mu_0}.$$

For the obtaining of (6.26) we have applied the Residue theorem for the function $\varphi_1(z) = ((e^{i\tau z} - 1)/\tau - iz) \dfrac{e^{itz}}{z - \lambda_0} \dfrac{1}{z - \mu_0}$ and an appropriate domain.

For the first integral on the right hand side of (6.25) it follows that

(6.27)

$$\int\limits_{-\infty}^{+\infty} (\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)) \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} \frac{(A - (\xi - i\delta)I)^{-1}h(x)}{\xi - i\delta - \mu_0} d\xi =$$
$$= \int\limits_{-\infty}^{+\infty} (\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)) \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} \frac{1}{\xi - i\delta - \mu_0} \frac{h(x)}{\alpha(x) - \xi + i\delta} d\xi -$$
$$- \int\limits_{-\infty}^{+\infty} (\frac{1}{\tau}(e^{i\tau(\xi - i\delta)} - 1) - i(\xi - i\delta)) \frac{e^{it(\xi - i\delta)}}{\xi - i\delta - \lambda_0} \frac{1}{\xi - i\delta - \mu_0} \cdot$$
$$\cdot \left(\int\limits_{-\infty}^{x} \frac{\alpha(\eta) + i}{\alpha(\eta) - \xi + i\delta} f(\eta) \Pi(\eta) \int\limits_{\eta}^{\overset{x}{\rightarrow}} e^{-i\frac{1 + (\xi - i\delta)\alpha(v)}{\alpha(v) - \xi + i\delta} b(v) dv} d\eta \frac{\Pi^*(x)(\alpha(x) - i)}{\alpha(x) - \xi + i\delta} \right) d\xi$$

For the first integral on the right hand side of (6.27) applying the Residue theorem for the function $\psi(z) = ((e^{i\tau z} - 1)/\tau - iz)\dfrac{e^{itz}}{z-\lambda_0}\dfrac{1}{z-\mu_0}\dfrac{1}{\alpha(x)-z}$ and a suitable domain we obtain

(6.28)

$$-\int_{-\infty}^{+\infty}(\frac{1}{\tau}(e^{i\tau(\xi-i\delta)}-1)-i(\xi-i\delta))\frac{e^{it(\xi-i\delta)}}{\xi-i\delta-\lambda_0}\frac{1}{\xi-i\delta-\mu_0}\frac{1}{\alpha(x)-\xi+i\delta}d\xi =$$

$$= -2\pi i\left(\left(\frac{1}{\tau}(e^{i\tau\lambda_0}-1)-i\lambda_0\right)\frac{e^{it\lambda_0}}{\lambda_0-\mu_0}\frac{1}{\alpha(x)-\lambda_0}+\right.$$

$$+\left(\frac{1}{\tau}(e^{i\tau\mu_0}-1)-i\mu_0\right)\frac{e^{it\mu_0}}{\mu_0-\lambda_0}\frac{1}{\alpha(x)-\mu_0}-$$

$$\left.-\left(\frac{1}{\tau}(e^{i\tau\alpha(x)}-1)-i\alpha(x)\right)\frac{e^{it\alpha(x)}}{\alpha(x)-\lambda_0}\frac{1}{\alpha(x)-\mu_0}\right)\to 0$$

as $\tau \to 0$.

The Lebesgue convergence theorem shows that the second integral on the right hand side of (6.27) tends to 0 as $\tau \to 0$.

Consequently from (6.28), (6.27), (6.26), (6.25), (6.24) it follows that

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}(\frac{1}{\tau}(e^{i\tau(\xi-i\delta)}-1)-i(\xi-i\delta))\frac{e^{it(\xi-i\delta)}}{\xi-i\delta-\lambda_0}(A-(\xi-i\delta)I)^{-1}g(x)d\xi$$

tends to 0 as $\tau \to 0$ according to the norm $\| \ \|_{\mathbf{L}^2}$. This relation together with (6.23), (6.22), (6.21) implies that

$$\frac{1}{\tau}(T_{t+\tau}f - T_t f) \longrightarrow iAT_t f \quad \text{as } \tau \to 0 \ \forall f \in \widehat{D}_0 \tag{6.29}$$

and (6.29) implies that $\dfrac{d}{dt}T_t f = iAT_t f$ for all $f \in \widehat{D}_0$ and (6.18) is proved.

Next the equalities (6.18) and (6.20) together with the closedness of the operator A give the equality (6.19) which proves the theorem. □

It has to mention that for the dissipative model A, defined by (1.21), the equality (6.18) implies that $\{T_t\}_{t\geq 0}$ is an uniformly bounded family of operators. Really, from the relations

$$\frac{d}{dt}\|T_t f\|^2_{\mathbf{L}^2} = -\operatorname{Im}(AT_t f, T_t) \leq 0$$

and the dissipativness of the model A it follows that

$$\|T_t f\|^2_{\mathbf{L}^2} \le \|T_0 f\|^2_{\mathbf{L}^2} \le \|f\|^2_{\mathbf{L}^2} \quad \forall f \in \widehat{D}_0, \tag{6.30}$$

and hence

$$\|T_t\|^2_{\mathbf{L}^2} \le 1. \tag{6.31}$$

Now we can extend the operator T_t by continuity on the whole space $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$. Then the proved properties of the operators T_t imply that the family $\{T_t\}_{t\ge 0}$ is a semigroup from the class (C_0).

The proved properties of the family $\{T_t\}_{t\ge 0}$ given by Theorem 6.2 and Theorem 6.3 allow us to define the dissipative curves generated by the unbounded operator A with the form (1.21) by $T_t f$ for each $f \in \widehat{D}_0$ and T_t are defined by (1.22), (6.17).

Before continuing with the asymptotics of these curves generated from A it has to mention that in the case when $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the conditions:

(i) $\alpha(x)$ is continuous unbounded strictly increasing in $\mathbb{R}$;

(ii) the inverse function $\sigma(u)$ of $\alpha(x)$ is absolutely continuous on $\mathbb{R}$;

(iii) $\sigma'(u)$ is a bounded function on $\mathbb{R}$,

the model A, defined by (1.21), after the change of the variable $x = \sigma(u)$ can be written in the form

$$Ag(u) = ug(u) + i \int_{-\infty}^{u} (\eta + i) g(\eta) \widehat{\Pi}(\eta) \overset{\rightarrow}{\int_{\eta}^{u}} e^{iv\widetilde{B}(v)dv} \widehat{\Pi}^*(u)(u-i)d\eta \tag{6.32}$$

(where $g \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n;\sigma(u))$, $\|\widehat{\Pi}(u)\| \in \mathbf{L}^2(\mathbb{R};\sigma(u))$ when the function $f(\sigma(u)) \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n;\sigma(u))$, $\|\Pi(\sigma(u))\| \in \mathbf{L}^2(\mathbb{R})$).

To avoid complications of the writing we can consider (as in the case of the operators from $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ – part 4) the model

$$Af(x) = xf(x) + i \int_{-\infty}^{x} (\xi + i) f(\xi) \Pi(\xi) \overset{\rightarrow}{\int_{\xi}^{x}} e^{ivB(v)dv} \Pi^*(x)(x-i)d\xi \tag{6.33}$$

(i.e. $\alpha(x) = x$) with a domain D_A which is dense in $\mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$.

It has to mention that the asymptotics of the curves generated by the model (6.32) can be obtained analogously to the asymptotics of the curves generated by the model (6.33) if we suppose additional conditions for $\sigma(u)$ (for example, $\sigma'(u) \in C_{\alpha_2}(\mathbb{R})$, $0 < \alpha_2 \le 1$).

Let the model A be defined by (6.33) and $\Pi(x)$, $B(x)$ be stated as above in this part. Let us denote

$$D_1 = \{f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n) : xf(x) \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)\}. \tag{6.34}$$

Next we need some appropriate denotations similar to the denotations, introduced in part 2 and part 4 for the models from $\widetilde{\mathbf{\Omega}}_{\mathbb{R}}$ and $\widetilde{\mathbf{\Lambda}}_{\mathbb{R}}$ and some preliminary properties, concerning the multiplicative integrals, describing the characteristic function of the operator A from the form (6.33).

Theorem 6.4. *Let the matrix function $B(x)$ is nonnegative and integrable on $\mathbb{R}$. Then for almost all $x \in [a;b]$ there exist the limit values of the multiplicative integral ($\tau > 0$)*

$$\begin{aligned} &s - \lim_{\tau\to 0} \overset{\overset{b}{\rightarrow}}{\int_a} e^{-i\frac{1+(x\pm i\tau)v}{v-x\mp i\tau}B(v)dv} = \\ &= s - \lim_{\varepsilon\to 0} \overset{\overset{x-\varepsilon}{\rightarrow}}{\int_a} e^{-i\frac{1+vx}{v-x}B(v)dv} e^{\pm\pi(1+x^2)B(x)} \overset{\overset{b}{\rightarrow}}{\int_{x+\varepsilon}} e^{-i\frac{1+vx}{v-x}B(v)dv}, \end{aligned} \tag{6.35}$$

where $-\infty \le a < b \le +\infty$.

The existence and the form of the limits (6.35) we have proved, using the ideas of the obtaining of the limits (2.13) in [27].

Let us denote the next operators

$$\widetilde{B}(x) = (1+x^2)B(x), \tag{6.36}$$

$$F_w(x - i\delta, u) = \overset{\overset{u}{\rightarrow}}{\int_w} e^{-i\frac{1+v(x-i\delta)}{v-x+i\delta}B(v)dv}, \tag{6.37}$$

$$F_w^-(x,u) = s - \lim_{\delta\to 0} \overset{\overset{u}{\rightarrow}}{\int_w} e^{-i\frac{1+v(x-i\delta)}{v-x+i\delta}B(v)dv}, \tag{6.38}$$

for all $w, u, x \in \mathbb{R}$ such that $-\infty \le w < u \le +\infty$ and

$$F_w^-(x,u) = s - \lim_{\delta\to 0} \overset{\overset{x-\delta}{\rightarrow}}{\int_w} e^{-i\frac{1+vx}{v-x}B(v)dv} e^{-\pi\widetilde{B}(x)} \overset{\overset{u}{\rightarrow}}{\int_{x+\delta}} e^{-i\frac{1+vx}{v-x}B(v)dv}, \tag{6.39}$$

$$R_w(x) = s-\lim_{\delta\to 0} \overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{-i\frac{1+vx}{v-x}B(v)dv} e^{\pi \widetilde{B}(x)} \left(\overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{-i\frac{1+vx}{v-x}B(v)dv} \right)^{-1}, \tag{6.40}$$

$$U_{2w}(x) = s-\lim_{\delta\to 0} \overset{\overrightarrow{x-\delta}}{\underset{w}{\int}} e^{-i\frac{1+vx}{v-x}B(v)dv} e^{iB(x)\int_w^{x-\delta} \frac{1+vx}{v-x}dv} e^{-iB(x)x(x-\delta-w)}, \tag{6.41}$$

$$P^-_{2w}(x,u) = R_w^{-1}(x)U_{2w}(x)e^{i\widetilde{B}(x)\ln\frac{x-w}{u-x}}, \tag{6.42}$$

$$U_3(x,u) = \lim_{\delta\to 0} e^{-iB(x)x(u-x-\delta)} e^{iB(x)\int_{x+\delta}^{u} \frac{1+vx}{v-x}dv} \overset{\overrightarrow{u}}{\underset{x+\delta}{\int}} e^{-i\frac{1+vx}{v-x}B(v)dv}, \tag{6.43}$$

$$Q^-_w(x,u) = P^-_{2w}(x,u)e^{i\widetilde{B}(x)\ln(u-x)}e^{-i\widetilde{B}(u)\ln(u-x)}, \tag{6.44}$$

$$V_{-\infty}(x) = \lim_{\delta\to 0} \overset{\overrightarrow{x-\delta}}{\underset{-\infty}{\int}} e^{-i\frac{1+vx}{v-x}B(v)dv} e^{i\widetilde{B}(x)\ln\delta} \tag{6.45}$$

for all w, u, x such that $-\infty \le w < x < u \le +\infty$.

The existence of these limits follows from the limit values of the considered multiplicative integrals, given by Theorem 6.4.

The next representation presents the resolvent $(A-\lambda I)^{-1}$ in a suitable form which we will use in the representation of the family $\{T_t\}_{t\ge 0}$.

Let $Q(x)$ be a measurable matrix function which satisfies the condition

$$\Pi(x)Q(x) = I,$$

$Q^*(x)$ is a smooth matrix function with $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$. Then for each f from the set

$$\begin{aligned} H_0 = \{f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n) : f' \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n), \|f(x)\| \le M_f, \\ \|f'(x)\| \le M_f, \lim_{x\to\pm\infty} f(x)Q^*(x) = 0\} \end{aligned} \tag{6.46}$$

and $\lambda : \operatorname{Im}\lambda \neq 0$ the resolvent $(A-\lambda I)^{-1}$ has the representation

$$(A-\lambda I)^{-1} f(x) = \int\limits_{-\infty}^{x} \widetilde{f}(w) \int\limits_{w}^{\overrightarrow{x}} e^{-i\frac{1+\lambda v}{v-\lambda}B(v)dv} dw \Pi^*(x) \frac{x-i}{x-\lambda}, \tag{6.47}$$

where

$$\widetilde{f}(w) = -i\frac{w}{w-i} f(w)\Pi(w) + \left(\frac{1}{w-i} f(w) Q^*(w)\right)'. \tag{6.48}$$

Let the matrix function $B(x)$ satisfies the conditions:

1) $\|B(x)\| \leq C$, $\|xB(x)\| \leq C$ $\forall x \in \mathbb{R}$;

2) $B(x) \in C_{\alpha_1}(\mathbb{R})$, $xB(x) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_1 \leq 1,\ 0 < \alpha_2 \leq 1)$ (i.e. $\|B(x_1) - B(x_2)\| \leq C|x_1 - x_2|^{\alpha_1}$, $\|x_1 B(x_1) - x_2 B(x_2)\| \leq C|x_1-x_2|^{\alpha_2}$).

Let $\alpha = \min\{\alpha_1, \alpha_2\}$. Then the next inequalities hold:

Lemma 6.5.

$$\|e^{iB(x)(1+x^2)\ln(x-\xi)} - e^{iB(\xi)(1+\xi^2)\ln(x-\xi)}\| \leq \widetilde{C}(1+|x|)|x-\xi|^{\alpha'} \tag{6.49}$$

for some constant $\widetilde{C} > 0$, for all $\alpha' : 0 < \alpha' < \alpha$ and for x, ξ such that $0 < x - \xi < 1$.

Lemma 6.6 *For each $\alpha' : 0 < \alpha' < \alpha$ there exists a constant $\widetilde{C} > 0$ such that*

$$\|e^{iB(\xi)(1+\xi^2)\ln(\xi-w)} - e^{iB(x)(1+x^2)\ln(x-w)}\| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{6.50}$$

for all $w, \xi, x : w < \xi < x$, $0 < x - w < 1$.

Lemma 6.7.

$$\|F_w^-(\xi;x) - Q_w^-(\xi;x)\| \leq \widetilde{C}(1+|x|)(x-\xi)^{\alpha'} \tag{6.51}$$

for some constant $\widetilde{C} > 0$, for all $w, \xi, x : w < \xi < x$, $0 < x - w < 1$ and $\forall \alpha' : 0 < \alpha' < \alpha$.

Lemma 6.8.

$$\|U_{2w}(x) - U_{2w}(\xi)\| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{6.52}$$

for some constant $\widetilde{C} > 0$*, for all* $w, \xi, x : w < \xi < x$*,* $0 < x - w < 1$ *and* $\alpha' = \alpha/(1+\alpha)$*.*

Lemma 6.9.

$$\|R_w^{-1}(\xi) - R_w^{-1}(x)\| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{6.53}$$

for some constant $\widetilde{C} > 0$*, for all* $w, \xi, x : w < \xi < x$*,* $0 < x - w < 1$ *and* $\alpha' = \alpha/(1+\alpha)$*.*

Lemma 6.10.

$$\|Q_w^{-}(x) - Q_w^{-}(\xi)\| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} \tag{6.54}$$

for some constant $\widetilde{C} > 0$*, for all* $w, \xi, x : w < \xi < x$*,* $0 < x - w < 1$ *and* $\alpha' = \alpha/(1+\alpha)$*.*

Lemma 6.11.

$$\left\| \int_w^{\overset{x}{\rightarrow}} e^{-i\frac{1+v\xi}{v-\xi}B(v)dv} - U_{2w}(\xi)e^{-iB(\xi)(1+\xi^2)\ln\frac{\xi-x}{\xi-w}} \right\| \leq \widetilde{C}(1+|x|)(\xi-x)^{\alpha'} \tag{6.55}$$

for some constant $\widetilde{C} > 0$*, for all* $w, \xi, x : w < x < \xi < x + \beta$*,* $\beta < 1$*, and for each* $\alpha' : 0 < \alpha' \leq \alpha \leq 1$*.*

Lemma 6.12.

$$\|U_3(x,u) - U_3(\xi,u)\| \leq \widetilde{C}(1+|x|)\left(\frac{x-\xi}{u-x}\right)^{\alpha'} \tag{6.56}$$

for some constant $\widetilde{C} > 0$*, for all* $\xi, x, u : \xi < x < u$*,* $0 < u - \xi < 1$*, and* $\alpha' = \alpha/(1+\alpha)$*.*

Lemma 6.13.

$$\|F_w^{-}(\xi,u) - F_w^{-}(x,u)\| \leq \widetilde{C}(1+|x|)\left(\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} + \left(\frac{x-\xi}{u-x}\right)^{\alpha'}\right), \tag{6.57}$$

$$\|U_w(x,u) - U_w(\xi,u)\| \leq \widetilde{C}(1+|x|)\left(\left(\frac{x-\xi}{\xi-w}\right)^{\alpha'} + \left(\frac{x-\xi}{u-x}\right)^{\alpha'}\right), \tag{6.58}$$

for some constant $\widetilde{C} > 0$, *for all* $w, \xi, x, u : w < \xi < x < u$, $0 < u - w < 1$, $\alpha' = \alpha/(1+\alpha)$, *where* $U_w(x,u)$ *is defined by the equality*

$$U_w(x,u) = R_w(x)F_w^-(x,u). \tag{6.59}$$

All of these inequalities follow from the properties of the multiplicative integrals using the ideas of proving of similar inequalities, proved in [27] and [12].

The next lemma presents a suitable representation of the curve $T_t f$ which allows to obtain the asymptotics of the curve $T_t f$ as $t \to +\infty$ when f belongs to a suitable subset of $\mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$.

Lemma 6.14. *The operator* T_t, *defined by* (1.22), *possesses the next representation*

(6.60)

$$T_t f(x) = T_t(A - \lambda_0 I)^{-1}(A - \mu_0 I)^{-1} h(x) =$$

$$= -\frac{1}{2\pi i} \int\limits_{\mathbb{R}\setminus\Delta} \frac{e^{it\xi}}{\xi - \lambda_0} \frac{1}{\xi - \mu_0} \frac{x - i}{x - \xi} \left(\int\limits_{-\infty}^{x} \widetilde{h}(\eta) F_\eta^-(\xi, x) d\eta \right) d\xi \Pi^*(x) -$$

$$- \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int\limits_{\Delta} \frac{e^{it\xi}}{(x - \xi)^{1-\varepsilon}} \frac{x - i}{\xi - \lambda_0} \frac{1}{\xi - \mu_0} \left(\int\limits_{-\infty}^{x} \widetilde{h}(\eta) F_\eta^-(\xi, x) d\eta \right) d\xi \Pi^*(x) +$$

$$+ \frac{e^{it\lambda_0} - e^{it\mu_0}}{\lambda_0 - \mu_0} g(x) + e^{it\lambda_0} f(x),$$

where $f(x) = (A - \lambda_0 I)^{-1}(A - \mu_0 I)^{-1} h(x)$, $h \in D_1 \cap H_0$, $\Delta = [x - \beta; x + \beta]$, β *is an arbitrary fixed number:* $0 < \beta < 1$, $\lambda_0 \neq \mu_0$, $\operatorname{Im} \lambda_0, \operatorname{Im} \mu_0 > 0$ *and* $\widetilde{h}(x)$ *is defined by*

$$\widetilde{h}(x) = \frac{-ix}{x - i} h(x) \Pi(x) + \left(\frac{1}{x - i} h(x) Q^*(x) \right)'. \tag{6.61}$$

Now we are in a position to give the asymptotics of the dissipative curve $T_t f(x)$ and these asymptotics are presented by Theorem 1.7.

P r o o f o f T h e o r e m 1.7. Let $h \in D_1 \cap H_0$ and $f = (A - \lambda_0 I)^{-1}(A - \mu_0 I)^{-1} h$ ($\operatorname{Im} \lambda_0 > 0, \operatorname{Im} \mu_0 > 0$). The representation (6.60) of the curve $T_t f(x)$ shows that its asymptotic behaviour as $t \to +\infty$ depends only on the behaviour of

$$\lim_{\varepsilon\to 0}\int\limits_{\Delta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\frac{x-i}{\xi-\lambda_0}\frac{1}{\xi-\mu_0}\left(\int\limits_{-\infty}^{x}\widetilde{h}(w)F_w^-(\xi,x)dw\right)d\xi\Pi^*(x) \tag{6.62}$$

where $\Delta=[x-\beta;x+\beta]$, β is a fixed number: $0<\beta<1$. The other addends in (6.60) tend to 0 as $t\to+\infty$ which follows directly. Next we use the methods and ideas as in [12, 13], but using suitable inequalities, concerning multiplicative integrals, presented in the previous lemmas of this part.

Direct calculations show that (as $t\to+\infty$):

$$\lim_{\varepsilon\to 0}\int\limits_{\Delta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\frac{x-i}{\xi-\lambda_0}\frac{1}{\xi-\mu_0}\left(\int\limits_{-\infty}^{x}\widetilde{h}(w)F_w^-(\xi,x)dw\right)d\xi\Pi^*(x)=$$

$$=\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\frac{x-i}{\xi-\lambda_0}\frac{1}{\xi-\mu_0}F_w^-(\xi,x)d\xi\right)dw\Pi^*(x)\sim$$

$$\sim\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}\Pi^*(x)=$$

$$\begin{aligned}&=\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi+\right.\\&\left.+\int\limits_{x}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}\Pi^*(x)=\\&=\lim_{\varepsilon\to 0}\left(\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw+\right.\\&+\int\limits_{x-\beta}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{w}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw+\end{aligned}\tag{6.63}$$

$$+\int\limits_{x-\beta}^{x}\widetilde{h}(w)\left(\int\limits_{w}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw+$$

$$+\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw\Bigg)\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}.$$

Now we will obtain separately the asymptotic behaviour as $t\to+\infty$ of the four integrals on the right hand side of the last equality in (6.63). At first for the second integral straightforward calculations show that

(6.64)

$$\lim_{\varepsilon\to 0}\int\limits_{x-\beta}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{w}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}=$$

$$=\lim_{\varepsilon\to 0}\int\limits_{x-\beta}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{w}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\overset{\overset{x}{\rightarrow}}{\int\limits_{w}}e^{-i\frac{1+\xi v}{v-\xi}B(v)dv}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}=$$

$$=\int\limits_{x-\beta}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{w}\frac{e^{it\xi}}{x-\xi}\overset{\overset{x}{\rightarrow}}{\int\limits_{w}}e^{-i\frac{1+\xi v}{v-\xi}B(v)dv}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}\to 0$$

as $t\to+\infty$ (using the Lebesgue lemma for the Fourier transform).

For the first integral in (6.63) we obtain consecutively

$$\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}=$$

$$=\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\overset{\overset{x-\beta}{\rightarrow}}{\int\limits_{w}}e^{-i\frac{1+\xi v}{v-\xi}B(v)dv}F_{x-\beta}^-(\xi,x)d\xi\right)dw\cdot$$

$$\cdot\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}\sim$$

$$\sim\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\overset{\overset{x-\beta}{\rightarrow}}{\int\limits_{w}}e^{-i\frac{1+vx}{v-x}B(v)dv}\left(\int\limits_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_{x-\beta}^-(\xi,x)d\xi\right)dw\cdot$$

$$\cdot\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}\sim$$

$$\sim\lim_{\varepsilon\to 0}\int_{-\infty}^{x}\widetilde{h}(w)\overset{\overset{x-\beta}{\to}}{\int_{w}}e^{-i\frac{1+vx}{v-x}B(v)dv}\left(\int_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}Q^-_{x-\beta}(\xi,x)d\xi\right)dw\cdot \tag{6.65}$$

$$\cdot\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}=$$

$$=\lim_{\varepsilon\to 0}\int_{-\infty}^{x}\widetilde{h}(w)\overset{\overset{x-\beta}{\to}}{\int_{w}}e^{-i\frac{1+vx}{v-x}B(v)dv}\cdot$$

$$\cdot\left(\int_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}Q^-_{x-\beta}(\xi)e^{-i\widetilde{B}(x)\ln(x-\xi)}d\xi\right)dw\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}\sim$$

$$\sim\lim_{\varepsilon\to 0}\int_{-\infty}^{x}\widetilde{h}(w)\overset{\overset{x-\beta}{\to}}{\int_{w}}e^{-i\frac{1+vx}{v-x}B(v)dv}Q^-_{x-\beta}(x)\cdot$$

$$\cdot\left(\int_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(x)\ln(x-\xi)}d\xi\right)dw\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0},$$

as $t\to+\infty$. In (6.65) we have used the inequality

$$\left\|\overset{\overset{x-\beta}{\to}}{\int_{w}}e^{-i\frac{1+vx}{v-x}B(v)dv}-\overset{\overset{x-\beta}{\to}}{\int_{w}}e^{-i\frac{1+v\xi}{v-\xi}B(v)dv}\right\|\le\frac{C_1}{\beta}|x-\xi|(1+\beta+|x|)\int_{w}^{x-\beta}\frac{1}{\xi-v}dv\le$$
$$\le\frac{C_1}{\beta}|x-\xi|(1+\beta+|x|)\frac{\beta^\varepsilon}{\delta}(x-\beta-w)^{1-\delta}\beta^\delta$$

($C_1>0$ is a suitable constant, $\delta:0<\delta<1$ is an arbitrary fixed sufficiently small number), Lemma 6.7, Lemma 6.10, the Lebesgue convergence theorem and the form of $Q^-_w(\xi,x)$:

$$Q^-_w(\xi,x)=Q^-_w(\xi)e^{-i\widetilde{B}(x)\ln(x-\xi)}. \tag{6.66}$$

For the third integral in (6.63) from Lemma 6.7, Lemma 6.10 and the form (6.66) of $Q_w^-(\xi,x)$ applying the Lebesgue convergence theorem and the Lebesgue lemma for the Fourier transform it follows that

(6.67)
$$\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{w}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}\sim$$
$$\sim\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)Q_w^-(x)\left(\int\limits_{w}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(x)\ln(x-\xi)}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}$$

as $t\to+\infty$.

For the last integral in (6.63) we have

(6.68)
$$\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}F_w^-(\xi,x)d\xi\right)dw\Pi^*(x)\frac{x-i}{x-\lambda_0}\frac{1}{x-\mu_0}=$$
$$=\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\int\limits_{w}^{\overset{x}{\rightarrow}}e^{-i\frac{1+v\xi}{v-\xi}B(v)dv}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}\sim$$
$$\sim\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}U_{2w}(\xi)}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(\xi)\ln\frac{\xi-x}{\xi-w}}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}\sim$$
$$\sim\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)U_{2w}(x)\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}e^{-i\widetilde{B}(\xi)\ln\frac{\xi-x}{\xi-w}}}{(x-\xi)^{1-\varepsilon}}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}\sim$$
$$\sim\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)U_{2w}(x)e^{i\widetilde{B}(x)\ln(x-w)}\cdot$$
$$\cdot\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(\xi)\ln(\xi-x)}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}\sim$$
$$\sim\lim_{\varepsilon\to 0}\int\limits_{-\infty}^{x}\widetilde{h}(w)U_{2w}(x)e^{i\widetilde{B}(x)\ln(x-w)}\cdot$$
$$\cdot\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(x)\ln(\xi-x)}d\xi\right)dw\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}$$

as $t \to +\infty$.

In the course of obtaining of the relations (6.68) we have applied consecutively Lemma 6.11, Lemma 6.8, Lemma 6.6, Lemma 6.5.

Now from (6.63), (6.64), (6.65), (6.67) and (6.68) it follows that

(6.69)
$$\lim_{\varepsilon\to 0}\int\limits_{x-\beta}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\cdot\frac{x-i}{\xi-\lambda_0}\cdot\frac{1}{\xi-\mu_0}\left(\int\limits_{-\infty}^{x}\widetilde{h}(w)F_w^-(\xi,x)dw\right)d\xi\Pi^*(x)\sim$$
$$\sim\lim_{\varepsilon\to 0}\left(\int\limits_{-\infty}^{x-\beta}\widetilde{h}(w)Q_w^-(x)\left(\int\limits_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(x)\ln(x-\xi)}d\xi\right)dw+\right.$$
$$+\int\limits_{x-\beta}^{x}\widetilde{h}(w)Q_w^-(x)\left(\int\limits_{x-\beta}^{x}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(x)\ln(x-\xi)}d\xi\right)dw+$$
$$+\int\limits_{-\infty}^{x}\widetilde{h}(w)U_{2w}(x)e^{i\widetilde{B}(x)\ln(x-w)}\cdot$$
$$\left.\cdot\left(\int\limits_{x}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}e^{-i\widetilde{B}(x)\ln(\xi-x)}d\xi\right)dw\right)\frac{x-i}{x-\lambda_0}\frac{\Pi^*(x)}{x-\mu_0}$$

as $t\to+\infty$, where $Q_w^-(x)=\overset{\overset{x-\beta}{\rightarrow}}{\int\limits_{w}}e^{-i\frac{1+vx}{v-x}B(v)dv}Q_{x-\beta}^-(x)$.

Next we consider the inner integrals on the right hand side of the relations (6.69) and after a suitable change of the variables we obtain

(6.70)
$$\int\limits_{w}^{x}e^{it\xi}(x-\xi)^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln(x-\xi)}d\xi=$$
$$=t^{-\varepsilon}(-i)^{\varepsilon}e^{itx}e^{i\widetilde{B}(x)\ln t}e^{-\frac{\pi}{2}\widetilde{B}(x)}\int\limits_{0}^{i(x-w)t}e^{-\theta}\theta^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln\theta}d\theta,$$

$$\int_{x-\beta}^{x} e^{it\xi}(x-\xi)^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln(x-\xi)}d\xi = \tag{6.71}$$
$$= t^{-\varepsilon}(-i)^{\varepsilon}e^{itx}e^{i\widetilde{B}(x)\ln t}e^{-\frac{\pi}{2}\widetilde{B}(x)}\int_{0}^{it\beta} e^{-\theta}\theta^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln\theta}d\theta,$$

$$\int_{x}^{x+\beta} e^{it\xi}(x-\xi)^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln(\xi-x)}d\xi = \tag{6.72}$$
$$= t^{-\varepsilon}(-1)^{\varepsilon-1}e^{itx}t^{-\varepsilon}i^{\varepsilon}e^{i\widetilde{B}(x)\ln t}e^{\frac{\pi}{2}\widetilde{B}(x)}\int_{0}^{-it\beta} e^{-\theta}\theta^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln\theta}d\theta.$$

The relation (6.69), the equalities (6.70), (6.71), (6.72) and the equality

$$Q_w^-(x)e^{-i\widetilde{B}(x)\ln(x-\xi)} = U_{2w}(x)e^{i\widetilde{B}(x)\ln(x-w)}e^{-i\widetilde{B}(x)\ln(x-\xi)}e^{-\pi\widetilde{B}(x)} \tag{6.73}$$

imply that the next relations hold as $t \to +\infty$

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{x-\beta}^{x+\beta}\frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}}\frac{x-i}{\xi-\lambda_0}\frac{1}{\xi-\mu_0}\left(\int_{-\infty}^{x}\widetilde{h}(w)F_w^-(\xi,x)dw\right)d\xi\Pi^*(x) \sim$$
$$\sim \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\left(\int_{-\infty}^{x-\beta}\widetilde{h}(w)Q_w^-(x)t^{-\varepsilon}(-i)^{\varepsilon}e^{itx}e^{i\widetilde{B}(x)\ln t}e^{-\frac{\pi}{2}\widetilde{B}(x)}\,\cdot\right.$$
$$\cdot\left(\int_{0}^{it\beta}e^{-\theta}\theta^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln\theta}d\theta\right)dw +$$
$$+\int_{x-\beta}^{x}\widetilde{h}(w)Q_w^-(x)t^{-\varepsilon}(-i)^{\varepsilon}e^{itx}e^{i\widetilde{B}(x)\ln t}e^{-\frac{\pi}{2}\widetilde{B}(x)}\,\cdot$$
$$\left(\int_{0}^{it\beta}e^{-\theta}\theta^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln\theta}d\theta - \int_{i(x-w)t}^{it\beta}e^{-\theta}\theta^{\varepsilon-1}e^{-i\widetilde{B}(x)\ln\theta}d\theta\right)dw + \tag{6.74}$$
$$+\int_{-\infty}^{x}\widetilde{h}(w)U_{2w}(x)e^{i\widetilde{B}(x)\ln(x-w)}(-1)^{\varepsilon-1}t^{-\varepsilon}i^{\varepsilon}e^{itx}e^{i\widetilde{B}(x)\ln t}e^{\frac{\pi}{2}\widetilde{B}(x)}\,.$$

$$\cdot \left(\int_0^{-it\beta} e^{-\theta} \theta^{\varepsilon - 1} e^{-i\widetilde{B}(x) \ln \theta} d\theta \right) dw \Bigg) \Pi^*(x) \frac{x-i}{x-\lambda_0} \cdot \frac{1}{x-\mu_0} =$$

$$= \frac{e^{itx}}{2\pi i} \int_{-\infty}^{x} \widetilde{h}(w) U_{2w}(x) e^{i\widetilde{B}(x) \ln(x-w)} dw e^{i\widetilde{B}(x) \ln t}.$$

$$\cdot \lim_{\varepsilon \to 0} \Bigg(e^{-\frac{3\pi}{2}\widetilde{B}(x)} \int_0^{it\beta} e^{-\theta} \theta^{\varepsilon-1} e^{-i\widetilde{B}(x) \ln \theta} d\theta -$$

$$- e^{\frac{\pi}{2}\widetilde{B}(x)} \int_0^{-it\beta} e^{-\theta} \theta^{\varepsilon-1} e^{-i\widetilde{B}(x) \ln \theta} d\theta \Bigg) \Pi^*(x) \frac{x-i}{x-\lambda_0} \cdot \frac{1}{x-\mu_0}.$$

Now straightforward calculations with the help of the properties of of the gamma-function $\mathbf{\Gamma}(\varepsilon I - i\widetilde{B}(x))$, presented in Lemma 2.1, Lemma 2.2, show that

$$\begin{aligned} &\Bigg\| \lim_{\varepsilon \to 0} \Bigg(e^{-\frac{3\pi}{2}\widetilde{B}(x)} \int_0^{it\beta} e^{-\theta} \theta^{\varepsilon-1} e^{-i\widetilde{B}(x) \ln \theta} d\theta - \\ &- e^{\frac{\pi}{2}\widetilde{B}(x)} \int_0^{-it\beta} e^{-\theta} \theta^{\varepsilon-1} e^{-i\widetilde{B}(x) \ln \theta} d\theta + \\ &+ 2e^{-\frac{\pi}{2}\widetilde{B}(x)} \sinh(\pi \widetilde{B}(x)) \mathbf{\Gamma}(\varepsilon I - i\widetilde{B}(x)) \Bigg) \Bigg\| \leq C \end{aligned} \tag{6.75}$$

for all $x \in \mathbb{R}$, $\forall t > 0$ sufficiently large, where $C > 0$ is a suitable constant.

On the other side direct calculations give the relations

$$\begin{aligned} &\frac{e^{itx}}{2\pi i} \int_{-\infty}^{x} \widetilde{h}(w) U_{2w}(x) e^{i\widetilde{B}(x) \ln(x-w)} dw e^{i\widetilde{B}(x) \ln t}. \\ &\cdot \lim_{\varepsilon \to 0} \Bigg(e^{-\frac{3\pi}{2}\widetilde{B}(x)} \int_0^{it\beta} e^{-\theta} \theta^{\varepsilon-1} e^{-i\widetilde{B}(x) \ln \theta} d\theta - \\ &- e^{\frac{\pi}{2}\widetilde{B}(x)} \int_0^{-it\beta} e^{-\theta} \theta^{\varepsilon-1} e^{-i\widetilde{B}(x) \ln \theta} d\theta + \\ &+ 2e^{-\frac{\pi}{2}\widetilde{B}(x)} \sinh(\pi \widetilde{B}(x)) \mathbf{\Gamma}(\varepsilon I - i\widetilde{B}(x)) \Bigg) \Pi^*(x) \frac{x-i}{x-\lambda_0} \cdot \frac{1}{x-\mu_0} \longrightarrow 0 \end{aligned} \tag{6.76}$$

as $t \to +\infty$ by the norm $\| \ \|$.

Consequently the relations (6.74), (6.75), (6.76) together with the equality (see Lemma 2.2)

$$\lim_{\varepsilon \to 0} \mathbf{\Gamma}(\varepsilon I - i\widetilde{B}(x)) \sinh \pi \widetilde{B}(x) = \pi i \mathbf{\Gamma}^{-1}(I + i\widetilde{B}(x))$$

imply that

(6.77)
$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{x-\beta}^{x+\beta} \frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}} \frac{x-i}{\xi - \lambda_0} \frac{1}{\xi - \mu_0} \left(\int\limits_{-\infty}^{x} \widetilde{h}(w) F_w^-(\xi, x) dw \right) d\xi \Pi^*(x) \sim$$
$$\sim -e^{itx} \int\limits_{-\infty}^{x} \widetilde{h}(w) U_{2w}(x) e^{i\widetilde{B}(x) \ln(x-w)} dw e^{i\widetilde{B}(x) \ln t} e^{-\frac{\pi}{2}\widetilde{B}(x)}.$$
$$\cdot \mathbf{\Gamma}^{-1}(I + i\widetilde{B}(x)) \Pi^*(x) \frac{x-i}{x-\lambda_0} \cdot \frac{1}{x - \mu_0}$$

as $t \to +\infty$. Now using the next representation of $U_{2w}(x)$

$$U_{2w}(x) = \int\limits_{-\infty}^{\overleftarrow{w}} e^{i\frac{1+vx}{v-x} B(v) dv} V_{-\infty}(x)(x-w)^{-i\widetilde{B}(x)},$$

where $V_{-\infty}(x)$ is defined by (6.45), we obtain
(6.78)
$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{x-\beta}^{x+\beta} \frac{e^{it\xi}}{(x-\xi)^{1-\varepsilon}} \frac{x-i}{\xi - \lambda_0} \frac{1}{\xi - \mu_0} \left(\int\limits_{-\infty}^{x} \widetilde{h}(w) F_w^-(\xi, x) dw \right) d\xi \Pi^*(x) \sim$$
$$\sim e^{itx} \int\limits_{-\infty}^{x} \widetilde{h}(w) \int\limits_{-\infty}^{\overleftarrow{w}} e^{i\frac{1+vx}{v-x} B(v) dv} dw V_{-\infty}(x) t^{i\widetilde{B}(x)} e^{-\frac{\pi}{2}\widetilde{B}(x)}.$$
$$\cdot \mathbf{\Gamma}^{-1}(I + i\widetilde{B}(x)) \Pi^*(x) \frac{x-i}{x-\lambda_0} \cdot \frac{1}{x - \mu_0}$$

as $t \to +\infty$, which finishes the proof of the theorem. $\square$

For the simplification of the writing let us denote the next operators:

(6.79)
$$\widehat{S} f(x) = \int\limits_{-\infty}^{x} \widetilde{h}(w) \int\limits_{-\infty}^{\overleftarrow{w}} e^{i\frac{1+vx}{v-x} B(v) dv} dw \frac{x-i}{x-\lambda_0} \cdot \frac{1}{x-\mu_0},$$

where $f(x)=(A-\lambda_0 I)^{-1}(A-\mu_0 I)^{-1}h(x)$, $h(x)\in D_1\cap H_0\cap S(\mathbb{R},\mathbb{C}^n)$, $\widetilde{h}$ is defined by (6.61),

$$T_+p = pV_{-\infty}(x)e^{-\frac{\pi}{2}\widetilde{B}(x)}\mathbf{\Gamma}^{-1}(I+i\widetilde{B}(x))\Pi^*(x), \quad \forall p\in\mathbb{C}^m, \tag{6.80}$$

$$\widehat{Z}(t,x)=\Pi(x)t^{i\widetilde{B}(x)}Q(x), \tag{6.81}$$

$$\widetilde{S}_+f(x)=T_+\widehat{S}f(x). \tag{6.82}$$

Using these denotations (6.79), (6.80), (6.81) the operator S_+, describing the asymptotics (1.23), takes the form

$$S_+f(x)=\widehat{Z}(t,x)T_+\widehat{S}f(x)=\widehat{Z}(t,x)\widetilde{S}_+f(x). \tag{6.83}$$

In the viewpoint of the next considerations it is suitable to consider the case when $\lambda_0=i$. Let us denote also the subspace

$$\begin{aligned}\widetilde{\mathcal{D}}_0=\{f\in\mathbf{L}^2(\mathbb{R};\mathbb{C}^n): f=(A-iI)^{-1}(A-\mu_0 I)^{-1}h,\\ h\in D_1\cap H_0\cap S(\mathbb{R},\mathbb{C}^n)\}\end{aligned} \tag{6.84}$$

The next theorem proves the boundedness of the operators $\dfrac{1}{x-i}\widehat{S}$ and S_+, describing the asymptotics (1.23).

Theorem 6.15. *If* $\|e^{-2\pi\widetilde{B}(x)}\|_{\mathbf{L}^2}<1$ *then the operators* $\dfrac{1}{x-i}\widehat{S}$ *and* S_+ *defined by* (6.79) *and* (6.83) *are bounded operators in the subspace* $\widetilde{\mathcal{D}}_0$.

Proof. From the form (6.83) of S_+, (6.80), (6.81) and the properties of the gamma-function (Lemma 2) it follows that

$$\begin{aligned}\|S_+f(x)\|^2_{\mathbf{L}^2}&=\|\widehat{Z}(t,x)T_+\widehat{S}f(x)\|^2_{\mathbf{L}^2}=\\ &=\|(\frac{1}{x-i}\widehat{S}f(x))T_+(x-i)\|^2_{\mathbf{L}^2}=\\ &=((\frac{1}{x-i}\widehat{S}f(x))V_{-\infty}(x)(I-e^{-2\pi\widetilde{B}(x)})\frac{1}{2\pi},(\frac{1}{x-i}\widehat{S}f(x))V_{-\infty}(x))=\\ &=\|\frac{1}{\sqrt{2\pi}}(\frac{1}{x-i}\widehat{S}f(x))V_{-\infty}(x)\sqrt{I-e^{-2\pi\widetilde{B}(x)}}\|^2_{\mathbf{L}^2}\end{aligned} \tag{6.85}$$

for each $f \in \widetilde{\mathcal{D}}_0$.

On the other hand from (6.30), the asymptotics (1.23) and the equality (6.85) we obtain

$$\left\| \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x-i} \widehat{S} f(x) \right) V_{-\infty}(x) \sqrt{I - e^{-2\pi \widetilde{B}(x)}} \right\|_{\mathbf{L}^2} \leq \|f\|_{\mathbf{L}^2}.$$

The last inequality together with the existence of the bounded inverse operator of $\sqrt{I - e^{-2\pi \widetilde{B}(x)}}$ implies that $\frac{1}{x-i} \widehat{S} f(x)$ is a bounded operator on the subspace $\widetilde{\mathcal{D}}_0$.

Hence from (6.85) it follows that S_+ is a bounded operator on $\widetilde{\mathcal{D}}_0$ and the proof is complete. □

The boundedness of the operators from the semigroup $\{T_t\}_{t\geq 0}$ in the subspace $\widetilde{\mathcal{D}}_0$, defined by (6.84), and Theorem 6.15 allow to extend T_t and S_+ by continuity onto $\mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$. In this way using the properties of $\{T_t\}_{t\geq 0}$ we can define an **exponential function** e^{itA} for $t \geq 0$ by the equality $e^{itA} = T_t$ and consider the dissipative continuous curves

$$e^{itA} f = T_t f, \qquad t \geq 0, \qquad \forall f \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n).$$

Now Theorem 1.7 and Theorem 6.15 imply that the next relation holds

$$\|e^{itA} f(x) - e^{itx} S_+ f(x)\|_{\mathbf{L}^2} \longrightarrow 0 \quad as \;\; t \to +\infty$$

for each $f \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$, where S_+ is defined by (6.83) (or (1.24)) for $\lambda_0 = i$.

The next theorem presents the behaviour of the corresponding correlation function $V(t+\tau, s+\tau) = (e^{i(t+\tau)A} f, e^{i(s+\tau)A} f)$ $(t > 0, s > 0)$ of the dissipative curve $e^{itA} f$ as $\tau \to +\infty$.

Theorem 6.16. *Let for the model A, defined by* (6.33)*, the next conditions hold:*

1) $\|B(x)\| \leq C$, $\|xB(x)\| \leq C$ $\forall x \in \mathbb{R}$;

2) $B(x) \in C_{\alpha_1}(\mathbb{R})$, $xB(x) \in C_{\alpha_2}(\mathbb{R})$ $(0 < \alpha_1 \leq 1,\ 0 < \alpha_2 \leq 1)$;

3) $\|B(x)\| \in \mathbf{L}(\mathbb{R})$, $\|xB(x)\| \in \mathbf{L}(\mathbb{R})$, $\|e^{-2\pi \widetilde{B}(x)}\|_{\mathbf{L}^2} < 1$;

4) $Q^*(x)$ *is a smooth matrix function on* $\mathbb{R}$ *and* $\|Q^{*\prime}(x)\| \in \mathbf{L}^2(\mathbb{R})$.

Then there exists the limit of the correlation function $\lim\limits_{\tau\to+\infty} V(t+\tau, s+\tau)$ *of the* *dissipative curve* $e^{itA}f$ *for each* $f \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^n)$ *and*

$$\lim_{\tau\to+\infty} V(t+\tau, s+\tau) = \int_{-\infty}^{+\infty} e^{i(t-s)x}(\widehat{S}f(x)T_+)(\widehat{S}f(x)T_+)^* dx =$$

(6.86)

$$= \int_{-\infty}^{+\infty} e^{i(t-s)x} \left\| \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x-i}\widehat{S}f(x) \right) V_{-\infty}(x)\sqrt{I - e^{-2\pi\widetilde{B}(x)}} \right\|^2 dx$$

for all $f \in \widetilde{\mathcal{D}}_0$ $(t, s > 0)$, *where* $V(t,s) = (e^{itA}f, e^{isA}f)$ *is the correlation function of the curve* $e^{itA}f$ *and the operators* $\widehat{S}$, T_+, $V_{-\infty}(x)$ *are defined by* (6.79), (6.80), (6.45) *correspondingly.*

The proof of this theorem follows as in the bounded case from the obtained asymptotics (1.23) and straightforward calculations.

One of the other applications of the asymptotics (1.23) of the curve $e^{itA}f$ for the model A, defined by (6.33), is a constructing of the scattering theory for the couple (A^*, A) as in the bounded case and in the unbounded case with equal dense domains of the model and its adjoint.

Now Theorem 1.8 gives the form of the wave operator as a weak limit.

P r o o f o f T h e o r e m 1.8. The equality (1.25) follows from (6.86). The equality (1.26) can be obtained following the ideas of the proof of Theorem 1.3 and Theorem 1.5 and using the equality

$$\lim_{y\to t} \frac{e^{i(y-t)A} - I}{y-t} f = itAf$$

which follows from the properties of the semigroup $\{T_t\}_{t\geq 0}$ (Theorem 6.3) and the equality

$$(\widetilde{S}_+^*\widetilde{S}_+Af, f) = (\widetilde{S}_+^*\mathcal{Q}\widetilde{S}_+f, f)$$

for each $f \in \widetilde{\mathcal{D}}_0$. The proof is complete. □

The equality (1.25) implies the existence of the wave operator W_- of the couple (A^*, A), defined by

$$(W_-(A^*, A)f, g) = \lim_{t\to-\infty}(e^{itA^*}e^{-itA}f, g)$$

and

$$W_-(A^*, A) = \widetilde{S}_+^*\widetilde{S}_+$$

as a weak limit.

We will prove the existence of the wave operator $W_-(A^*,A)$ as a strong limit. For this proof we need the form of $\Phi e^{-itA}g(x)$ $(t<0,\ (A-iI)^{-1}g(x)\in \widetilde{\mathcal{D}}_0)$. The definition (1.22) of the operators e^{-itA} $(t<0)$ from the considered semigroup, the form (6.7) of the operator Φ and straightforward calculations show that $\Phi e^{-itA}g(x)$ has the representation

$$\Phi e^{-itA}g(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{-it\xi}\widetilde{G}f(\xi)d\xi \quad (t<0), \tag{6.87}$$

where

$$\begin{aligned}\widetilde{G}f(\xi)=-\frac{1}{\sqrt{2\pi}}\frac{1}{\mu_0-i}\Bigg(&\frac{1}{\xi-i}\int\limits_{-\infty}^{+\infty}\widetilde{h}(w)\int\limits_{w}^{\overset{+\infty}{\rightarrow}}e^{B(v)dv}dw-\\ &-\frac{1}{\xi-\mu_0}\int\limits_{-\infty}^{+\infty}\widetilde{h}(w)\int\limits_{w}^{\overset{+\infty}{\rightarrow}}e^{-i\frac{1+\mu_0 v}{v-\mu_0}B(v)dv}dw+\\ &+\left(\frac{1}{\xi-\mu_0}-\frac{1}{\xi-i}\right)\int\limits_{-\infty}^{+\infty}\widetilde{h}(w)F_w^-(\xi,+\infty)dw\Bigg).\end{aligned} \tag{6.88}$$

Now Theorem 1.9 presents the existence of the wave operator $W_-(A^*,A)$ as a strong limit.

P r o o f o f T h e o r e m 1.9. Let us denote

$$W(t)=e^{itA^*}e^{-itA}=T_{-t}^*T_{-t}$$

when $t<0$. Let $f\in\widetilde{\mathcal{D}}_0$ (i.e. $f=(A-iI)^{-1}g=(A-iI)^{-1}(A-\mu_0 I)^{-1}h$, $h\in D_1\cap H_0\cap S(\mathbb{R},\mathbb{C}^n)$). From the properties of the semigroup $\{T_t\}_{t\geq 0}$ we have

$$-\frac{d}{dt}T_t^*T_tf=$$

$$=(A^*+iI)T_t^*(i(A-iI)^{-1}-i(A^*+iI)^{-1}+2(A^*+iI)^{-1}(A-iI)^{-1})T_tg=$$

$$=(A^*+iI)T_t^*B_iT_tg=(A^*+iI)T_t^*\Phi^*\Phi T_tg,$$

$(g=(A-\mu_0)^{-1}h)$ where we have used the equalities (6.6), (6.9) and the form (6.7) of the operator Φ.

For arbitrary numbers $t_1,t_2>0$ we obtain

$$\|T_{t_2}^*T_{t_2}f(x) - T_{t_1}^*T_{t_1}f(x)\|_{\mathbf{L}^2}^2 = \left\| \int_{t_1}^{t_2} (A^* + iI)T_\tau^*\Phi^*\Phi T_\tau g(x) d\tau \right\|_{\mathbf{L}^2}^2 . \tag{6.89}$$

But the auxiliary selfadjoint operator B_i, defined by (6.5) for the dissipative operator A , takes the form (see [18])

$$B_i f = \sum_{\alpha=1}^{m} (f, g_\alpha) g_\alpha = \sum_{\alpha=1}^{m} (f, \Phi^* e_\alpha)\Phi^* e_\alpha, \tag{6.90}$$

where $\{e_\alpha\}_1^m$ is an orthonormal basis in $\mathbb{C}^m$ and $g_\alpha = \Phi^* e_\alpha$. Then from (6.90) and from the equality (6.89) after straightforward calculations we obtain the next relations

$$\begin{aligned} &\|T_{t_2}^*T_{t_2}f(x) - T_{t_1}^*T_{t_1}f(x)\|_{\mathbf{L}^2}^2 = \\ &= \int_{-\infty}^{+\infty} \| \sum_{\alpha=1}^{m} \int_{t_1}^{t_2} (\Phi T_\tau g(x), e_\alpha)(A^* + iI)T_\tau^*\Phi^* e_\alpha d\tau \|^2 dx \leq \\ &\leq M \sum_{\alpha=1}^{m} \int_{t_1}^{t_2} |(\Phi T_\tau g(x), e_\alpha)|^2 d\tau \int_{t_1}^{t_2} \|(A^* + iI)T_\tau^*\Phi^* e_\alpha\|_{\mathbf{L}^2}^2 d\tau \end{aligned} \tag{6.91}$$

(where $M > 0$ is a suitable constant).

In the case when $t_1, t_2 < 0$, $\tau < 0$ $(t_1 < t_2)$ the inequality (6.91) has the form

$$\begin{aligned} &\|W(t_2)f(x) - W(t_1)f(x)\|_{\mathbf{L}^2}^2 \leq \\ &\leq M \sum_{\alpha=1}^{m} \int_{t_1}^{t_2} |(\Phi e^{-i\tau A} g(x), e_\alpha)|^2 d\tau \int_{t_1}^{t_2} \|(A^* + iI)e^{i\tau A^*}\Phi^* e_\alpha\|_{\mathbf{L}^2}^2 d\tau. \end{aligned} \tag{6.92}$$

Now the form (6.88) of $\widetilde{G}f(\xi)$ shows that $\|\widetilde{G}f(\xi)\| \in \mathbf{L}^2(\mathbb{R})$ and consequently the function

$$\widehat{G}(\tau) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-i\tau\xi} \widetilde{G}f(\xi) d\xi \qquad (\tau \in \mathbb{R}) \tag{6.93}$$

belongs to $\mathbf{L}^2(\mathbb{R};\mathbb{C}^m)$ as a Fourier transform of the function $\widetilde{G}f(\xi)$. This implies that $\Phi e^{-i\tau A} g(x) = \widehat{G}(\tau)\chi_{(-\infty,0)}(\tau) \in \mathbf{L}^2(\mathbb{R};\mathbb{C}^m)$ and $\|\Phi e^{-i\tau A} g(x)\| \in \mathbf{L}^2((-\infty;0])$. Hence

$$|(\Phi e^{-i\tau A} g(x), e_\alpha)| \in \mathbf{L}^2((-\infty;0]). \tag{6.94}$$

The relation (6.94) implies that there exists the limit

$$\lim_{t_1 \to -\infty} \int\limits_{t_1}^{t_2} |(\Phi e^{-i\tau A} g(x), e_\alpha)|^2 d\tau. \tag{6.95}$$

Now on the one hand we have

$$\|(A^* + iI)e^{i\tau A^*}\Phi^* e_\alpha\|_{\mathbf{L}^2} \le \|\Phi e^{-i\tau A}(A - iI)\|_{\mathbf{L}^2} \quad (\tau \le 0).$$

For the function

$$\psi(\tau) = \|\Phi e^{-i\tau A}(A - iI)\|_{\mathbf{L}^2} = \sup_{\|f\|_{\mathbf{L}^2}=1} \|\Phi e^{-i\tau A}(A - iI)f\|$$

in $(-\infty;0]$ there exists a sequence

$$\psi_n(\tau) = \|\Phi e^{-i\tau A}(A - iI)f_n\|, \quad f_n \in \widetilde{\mathcal{D}}_0, \ \ \|f_n\|_{\mathbf{L}^2} = 1, \tag{6.96}$$

$(\tau \in (-\infty;0])$ such that

$$\psi_n(\tau) \to \psi(\tau) \ \ as \ \ n \to +\infty, \ \ \forall \tau \in (-\infty;0] \tag{6.97}$$

and $\psi_n(\tau) \in \mathbf{L}^2((-\infty;0])$.

On the other hand for the function $\|T_t f\|^2_{\mathbf{L}^2}$ ($f \in \widetilde{\mathcal{D}}_0$ and $t \ge 0$) from (6.18) after straightforward calculations we obtain

$$\frac{d}{dt}\|T_t f\|^2_{\mathbf{L}^2} = -((iR_i - iR_i^* + 2R_i^* R_i)T_t g, T_t g) = \tag{6.98}$$
$$= -(B_i T_t g, T_t g) = -(\Phi^*\Phi T_t g, T_t g) = -\|\Phi T_t f\|^2_{\mathbf{L}^2},$$

where $R_i = (A - iI)^{-1}$, B_i is defined by (6.5) and (6.9) and Φ is defined by (6.7). Then from (6.98) it follows that

$$\int\limits_0^t \|\Phi T_\tau g(x)\|^2 d\tau = \|f\|^2_{\mathbf{L}^2} - \|T_t f\|^2_{\mathbf{L}^2}. \tag{6.99}$$

From (6.98) for the nonnegative decreasing function $\|T_t f\|^2_{\mathbf{L}^2}$ ($f \in \widetilde{\mathcal{D}}_0$, $t \geq 0$) it follows that there exists the limit $\lim\limits_{t\to-\infty} \|e^{-itA} f(x)\|^2_{\mathbf{L}^2}$. From the existence of this limits and from (6.99) it follows that there exists the integral $\int\limits_{-\infty}^0 \|\Phi e^{-i\tau A} g(x)\|^2 d\tau = \|f\|^2_{\mathbf{L}^2} - \lim\limits_{\tau\to-\infty} \|e^{-i\tau A} f(x)\|^2_{\mathbf{L}^2}$ and the next inequality holds

$$\int\limits_{-\infty}^0 \|\Phi e^{-i\tau A} g(x)\|^2 d\tau \leq 2\|f\|^2_{\mathbf{L}^2}. \tag{6.100}$$

From the inequality (6.100) and the form (6.96) of $\psi_n(\tau)$ we have

$$\int\limits_{-\infty}^0 \psi_n^2(\tau) d\tau = \int\limits_{-\infty}^0 \|\Phi e^{-i\tau A}(A - iI) f_n\|^2 \leq 2\|f_n\|^2_{\mathbf{L}^2} = 2 \tag{6.101}$$

for all $n \in \mathbb{N}$. The last inequality (6.101) together with (6.97) implies that $\psi^2(\tau) \in \mathbf{L}((-\infty; 0])$. Then the inequalities

$$\|(A^* + iI)e^{i\tau A^*}\Phi^* e_\alpha\|^2_{\mathbf{L}^2} \leq \|(A^* + iI)e^{i\tau A^*}\Phi^*\|^2 . \|e_\alpha\|^2 =$$
$$= \|\Phi e^{-i\tau A}(A - iI)\|^2_{\mathbf{L}^2} = \psi^2(\tau)$$

show that

$$\|(A^* + iI)e^{i\tau A^*}\Phi^* e_\alpha\|^2_{\mathbf{L}^2} \in \mathbf{L}((-\infty; 0]). \tag{6.102}$$

From (6.95) and (6.102) we obtain

$$\|W(t_2)f(x) - W(t_1)f(x)\|^2_{\mathbf{L}^2} \to 0 \quad as \;\; t_1, t_2 \to -\infty$$

for all $f \in \widetilde{\mathcal{D}}_0$. Consequently, there exists the limit

$$\lim_{t \to -\infty} W(t)f = \lim_{t \to -\infty} e^{itA^*} e^{-itA} f \quad \forall f \in \widetilde{\mathcal{D}}_0.$$

But $\widetilde{\mathcal{D}}_0$ is dense in $\mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$ and $e^{itA^*}e^{-itA}$ is an uniformly bounded family of operators and hence there exists the limit

$$\lim_{t \to -\infty} e^{itA^*} e^{-itA} f \quad \forall f \in \mathbf{L}^2(\mathbb{R}; \mathbb{C}^n)$$

(see, for example, Lemma III 3.5 [10]) and the proof is complete. □

In order to conclude this paper it has to mention that presented results in this paper consider the class of operators with an absolutely continuous real spectrum. In the general case of the operators from the considered classes with an arbitrary real spectrum these results can be extended using the decomposition of the spectrum of an absolutely continuous spectrum and a singular spectrum. We can presume that the existence of the singular component of the spectrum does not change the asymptotics of the corresponding continuous curves. Probably the extension will have effect on the subspaces of the initial conditions. The considerations of these questions is forthcoming.

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