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ON THE RANGE AND THE KERNEL OF DERIVATIONS

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Abstract. Let H be a separable infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H into itself. Given $A \in L(H)$, the derivation $\delta_A : L(H) \longrightarrow L(H)$ is defined by $\delta_A(X) = AX - XA$. In this paper we prove that if A is an n-multicyclic hyponormal operator and T is hyponormal such that $AT = TA$, then $\|\delta_A(X)+T\| \geq \|T\|$ for all $X \in L(H)$. We establish the same inequality if A is a finite operator and commutes with normal operator T. Some related results are also given.

1. Introduction. Let H be an infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators acting on H. If $A \in L(H)$, then the inner derivation induced by A is the operator δ_A defined on $L(H)$ by $\delta_A(X) = AX - XA$. By finite operator we shall mean a bounded linear operator A on H such that

$$\|\delta_A(X) + I\| \geq 1 \tag{1}$$

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for every $X \in L(H)$. As stated in [11] J. P. Williams proved that the class of finite operators contains every normaloid (i.e., the spectral radius $r(A)$ of A equals the norm of A), every operator with compact direct summand and the entire C^*-algebra generated by each of its members. The purpose of this paper is to investigate this class of operators to give natural generalizations of the norm inequality (1). The basic tools in the main results is to use Anderson's inequality for normal operators [1] and the Berberian extension Theorem [12, p. 15]. The present paper is organized as follows. In Theorem 2.2 we initiate a new approach to extend this results to certain intertwining nonnormal operators A and T where A is an n-multicyclic hyponormal operator and requiring that T is hyponormal. The point of view about finite operators is developed in Theorem 2.3, in which we give a natural generalization of the inequality (1). Using a very simple argument we show in Theorem 2.4 that if A satisfies a quadratic polynomial, then A is a finite operator and that $A^* \notin \overline{R(\delta_A)}^W$, where $\overline{R(\delta_A)}^W$ is the weak closure of the range $R(\delta_A)$ of δ_A.

In addition to the notation already introduced, we shall use the following further notation. Let $K(H)$ be the ideal of all compact operators in $L(H)$ and let $\mathcal{C}(H)$ denote the Calkin algebra $L(H)/K(H)$. For $X \in L(H)$, let $[X]$ denote the projection of $L(H)$ onto the Calkin algebra. We shall denote the kernel, the orthogonal complement of the kernel, the range of X by $\ker(X)$, $\ker(X)^{\perp}$ and $R(X)$ respectively. The spectrum, the approximate point spectrum and the point spectrum of X will be denoted by $\sigma(X)$, $\sigma_{ap}(X)$ and $\sigma_p(X)$, and the restriction of X to an invariant subspace M will be denoted by $X|M$.

Given $A \in L(H)$, there exists a Hilbert space $H^\circ \supset H$ and an isometric $*$-isomorphism $A \longrightarrow A^\circ$ such that $\sigma(A) = \sigma(A^\circ)$ and $\sigma_{ap}(A) = \sigma_{ap}(A^\circ) = \sigma_p(A^\circ)$. This is the Berberian extension Theorem [12].

2. Main results.

Definition 2.1 [11]. *Let $A \in L(H)$. We say that A is a finite operator if,*

$$\|AX - XA + I\| \geq 1,$$

for all $X \in L(H)$.

Definition 2.2. *Let $A \in L(H)$. The operator A is said to be n-multicyclic if there are n vectors $g_1, g_2, \cdots, g_n \in H$, called generating vectors, such that $\{f(A)g_i : f \in R(\sigma(A)),\ 1 \leq i \leq n\}$ has span dense in H, where $R(\sigma(A))$ denotes the rational functions analytic on $\sigma(A)$.*

Theorem 2.1 [2]. *If A is an n-multicyclic hyponormal operator, then $[A^*, A]$ is in trace class, and $tr([A^*, A]) \leq \frac{n}{\pi}\omega(\sigma(A))$, where ω is planar Lebesgue measure.*

Theorem 2.2. *Let $A \in L(H)$. If A is an n-multicyclic hyponormal operator, then for every hyponormal operator T such that $AT = TA$, we have*

$$\|AX - XA + T\| \geq \|T\|$$

for all $X \in L(H)$.

Proof. Let T be a hyponormal operator in $L(H)$ such that $AT = TA$. We have $r(T) = \|T\|$, then it is enough to show that

$$\|AX - XA + T\| \geq |\lambda|$$

for all $X \in L(H)$ and all $\lambda \in \sigma(T)$. It is well known that T enjoys the property that $\sigma(T) = \sigma_p(T) \cup \sigma_e(T)$ (see [5]).
Let $\lambda \in \sigma(T)$, we will consider two cases for the location of λ.

Case 1. Assume that $\lambda \in \sigma_p(T)$. We shall divide this case into two different steps.

(i) If $\lambda \in \sigma_p(T)$ such that $\dim\ker(T-\lambda) < \infty$. Since A commutes with T, it follows that the subspace $E_\lambda = \ker(T - \lambda)$ is invariant by T and A. Moreover, A/E_λ is normal hence E_λ reduces A and T simultaneously [9, p. 514]. Then with respect to the orthogonal decomposition $H = E_\lambda \oplus E_\lambda^\perp$, the operators A and T can be respectively represented as

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} \lambda & 0 \\ 0 & * \end{pmatrix}.$$

Let $X \in L(H)$ have the matrix representation $X = \begin{pmatrix} Y & Z \\ R & S \end{pmatrix}$, we get

$$\|AX - XA + T\| = \left\| \begin{pmatrix} BY - YB + \lambda & * \\ * & * \end{pmatrix} \right\|.$$

Since the norm of an operator matrix always dominates the norm of its diagonal part [7, p. 82], it follows that

$$\|AX - XA + T\| \geq \|BY - YB + \lambda\|$$

A is a finite operator because A is hyponormal [11], hence B thus. Then we obtain

$$\|BY - YB + \lambda\| \geq |\lambda|$$

Consequently we have

$$\|AX - XA + T\| \geq |\lambda|$$

for all $X \in L(H)$, and all $\lambda \in \sigma_p(T)$ such that $\dim\ker(T - \lambda) < \infty$.

(ii) If $\lambda \in \sigma_p(T)$ such that $\dim\ker(T - \lambda) = \infty$. Since T is a hyponormal operator, then $\dim\ker(T - \lambda)^* = \infty$. It follows that $T - \lambda$ is not a Fredholm operator, hence $\lambda \in \sigma_e(T)$ (see the case 2.).

Case 2. Assume that $\lambda \in \sigma_e(T)$. Also, we will divide this case into two steps.

(i) T has no isolated eigenvalues of finite multiplicity.
The condition $AT = TA$ implies that $[A][T] = [T][A]$. Since A is an n-multicyclic hyponormal operator, it follows from the Theorem 2.1 that $[A]$ is normal. Anderson's result [1] applied to the Calkin algebra insures that

$$\|AX - XA + T\| \geq \|[A][X] - [X][A] + [T]\| \geq \|[T]\|$$

for all $X \in L(H)$. On the other hand, since T is hyponormal and has no isolated eigenvalues of finite multiplicity, one obtains from Remark [5, p. 186] that $\|[T]\| = r([T])$. Hence

$$\|[A][X] - [X][A] + [T]\| \geq |\lambda|$$

for all $X \in L(H)$. It follows that

$$\|AX - XA + T\| \geq |\lambda|$$

for all $X \in L(H)$, as desired.

(ii) If T has isolated eigenvalues of finite multiplicity. We consider the subspace $E = \bigoplus_{\mu\in\beta(T)} \ker(T-\mu)$, where $\beta(T)$ is the set of all isolated eigenvalues of T with finite multiplicity. Since T/E is a normal operator then E reduces T. With respect to the decomposition $H = E \oplus E^{\perp}$, we have

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

It follows from an application of Anderson's result [1] and the Theorem 2.1 that

$$\|AX - XA + T\| \geq \|[A][X] - [X][A] + [T]\| \geq \|[T]\|.$$

Moreover, $\lambda \in \sigma_e(T)$ if and only if $\lambda \in \sigma_e(T_2)$. By hypothesis we have $\lambda \in \sigma_e(T) = \sigma_e(T_2)$. It is easy to check that

$$\inf\left\{\left\|\begin{pmatrix} K_1 + T_1 & K_2 \\ K_3 & T_2 + K_4 \end{pmatrix}\right\|,\ K_1, K_2, K_3, K_4 \text{ compacts}\right\} \geq$$
$$\geq \inf\left\{\|T_2 + K_4\|,\ K_4 \text{ compact}\right\}$$

Then it follows immediately

$$\|[T]\| \geq \|[T_2]\|.$$

Since T_2 has no isolated eigenvalues of finite multiplicity, then by using the Remark [5, p. 186] we have $\|AX - XA + T\| \geq |\lambda|$. Whence

$$\|AX - XA + T\| \geq |\lambda|$$

For all $X \in L(H)$ and all $\lambda \in \sigma_e(T)$. Finally, we conclude that

$$\|AX - XA + T\| \geq |\lambda|$$

For all $X \in L(H)$ and all $\lambda \in \sigma(T)$, and the proof is complete. □

As a special case we get the following Corollary.

Corollary 2.1. *Let $A, T \in L(H)$, such that A quasi-normal operator, T hyponormal and $AT = TA$. Then*

$$\|AX - XA + T\| \geq \|T\|$$

for all $X \in L(H)$.

Proof. Since A is a quasi-normal operator, it follows from [6] that $A = N + K$, where N is a normal and K is a compact. Hence, by using the same argument as in the above theorem, we get the desired inequality. □

Theorem 2.3. *Let A and T be commuting operators such that A is a finite operator and T is normal. Then*

$$\|AX - XA + T\| \geq \|T\|$$

for all $X \in L(H)$.

Proof. Let $\lambda \in \sigma_p(T)$ and let E be the subspace $E = \ker(T - \lambda)$. Since A commutes with T it follows from Fuglede-Putnam's theorem [8] that E reduces

A and T simultaneously. Hence, with respect to the decomposition $H = E \oplus E^{\perp}$, we have

$$A = \begin{pmatrix} B & 0 \\ 0 & * \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} \lambda & 0 \\ 0 & * \end{pmatrix}.$$

Let $X \in L(H)$ have the representation $X = \begin{pmatrix} Y & * \\ * & * \end{pmatrix} \in L(H)$. Then,

$$\|AX - XA + T\| = \left\| \begin{pmatrix} BY - YB + \lambda & * \\ * & * \end{pmatrix} \right\| \geq \|BY - YB + \lambda\|$$

Since B is a finite operator, it follows that

$$\|AX - XA + T\| \geq |\lambda|,$$

for all $X \in L(H)$ and all $\lambda \in \sigma_p(T)$.

Using the Berberian extension Theorem, we have that A° is finite , T° is normal and $A^\circ T^\circ = T^\circ A^\circ$. Since, $\sigma_p(T^\circ) = \sigma_{ap}(T^\circ) = \sigma_{ap}(T) = \sigma(T)$, it follows from the first part that

$$\|AX - XA + T\| \geq |\lambda|$$

for all $X \in L(H)$ and all $\lambda \in \sigma(T)$. Hence,

$$\|AX - XA + T\| \geq \|T\|$$

for all $X \in L(H)$. This completes the proof. $\square$

Theorem 2.4. *Let $A \in L(H)$. If A satisfies some quadratic polynomial, then A is a finite operator and $A^* \notin \overline{R(\delta_A)}^W$.*

P r o o f. Suppose that A satisfies $A^2 - 2\alpha A + \beta = 0$, hence $(A - \alpha)^2$ is a normal operator. Then, by Putnam's result [10] we may write $A - \alpha = N \oplus M$, where N is normal and $M = \begin{pmatrix} B & C \\ 0 & -C \end{pmatrix}$, with B normal and C is an injective positive operator such that $BC = CB$. Therefore, $A = (N + \alpha I) \oplus (M + \alpha I)$. Then for linear operator $X = \begin{pmatrix} Y & Z \\ R & S \end{pmatrix}$ we have

$$AX - XA + I = \begin{pmatrix} NY - YN + I & * \\ * & * \end{pmatrix}.$$

Since the norm of operator matrix always dominates the norm of its diagonal part [7, p. 82] one obtains

$$\|AX - XA + I\| \geq \|NY - YN + I\|.$$

Hence it follows from Williams's result [11] on normal operators that

$$\|AX - XA + I\| \geq 1$$

for all $X \in L(H)$. Let us assume that $A^* \in \overline{R(\delta_A)}^W$. An easy calculation leads us to $A^*A \in \overline{R(\delta_A)}^W$. Since $\overline{R(\delta_A)}^W$ contains nonzero positive operator [3], it follows that $A = 0$. $\square$

Remark 2.1. The above theorem is due to J. P. Williams [11], however we proved it by other method that this used.

Another interesting class of operators for which the Theorem 2.3 is satisfied is the class of operators A such that A^*A and $A + A^*$ commute. It is well known that this class has the property that $r(A) = \|A\|$ (see [4]).

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