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RECURSIVE METHODS FOR CONSTRUCTION OF BALANCED n-ARY BLOCK DESIGNS

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Abstract. This paper presents a recursive method for the construction of balanced n-ary block designs.

This method is based on the analogy between a balanced incomplete binary block design ($\mathcal{B.I.E.B}$) and the set of all distinct linear sub-varieties of the same dimension extracted from a finite projective geometry. If $\mathcal{V}_1$ is the first $\mathcal{B.I.E.B}$ resulting from this projective geometry, then by regarding any block of $\mathcal{V}_1$ as a projective geometry, we obtain another system of $\mathcal{B.I.E.B}$. Then, by reproducing this operation a finite number of times, we get a family of blocks made up of all obtained $\mathcal{B.I.E.B}$ blocks. The family being partially ordered, we can obtain an n-ary design in which the blocks are consisted by the juxtaposition of all binary blocks completely nested. These n-ary designs are balanced and have well defined parameters. Moreover, a particular balanced n-ary class is deduced with an appreciable reduction of the number of blocks.

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Key words: Balanced incomplete binary blocks, n-ary designs, finite projective geometry, finite linear sub-variety.

Introduction. In this article, we propose a new method for the construction of balanced n-ary block designs. Introduced by Tocher [13], these designs generalize the construction of $\mathcal{B}.\mathcal{I}.\mathcal{E}.\mathcal{B}$. Tocher obtained some balanced ternary designs from trial and error. After that, other construction methods of n-ary blocks were suggested using a set of mutually orthogonal Latin squares [9], α-resolvable balanced incomplete block designs [3] or the method of differences [12]. Other methods of construction of balanced ternary designs can be found in [2, 7, 8, 11] and balanced n-ary designs in [1, 5, 10]. We suggest here a method based on the analogy between a balanced incomplete binary block design and the set of all distinct linear sub-varieties of the same dimension extracted from a finite projective geometry by using a Galois fields. It consists of a recursive diagram resulting from a projective geometry from which we extract the set of all distinct linear sub-varieties of the same dimension. Again, we reproduce this operation with each sub-variety considered as a projective geometry of a lower dimension. This repeated operation a finite number of times for each obtained sub-variety, allows the construction of an n-ary design which blocks are consisted by the juxtaposition of all binary blocks completely nested. This design is balanced and each treatment can occur $0, 1, \ldots$ or $n-1$ times in each block. The parameters of this design are well defined and take a very simple form when dimensions of the different extracted linear sub-varieties are in the form $m_j = m - j$, $j = \overline{1, n-1}$. With the same approach, we deduce a particular class of n-ary designs by imposing that each treatment occurs $0, 1, q_1, \ldots, q_s$ or $n-1$ times in each block of the final design, the integers $q_1, \ldots, q_s$ must be less than $n-1$. These designs are characterized by a relative reduction of the number of blocks, in particular the n-ary designs which each treatment occurs at most 1 or $n-1$ times in each block.

I. Description of the method.

Definition 1. *An n-ary block design is an arrangement of ν treatments into b blocks, each of size k, such that every treatment is repeated r times and occurs $0, 1, 2, \ldots$ or $n-1$ times in each block.*

Let δ_{ij} be Kronecker's symbol, n_{ij} the number of times the i^{th} treatment occurs in the j^{th} block and $N = (n_{ij})_{(v,b)}$ the incidence matrix of the design.

The design is said to be balanced if the product of any two rows of the incidence matrix N of the design is in the form: $(\mu - \lambda) .\delta_{il} + \lambda$, where $\mu = \sum_{j=1}^{b} n_{ij}^2$

and $\lambda = \sum_{j=1}^{b} n_{ij}.n_{lj}$ *are independent of the rows* i *and* l $(i \neq l)$.

In particular, balanced incomplete binary blocks designs are characterized by the parameters (ν, b, k, r, λ) where λ is the number of occurrences which two treatments are in the design.

One of construction methods of a $\mathcal{B.I.E.B}$ design consists of its identification with a system of linear sub-varieties of an m-dimensional projective geometry $\mathcal{PG}(m, p^\eta)$ defined on a Galois fields of p^η elements (cf. Dugué [4]). This analogy consists to represent a treatment as a point of this geometry and a block as an h-dimensional linear sub-variety $(h < m)$, allowing to make the deduction of the associated $\mathcal{B.I.E.B}$ parameters easier.

Description of the method

Let $\mathcal{V}_m$ be an m-dimensional projective geometry, the method consists first to build the set of all m_1-dimensional linear sub-varieties $(m_1 < m)$, which a system of $\mathcal{B.I.E.B}$ (said of the 1st generation) noted $\{\mathcal{V}(i_1) : 1 \leq i_1 \leq b_1\}$ corresponds. Then, we consider each sub-variety $\mathcal{V}(i_1)$ of this system as an m_1-dimensional projective geometry, and we build all the m_2-dimensional distinct linear sub-varieties $\{\mathcal{V}(i_1, i_2) : 1 \leq i_2 \leq b_2\}$ $(m_2 < m_1)$, contained in the sub-variety $\mathcal{V}(i_1)$. This system is identified as $\mathcal{B.I.E.B}$ design (said of 2nd generation). Following this first operation, if we juxtapose all the nested sub-varieties $\mathcal{V}(i_1)$ and $\mathcal{V}(i_1, i_2)$, we obtain a system of ternary blocks

$$\{\mathcal{V}(i_1) \vee \mathcal{V}(i_1, i_2) : 1 \leq i_2 \leq b_2 \text{ and } 1 \leq i_1 \leq b_1\}$$

where $\mathcal{V}(i_1) \vee \mathcal{V}(i_1, i_2)$ is the juxtaposition of the sub-variety $\mathcal{V}(i_1, i_2)$ with its ascending $\mathcal{V}(i_1)$. On the other hand, if we defer the operation of juxtaposition to a later step, and we consider again each sub-variety $\mathcal{V}(i_1, i_2)$ as an m_2-dimensional projective geometry, we obtain in the same way a system of m_3-dimensional distinct sub-varieties $\{\mathcal{V}(i_1, i_2, i_3) : 1 \leq i_3 \leq b_3\}$ $(m_3 < m_2)$, which determines a system of $\mathcal{B.I.E.B}$ design (said of 3rd generation). In this step, if we juxtapose all the strictly nested sub-varieties $\mathcal{V}(i_1), \mathcal{V}(i_1, i_2)$ and $\mathcal{V}(i_1, i_2, i_3)$, we obtain a system of balanced quaternary blocks made up of blocks $\mathcal{V}(i_1) \vee \mathcal{V}(i_1, i_2) \vee \mathcal{V}(i_1, i_2, i_3)$ where $1 \leq i_3 \leq b_3$, $1 \leq i_2 \leq b_2$ and $1 \leq i_1 \leq b_1$. Similarly, we obtain a balanced n-any design by repeating $(n-1)$ times this extraction operation, and by juxtaposing each final block $\mathcal{V}(i_1, \ldots, i_{n-1})$ with all the stock blocks from where it derives $\{\mathcal{V}(i_1, \ldots, i_j) : 1 \leq j \leq n-2\}$. For example, a block of this n-ary design is in the form $\mathcal{V}(i_1) \vee \cdots \vee \mathcal{V}(i_1, \ldots, i_j) \vee \cdots \vee \mathcal{V}(i_1, \ldots, i_{n-1})$. By using the properties of these sub-varieties, we determine the parameters $\nu, b^{(n)}, k^{(n)}, r^{(n)}, \mu^{(n)}$ and $\lambda^{(n)}$ of this n-ary block design denoted by $\mathcal{P}_n\left(\nu, b^{(n)}, k^{(n)}, r^{(n)}, \mu^{(n)}, \lambda^{(n)}\right)$. The parameters $b^{(n)}$ and $k^{(n)}$ are easily deduced and given by

$$b^{(n)} = \prod_{j=1}^{n-1} b_j, \qquad k^{(n)} = \sum_{j=1}^{n-1} k_j,$$

where b_j (resp. k_j) is the number of blocks of each $\mathcal{B.I.E.B}$ of the j^{th} generation (resp. the size of b_j). The determination of parameters $r^{(n)}, \mu^{(n)}$ and $\lambda^{(n)}$ requires the following result:

Proposition 1. *i) The number of distinct m_1-dimensional linear sub-varieties contained in $\mathcal{V}_m$ not passing through a given point is : $b_1 - r_1$.*
ii) The number of distinct m_1-dimensional linear sub-varieties contained in $\mathcal{V}_m$ passing through a point t_1 and not through another t_2 is : $r_1 - \lambda_1$.
iii) The number of distinct m_1-dimensional linear sub-varieties contained in $\mathcal{V}_m$ not passing neitheir through t_1 nor through t_2 is : $b_1 - 2r_1 + \lambda_1$.

Proof. The results *i)* and *ii)* are directly obtained from the definition of the $\mathcal{B.I.E.B}'$s parameters corresponding to the system of the m_1-dimensional distinct sub-varieties $\{\mathcal{V}(i_1) : 1 \leq i_1 \leq b_1\}$. Concerning the conclusion *iii)*, seeing that the number of m_1-dimensional linear sub-varieties containing t_1 or t_2 is equal to the power of the party:

$$\{i_1 : \{t_1, t_2 \in \mathcal{V}(i_1)\} \text{ or } \{t_1 \in \mathcal{V}(i_1), t_2 \notin \mathcal{V}(i_1)\} \text{ or } \{t_1 \notin \mathcal{V}(i_1), t_2 \in \mathcal{V}(i_1)\}\},$$

(i.e. $\lambda_1 + 2(r_1 - \lambda_1)$). So it is then easy to deduce the number of m_1-dimensional linear sub-varieties not containing neitheir t_1 nor t_2. $\square$

Theorem 1. *The designs $\mathcal{P}_n\left(\nu, b^{(n)}, k^{(n)}, r^{(n)}, \mu^{(n)}, \lambda^{(n)}\right)$ are balanced n-ary designs with the parameters:*

$$r^{(n)} = \sum_{j=0}^{n-1} j.\left(b_{j+1} - r_{j+1}\right) \times \left[\prod_{l=j+2}^{n-1} b_l\right] \times \left[\prod_{l=1}^{j} r_l\right],$$

$$\mu^{(n)} = \sum_{j=0}^{n-1} j^2.\left(b_{j+1} - r_{j+1}\right) \times \left[\prod_{l=j+2}^{n-1} b_l\right] \times \left[\prod_{l=1}^{j} r_l\right],$$

and

$$\lambda^{(n)} = \sum_{j=1}^{n-2} 2j.\left[r_{j+1} - \lambda_{j+1}\right].\prod_{l=1}^{j} \lambda_l \times \sum_{i=j+1}^{n-1} i.\left(b_{i+1} - r_{i+1}\right).\prod_{l=j+2}^{i} r_l.\prod_{l=i+2}^{n-1} b_l$$
$$+ \sum_{j=1}^{n-1} j^2.\left[b_{j+1} - 2r_{j+1} + \lambda_{j+1}\right].\prod_{l=1}^{j} \lambda_l.\prod_{l=j+2}^{n-1} b_l$$

with $\sum_{l=j+2}^{i} r_l = 1$ *if* $j+1 \geq i$, $\prod_{l=q}^{n-1} b_l = 1$ *if* $q \geq n$, $b_n - r_n = 1$ *and* $b_n - 2r_n + \lambda_n = 1$ *where* r_j *(resp.* λ_j*) is the number of repetitions of a treatment (resp. the number of occurrences of any two treatments) in a* $\mathcal{B.I.E.B}$ *of the* j^{th} *generation.*

Proof. The final design $\mathcal{P}_n$ is an n-ary design. Indeed, if an arbitrary treatment t belongs to the sub-variety $\mathcal{V}(i_1, \ldots, i_j)$ where $j \leq n-2$, then from one side, t belongs to all the ascending $\mathcal{V}(i_1, \ldots, i_l)$ $(1 \leq l \leq j-1)$ of this sub-variety, and from the other side, it is transmitted to certain of its descendants $\mathcal{V}(i_1, \ldots, i_j, i_{j+1}, \ldots, i_{n-1})$, which shows that this treatment will occur $(n-1)$ times in certain blocks of the design $\mathcal{P}_n$. However, if this treatment isn't transmitted to a descendant $\mathcal{V}(i_1, \ldots, i_j, i_{j+1})$ of $\mathcal{V}(i_1, \ldots, i_j)$, then t is missing from all its descendants $\mathcal{V}(i_1, \ldots, i_l)$ $(j+1 \leq l \leq n-1)$, and then this treatment will occur exactly j times in the final block. On the other hand, if this treatment is missing from a sub-variety $\mathcal{V}(i_1)$, it will be missing from all its descendants and could not occur in any block resulting from $\mathcal{V}(i_1)$. This confirms that the system $\mathcal{P}_n$ is an n-ary design. Determination of the parameters of $\mathcal{P}_n$.

Concerning the parameter $r^{(n)} = \sum_{j=1}^{b^{(n)}} n_{ij}$, we can rewrite it in the form

$$r^{(n)} = \sum_{j=0}^{n-1} \sum_{l \in \mathbf{I}_j} n_{il}$$

where $\mathbf{I}_j = \left\{ l \in \left\{1, \ldots, b^{(n)}\right\} / n_{il} = j \right\}$ for each $j \in \{0, 1, \ldots, n-1\}$, and has as power the number of blocks where a treatment t exactly occurs j times, (the parties $\mathbf{I}_j$ are disjointed and their union is $\left\{1, \ldots, b^{(n)}\right\}$). For an arbitrary treatment t, we have to evaluate the number of blocks of the design $\mathcal{P}_n$ where this treatment occurs j times.

The number of blocks where the treatment t is missing can be described by the party $\mathcal{A}(t, 0) = \{(i_1) \in \{1, \ldots, b_1\} / \ t \notin \mathcal{V}(i_1)\}$, and the blocks where this treatment occurs $n-1$ times are described by the party

$\mathcal{A}(t, n-1) = \{(i_1, \ldots, i_{n-1}) / \ t \in \mathcal{V}(i_1, \ldots, i_{n-1})\}$, which confirms that the treatment t belongs to all the ascending of the sub-variety $\mathcal{V}(i_1, \ldots, i_{n-1})$. The blocks where the treatment t occurs j times $(1 \leq j \leq n-2)$, can be described by the party:

$$\mathcal{A}(t, j) = \left\{ \begin{array}{c} (i_1, \ldots, i_j) \in \prod_{l=1}^{j} \{1, \ldots, b_l\} \ / \text{ there exists } i_{j+1} \in \{1, \ldots, b_{j+1}\}, \\ t \in \mathcal{V}(i_1, \ldots, i_j) \text{ and } t \notin \mathcal{V}(i_1, \ldots, i_{j+1}) \end{array} \right\}.$$

The hypothesis $t \notin \mathcal{V}(i_1, \ldots, i_{j+1})$ implies that this treatment can't occur in all the descendants of $\mathcal{V}(i_1, \ldots, i_{j+1})$, whereas the hypothesis $t \in \mathcal{V}(i_1, \ldots, i_j)$ implies that this treatment t necessarily belongs to all the ascending of $\mathcal{V}(i_1, \ldots, i_j)$, thus this treatment will exactly occur j times in certain blocks of $\mathcal{P}_n$:

$$\mathcal{V}(i_1) \vee \cdots \vee \mathcal{V}(i_1, \ldots, i_j) \vee \mathcal{V}(i_1, \ldots, i_{j+1}) \vee \cdots \vee \mathcal{V}(i_1, \ldots, i_{n-1}).$$

An easy calculation allows the evaluation of the power of each of these parties, so we can deduce the value of the parameter $r^{(n)}$, and similarly $\mu^{(n)}$.

Concerning the parameter $\lambda^{(n)}$, this one can be rewritten in the form:

$$\lambda^{(n)} = \sum_{j=0}^{n-1} \sum_{l' \in \mathbf{I}_{j,j}} n_{il'}.n_{ll'} + 2 \sum_{j=0}^{n-2} \sum_{j'=j+1}^{n-1} \sum_{l' \in \mathbf{I}_{j,j'}} n_{il'}.n_{ll'},$$

where for $j' \geq j+1$, $\mathbf{I}_{j,j'} = \left\{l' \in \left\{1, \ldots, b^{(n)}\right\} / n_{il'} = j \text{ and } n_{ll'} = j'\right\}$ describes the set of blocks where the treatments t and t' occur exactly j times together. Let's evaluate the number of blocks of the final design where these two treatments occur j times together for $j = 0, 1, \ldots, n-1$.

i) For $j' \geq 0$, the party $\mathcal{B}(t, t'; 0, j')$ of $\{1, \ldots, b_1\}$ such as $t' \notin \mathcal{V}(i_1)$ and either $t \notin \mathcal{V}(i_1)$, either $t \in \mathcal{V}\left(i_1, \ldots, i_{j'}\right) \backslash \mathcal{V}\left(i_1, \ldots, i_{j'+1}\right)$, describes the blocks where the treatments don't occur together.

ii) For $j' \geq j$ and $j = \overline{1, n-2}$, we consider the party $\mathcal{B}(t, t'; j, j')$ of $\prod_{l=1}^{j} \{1, \ldots, b_l\}$, defined by :
$(i_1, \ldots, i_j) \in \mathcal{B}(t, t'; j, j') \iff \exists \left(i_{j+1}, i_{j'}\right) \in \{1, \ldots, b_{j+1}\} \times \left\{1, \ldots, b_{j'}\right\}$ and such that, either $\{t, t' \in \mathcal{V}(i_1, \ldots, i_j)$ and $t, t' \notin \mathcal{V}(i_1, \ldots, i_{j+1})\}$, either $\left\{t' \in \mathcal{V}(i_1, \ldots, i_j) \text{ and } t' \notin \mathcal{V}(i_1, \ldots, i_{j+1}),\ t \in \mathcal{V}\left(i_1, \ldots, i_j, \ldots, i_{j'}\right) \text{ and } t \notin \mathcal{V}\left(i_1, \ldots, i_{j'+1}\right)\right\}$. This party describes the set of the blocks of the design $\mathcal{P}_n$ where t and t' occur exactly j times together.

iii) For $j = j' = n-1$, the party $\mathcal{B}(t, t'; n-1, n-1)$ characterized by the $n-1$ tuples $(i_1, \ldots, i_{n-1})$ such that $t, t' \in \mathcal{V}(i_1, \ldots, i_{n-1})$, describes the blocks where the treatments t and t' occur $(n-1)$ times together. Using the result of the proposition 1, we determine the power of each of these parties, and then we deduce the value of the parameter $\lambda^{(n)}$.

The parameters $\lambda^{(n)}$ and $\mu^{(n)}$ are constant, this confirms that the n-ary design $\mathcal{P}_n$ is balanced. $\square$

If the dimensions m_j of the sub-varieties $\mathcal{V}(i_1, \ldots, i_j)$ of the j^{th} generation are in the form $m_j = m - j : j = \overline{1, n-1}$ and $n - 1 < m$, then the $\mathcal{B.I.E.B}'$s parameters of the j^{th} generation are reduced to:

$$b_j = 1 + s + \cdots + s^{m-(j-1)} \quad \text{where} \quad s = p^n \quad \text{(the power of Galois fields)}$$

and

$$r_j = k_j = \lambda_{j-1} = b_{j+1},$$

which allows to write the parameters of the design $\mathcal{P}_n$ in a simpler form.

Corollary 1. *If for each $j \in \{1, \ldots, n-1\}$ the dimension m_j of the sub-variety $\mathcal{V}(i_1, \ldots, i_j)$ of the j^{th} generation is equal to $m_j = m - j$, then the parameters of the n-ary design $\mathcal{P}_n$ are in the form:*

$$r^{(n)} = \prod_{l=1}^{n-2} r_l . \sum_{j=0}^{n-2} j.\left(r_j - r_{j+1}\right) + (n-1) \prod_{l=1}^{n-1} r_l,$$

$$\mu^{(n)} = \prod_{l=1}^{n-2} r_l . \sum_{j=0}^{n-2} j^2 .\left(r_j - r_{j+1}\right) + (n-1)^2 . \prod_{l=1}^{n-1} r_l,$$

and

$$\begin{aligned}\lambda^{(n)} = &\prod_{l=1}^{n-3} \lambda_l . \sum_{j=1}^{n-3} 2j.\left(\lambda_j - \lambda_{j+1}\right)\left\{\sum_{i=j+1}^{n-1} i.\left(\lambda_{i-1} - \lambda_i\right)\right\} \\ &+ \prod_{l=1}^{n-3} \lambda_l . \sum_{j=1}^{n-2} j^2 \lambda_j\left(\lambda_{j-1} - 2\lambda_j + \lambda_{j+1}\right) \\ &+ (n-1)\left[2(n-2).\left(\lambda_{n-2} - \lambda_{n-1}\right) + (n-1).\lambda_{n-1}\right]. \prod_{l=1}^{n-2} \lambda_l,\end{aligned}$$

where $\lambda_0 = b_2$.

Example. In a $\mathcal{PG}(3,2)$ there are 15 distinct 2-dimensional sub-varieties. Each sub-variety corresponds to a block entirely determined by one of the equations:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0 \bmod (2)$$

the $a_i \in \mathcal{GF}(2)$ (the Galois fields of 2 elements), and each point p is defined by its 4 components (x_1, x_2, x_3, x_4). The parameters of the resulting $\mathcal{B.I.E.B}$ system are:

$$v = b_1 = 15, r_1 = k_1 = 7 \quad \text{and} \quad \lambda_1 = 3.$$

Again, each block which is considered as a 2-dimensionnal linear sub-variety, provides a new $\mathcal{B.I.E.B}(7,7,3,3,1)$ system of the 2^{nd} generation, these blocks are entirely determined by the system of equations:

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0 \bmod(2) \\ \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 = 0 \bmod(2) \end{cases}, \text{where the coefficients } a_i \text{ and } \alpha_i \in \mathcal{GF}(2).$$

For example, the block b_1:$\{p_2, p_3, p_4, p_8, p_9, p_{10}, p_{14}\}$ provides the $\mathcal{B.I.E.B}(7,7,3,3,1)$ defined by the system of equations:

$$\begin{cases} x_1 = 0 \bmod(2) \\ \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0 \bmod(2) \end{cases}.$$

In a similar way, we determine the other $\mathcal{B.I.E.B}\,(7,7,3,3,1)$ sets of the 2nd generation. Then, by juxtaposing each block b_j with each one of its descendants $b_{j,l}$ $l = \overline{1,7}$ and $j = \overline{1,15}$, we obtain the ternary design $\mathcal{P}_3$ characterized by the parameters : $\nu = 15$, $b^{(3)} = 105$, $r^{(3)} = 70$, $k^{(3)} = 10$ and $\lambda^{(3)} = 42$. The entries of the matrix $N.{}^tN$ are $\mu = 112$ on the diagonal and $\lambda = 42$ otherwise, where N is the incidence matrix.

Construction of particular n-ary designs

Generally, the design $\mathcal{P}_n$ contains an important number of blocks for large values of m. However, using certain restrictions, it is possible to substantially reduce the number of blocks, by imposing for example that each treatment occurs $0, 1, q_1, \ldots, q_s$ or $n-1$ times, where the (q_i) are strictly increasing. This is equivalent to extract the particular balanced n-ary design from the design $\mathcal{P}_n$. In a precise way, we have:

Proposition 2. *For each sequence $(q_1, \ldots, q_s)$ of integers such that $1 = q_0 < q_1 < \cdots < q_s < n-1$, there exists a balanced n-ary design $\mathcal{Q}_n$ in which each treatment occurs $0, 1, q_1, \ldots, q_s$ or $n-1$ times. This design is entirely determined by the parameters $\left(\nu, b'^{(n)}, r'^{(n)}, k^{(n)}, \lambda'^{(n)}\right)$ where $k^{(n)}$ is the same as above and*

$$b'^{(n)} = (b_1 - r_1)\,.\prod_{j=2}^{n-1} b_j + \sum_{l=0}^{s}\prod_{j=1}^{q_l} r_j.\,(b_{q_l+1} - r_{q_l+1})\,.\prod_{j=q_l+2}^{n-1} b_j + \prod_{j=1}^{n-1} r_j,$$

$$r'^{(n)} = \sum_{l=0}^{s} q_l \prod_{j=1}^{q_l} r_j.\,(b_{q_l+1} - r_{q_l+1})\,.\prod_{j=q_l+2}^{n-1} b_j + (n-1)\prod_{j=1}^{n-1} r_j,$$

$$\mu'^{(n)} = \sum_{l=0}^{s} q_l^2 \prod_{j=1}^{q_l} r_j.\,(b_{q_l+1} - r_{q_l+1})\,.\prod_{j=q_l+2}^{n-1} b_j + (n-1)^2\prod_{j=1}^{n-1} r_j$$

and

$$\begin{aligned}\lambda'^{(n)} = &\sum_{\tau=0}^{s} q_\tau^2.\prod_{j=1}^{q_\tau} \lambda_j.\,(b_{q_\tau+1} - 2r_{q_\tau+1} + \lambda_{q_\tau+1})\,.\prod_{j=q_\tau+2}^{n-1} b_j \\ &+ 2\sum_{\tau=0}^{s} q_\tau.q_{\tau'}.\prod_{j=1}^{q_\tau} \lambda_j.\,(r_{q_\tau+1} - \lambda_{q_\tau+1})\,.\prod_{j=q_\tau+2}^{q_{\tau'}} r_j.\,\left(b_{q_{\tau'}+1} - r_{q_{\tau'}+1}\right).\prod_{j=q_{\tau'}+2}^{n-1} b_j \\ &+ 2\,(n-1)\sum_{\tau=0}^{s} q_\tau \prod_{j=1}^{q_\tau} \lambda_j.\,(r_{q_\tau+1} - \lambda_{q_\tau+1})\,.\prod_{j=q_\tau+2}^{n-1} r_j + (n-1)^2\prod_{j=1}^{n-1} \lambda_j.\end{aligned}$$

P r o o f. A block of the design $\mathcal{Q}_n$ is in the form $\mathcal{V}(i_1)\vee\cdots\vee\mathcal{V}(i_1,\ldots,i_{n-1})$, in which an arbitrary treatment occurs $0,1,q_1,\ldots,q_s$ or $n-1$ times. These blocks are entirely described by one of the following parties:

$$A(t,0)=\{i_1\in\{1,\ldots,b_1\}/t\notin\mathcal{V}(i_1)\},$$

and for $l:0\leq l\leq s$, with $q_0=1$,

$$A(t,q_l)=\left\{\begin{array}{c}(i_1,\ldots,i_{q_l})\in\prod_{u=1}^{q_l}\{1,\ldots,b_u\}/\exists i_{q_l+1}\in\{1,\ldots,b_{q_l+1}\},\\ t\in\mathcal{V}(i_1,\ldots,i_{q_l})\text{ and }t\notin\mathcal{V}(i_1,\ldots,i_{q_l+1})\end{array}\right\},$$

and

$$A(t,n-1)=\{(i_1,\ldots,i_{n-1})/t\in\mathcal{V}(i_1,\ldots,i_{n-1})\}.$$

An easy calculation provides the number $b'^{(n)}$ of all these blocks on the one hand, and on the other hand, so that a treatment t don't belong to a block of the design $\mathcal{Q}_n$, it's necessary that this treatment is missing from the subvariety $\mathcal{V}(i_1)$ (i.e. $i_1\in A(t,0)$). In contrast, it is sufficient that $(i_1,\ldots,i_{q_l})\in A(t,q_l)$ so that it occurs q_l times, and it is necessary to retain only the blocks $\mathcal{V}(i_1)\vee\cdots\vee\mathcal{V}(i_1,\ldots,i_{n-1})$, for which $t\in\mathcal{V}(i_1,\ldots,i_{n-1})$, so that it exactly occurs $(n-1)$ times. This confirms that this design is an n-ary design.
The values of the parameters $r'^{(n)}$ and $\mu'^{(n)}$ are easily deduced. Concerning the parameter $\lambda'^{(n)}$, considering the configuration of the design $\mathcal{Q}_n$, we note that two arbitrary treatments t and t' occur $0,1,q_1,\ldots,q_s$ or $n-1$ times together in a block of this design. These blocks are entirely described by one of the following parties:

(*a*) $$B(t,t';0,0)=\{i_1\in\{1,\ldots,b_1\}/t\text{ or }t'\notin\mathcal{V}(i_1)\},$$

(*b*) for $l=0,1,\ldots,s$ with $q_0=1$,

$$B(t,t';q_l,q_l)=\left\{\begin{array}{c}\{i_1,\ldots,i_{q_l}\}\in\prod_{u=1}^{q_l}\{1,\ldots,b_u\}/\exists i_{q_l+1}\in\{1,\ldots,b_{q_l+1}\},\\ t,t'\in\mathcal{V}(i_1,\ldots,i_{q_l})\text{ and }t,t'\notin\mathcal{V}(\{i_1,\ldots,i_{q_l+1}\})\end{array}\right\},$$

(*c*) for $0\leq l'<l\leq s$,

$$B(t,t';q_l,q_{l'})=\left\{\begin{array}{c}\{i_1,\ldots,i_{q_{l'}}\}\in\prod_{u=1}^{q_{l'}}\{1,\ldots,b_u\}/\exists i_{q_{l'}+1}\in\{1,\ldots,b_{q_{l'}+1}\},\\ t'\in\mathcal{V}(i_1,\ldots,i_{q_{l'}})\text{ and }t'\notin\mathcal{V}(i_1,\ldots,i_{q_{l'}+1})\text{ and}\\ t\in\mathcal{V}(i_1,\ldots,i_{q_l})\text{ and }t\notin\mathcal{V}(i_1,\ldots,i_{q_l+1})\end{array}\right\},$$

(d) for $0 < l \leq s$,

$$B\left(t,t';n-1,q_l\right) = \left\{ \begin{array}{c} \{i_1,\ldots,i_{q_l}\}/\exists i_{q_l+1} \in \{1,\ldots,b_{q_l+1}\}, t' \in \mathcal{V}(i_1,\ldots,i_{q_l}) \\ \text{and } t' \notin \mathcal{V}(i_1,\ldots,i_{q_l+1}) \text{ and } t \in \mathcal{V}(i_1,\ldots,i_{n-1}) \end{array} \right\},$$

and

$$(e) \qquad B\left(t,t';n-1,n-1\right) = \left\{\{i_1,\ldots,i_{n-1}\}/t,t' \in \mathcal{V}(i_1,\ldots,i_{n-1})\right\}.$$

So, the number of blocks $\mathcal{V}(i_1) \vee \cdots \vee \mathcal{V}(i_1,\ldots,i_{n-1})$ where for example t and t' occur q_τ times together $(0 \leq \tau \leq s)$ is the sum of the powers of the parties $B(t,t';q_\tau,q_\tau)$, $B(t,t';q_{\tau'},q_\tau)$, $(\tau' > \tau)$ and $B(t,t';n-1,q_\tau)$ respectively multiplied by the coefficients $1, 2$ and 2, taking into account the symmetrical role of the two treatments t and t'. Then, an elementary calculation provides the value of $\lambda'^{(n)}$. Moreover, these parameters are independent of the treatments; this confirms that the design $\mathcal{Q}_n$ is balanced. □

A particular n-ary design resulting from the previous design $\mathcal{Q}_n$ in the n-ary design in which each treatment occurs $0, 1$ or $(n-1)$ times, which corresponds to omit the sequence $(q_1,\ldots,q_s)$.

Corollary 2. *There exists a balanced n-ary design $\mathcal{R}_n$ in which each treatment occurs $0, 1$ or $(n-1)$ times. This design is entirely determined by the parameters* $\left(\nu, b''^{(n)}, r''^{(n)}, k^{(n)}, \lambda''^{(n)}\right)$ *where $k^{(n)}$ is the same as above and*

$$b''^{(n)} = (b_1 - r_1) \,.\, \prod_{j=2}^{n-1} b_j + r_1 (b_2 - r_2) \,.\, \prod_{j=3}^{n-1} b_j + \prod_{j=1}^{n-1} r_j,$$
$$r''^{(n)} = r_1 (b_2 - r_2) \,.\, \prod_{j=3}^{n-1} b_j + (n-1) \,.\, \prod_{j=1}^{n-1} r_j,$$
$$\mu''^{(n)} = r_1 (b_2 - r_2) \,.\, \prod_{j=3}^{n-1} b_j + (n-1)^2 \,.\, \prod_{j=1}^{n-1} r_j,$$

and

$$\lambda''^{(n)} = \lambda_1 (b_2 - 2r_2 + \lambda_2) \,.\, \prod_{j=3}^{n-1} b_j + 2(n-1) \,.\lambda_1 (r_2 - \lambda_2) \prod_{j=3}^{n-1} r_j + (n-1)^2 \prod_{j=1}^{n-1} \lambda_j.$$

The number of blocks in the design $\mathcal{R}_n$ is relatively smaller than that of design $\mathcal{P}_n$.

Finally, we finish by the following result which is the analogy of the Corollary 1:

Corollary 3. *If the dimension of the sub-varieties* $\mathcal{V}(i_1,\ldots,i_j)$ *of the* j^{th} *generation is equal to* $m_j = m - j$*, then the parameters of the* n*-ary design* $\mathcal{Q}_n^*$ *are in the form:*

$$b^{*(n)} = (b_1 - b_2)\prod_{j=2}^{n-1} b_j + \prod_{j=2}^{n-1} b_j \sum_{l=0}^{s}(b_{q_l+1} - b_{q_l+2}) + \prod_{l=1}^{n-1} b_{j+1},$$

$$r^{*(n)} = \prod_{j=1}^{n-2} r_j.\sum_{l=0}^{s} q_l.(r_{q_l} - r_{q_l+1}) + (n-1)\prod_{j=1}^{n-1} r_j,$$

$$\mu^{*(n)} = \prod_{j=1}^{n-2} r_j.\sum_{l=0}^{n-2} q_l^2.(r_{q_l} - r_{q_l+1}) + (n-1)^2.\prod_{j=1}^{n-1} r_j,$$

and

$$\begin{aligned}\lambda^{*(n)} &= \prod_{j=1}^{n-3}\lambda_j.\sum_{\tau=0}^{s} q_\tau^2.\lambda_{q_\tau}(\lambda_{q_\tau-1} - 2\lambda_{q_\tau} + \lambda_{q_\tau+1})\\ &+ 2\prod_{j=1}^{n-3}\lambda_j.\sum_{\tau=0}^{s} q_\tau.q_{\tau'}.(\lambda_{q_\tau} - \lambda_{q_\tau+1}).\left(\lambda_{q_{\tau'}-1} - \lambda_{q_{\tau'}}\right)\\ &+ 2(n-1)\prod_{j=1}^{n-2}\lambda_j.\sum_{\tau=0}^{s} q_\tau.(\lambda_{q_\tau} - \lambda_{q_\tau+1}) + (n-1)^2.\prod_{j=1}^{n-1}\lambda_j\end{aligned}$$

where $\lambda_0 = b_2$ *and* $b_n = r_{n-1}$.

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