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ON LOCAL UNIFORM TOPOLOGICAL ALGEBRAS

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Abstract. Every unital *"combinatorially regular"* commutative uniform complete locally m-convex algebra is local.

1. Introduction and preliminaries. The problem of whether a given *topological algebra* (even a unital commutative Banach algebra) is *local*, is a classical subject. Up to now, we have the general theory of local (topological) algebras for particular classes of topological algebras, which however are *regular*. Otherwise, we have also an important topological algebras, which are *local* but *not regular*, as for instance, the algebra of *holomorphic functions*. Yet, the problem of finding when a given topological algebra is local is of a particular importance, even for Quantum Field Theory, where its *"central message"* is that

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all information, we can get is *"Strictly local"* (R. Haag [7], see also A. Mallios [13]).

In this connection, it is proved in [19] that *every regular locally m-convex uniform is local.* Now, our main objective by the ensuing paper is to study the *"locality"* of uniform algebras, not necessarily regular, by applying in our case the, more general, notion of a *"combinatorially regular"* (topological) algebra (see also [1, p. 25, Definition 2.1]), initiated by R. Arenes (1962) for Banach algebras. Thus generalizing also our previous results in [19].

In the first part, we are going to discuss the necessary preliminary material, along with fixing up the relevant terminology.

Within this context, we mean by a *topological algebra* E a topological $\mathbb{C}$-vector space being also an algebra, with separately continuous ring multiplication, having non-empty *spectrum* $\mathfrak{M}(E)$ endowed with the Gel'fand topology. The respective *Gel'fand map* of E is given by

$$\mathcal{G} : E \longrightarrow \mathcal{C}(\mathfrak{M}(E)) : x \longmapsto \mathcal{G}(x) \equiv \hat{x} : \mathfrak{M}(E) \longrightarrow \mathbb{C}$$
$$: f \longmapsto \hat{x}(f) := f(x).$$

The image of $\mathcal{G}$, denoted by $E^{\wedge}$, is called the *Gel'fand transform algebra* of E and is topologized as a *locally m-convex algebra* by the inclusion

$$E^{\wedge} \subseteq \mathcal{C}_c(\mathfrak{M}(E)),$$

where the algebra $\mathcal{C}(\mathfrak{M}(E))$ carries the topology "c" of compact convergence in $\mathfrak{M}(E)$ [10, p. 19, Example 3.1]. A topological algebra E is called *semi-simple* if the respective Gel'fand map $\mathcal{G}$ is one-to-one.

A *uniform* algebra is a locally m-convex (Hausdroff) topological algebra E, whose topology is given by a net $(p_\alpha)_{\alpha \in I}$ of semi-norms, such that

$$p_\alpha(x^2) = p_\alpha(x)^2,$$

$x \in E$, for every $\alpha \in I$.

Given an algebra E, and let $\alpha \in \mathcal{C}_c(\mathfrak{M}(E))$, we say that α *locally belongs* to $E^{\wedge}$, if for each $f \in \mathfrak{M}(E)$, there exists a *neighborhood* U of f and an *element* $x \in E$, with $\hat{x}|_U = \alpha|_U$.

We shall say that E is *local*, if every *continuous* $\mathbb{C}$-*valued function* α on $\mathfrak{M}(E)$, which *locally belongs* to $E^{\wedge}$, is realized by an *element* of $\in E$ (viz. there exists an element $x \in E$, such that $\alpha = \hat{x}$ on $\mathfrak{M}(E)$).

Suppose E be a topological algebra whose spectrum is $\mathfrak{M}(E)$ and let A be any subset of E. The set

$$h(A) = \{f \in \mathfrak{M}(E) : A \subseteq ker(f)\}, \tag{1.1}$$

is called the hull of A in $\mathfrak{M}(E)$. On the other hand, if B is any subset of $\mathfrak{M}(E)$, the set

$$k(B) = \{x \in E : B \subseteq Z(\hat{x})\}, \tag{1.2}$$

with $Z(\hat{x}) = \{f \in \mathfrak{M}(E) : \hat{x}(f) = f(x) = 0\}$, is called the kernel of B in E. Thus, a set $B \in \mathfrak{M}(E)$ is said to be a hull, if

$$B = h(k(B)). \tag{1.3}$$

While, a set $A \subseteq E$ is said to be a kernel, if one has

$$A = k(h(A)) \tag{1.4}$$

Thus, the hulls of $\mathfrak{M}(E)$, i.e. the sets $B = h(k(B))$, may be taken as the closed sets of a topology on $\mathfrak{M}(E)$, the later is called hull-kernel topology (or for short hk-topology), and denoted by τ_{hk}.

Now, a topological algebra E is said to be *combinatorially regular* [1, p. 25, Definition 2.1], if for every *compact* K of $\mathfrak{M}(E)$; whenever a *finite (weak) open cover*, $\{U_1, \ldots, U_n\}$, of K is given, there exists *regularly closed* sets $\{F_1, \ldots, F_n\}$ which cover $\mathfrak{M}(E)$, and *regularly open* $\{V_1, \ldots, V_n\}$; such that

$$F_i \subseteq V_i \subseteq U_i \quad , \quad \forall\, i = 1, \ldots, n. \tag{1.5}$$

"Regularly" refers to the *hull-kernel topology.* (see [10, p. 331, Definition 1.2]).

While, we say that E is *regular* algebra, whenever for each *(weakly) closed* set F of $\mathfrak{M}(E)$ and $f \notin F$, there exists an *element* $x \in E$ such that

$$\hat{x}(f) = 1 \ \ and \ \ \hat{x} = 0 \ on\ F,$$

[10, p. 332, Definition 1.2]. Furthermore, we note that E is *regular* algebra if, and only if, the *hull-kernel topology* coincides with the *Gel'fand topology* on $\mathfrak{M}(E)$ [10, p. 332, Theorem 2.1]; so every *regular* algebra is *combinatorially regular.*

Finally, we give an important result, which we are going to apply in the sequel. One has

Lemma 1.1. (Aren's-Michael decomposition, [10, p. 88, Theorem 3.1]). *Every complete locally m-convex algebra E is (within a topological algebraic isomorphism) the projective limit of a Banach algebras, that is*

$$E = \varprojlim E_\alpha, \tag{1.6}$$

denoted by E_α the Banach algebras corresponding to a given local basis of neighborhoods of zero, $\Gamma = (U_\alpha)_{\alpha \in I}$ [10].

2. Main results. Through this section, E is a unital commutative locally m-convex algebra, whose topology is given by a net $(p_\alpha)_{\alpha\in I}$ of semi-norms. We recall that, the factors of the Arens-decomposition E_α, as already mentioned in (1.6), are given, more precisely, by

$$E_\alpha = \widehat{E/N_\alpha}. \tag{2.1}$$

Where the second term is the completion of E/N_α, and $N_\alpha = ker(p_\alpha) = \{x \in E : p_\alpha(x) = 0\}$ is a closed ideal of E.

Before we proceed further, we first need the following two lemmas concerning the *"hull-kernel"* notion, both of them are often used in the sequel. that is, for a given topological algebra E with a spectrum $\mathfrak{M}(E)$ and I a closed ideal of E, we denote by

$$\phi : E \to E/I \tag{2.2}$$

the canonical quotient map. While, the transposition of the latter is denoting by

$$\rho \equiv {}^t\phi : \mathfrak{M}(E/I) \to \mathfrak{M}(E). \tag{2.3}$$

Lemma 2.1. *Let E be a topological algebra whose spectrum is $\mathfrak{M}(E)$ and I a closed ideal of E. Furthermore, let A be any set in E and B a set of $\mathfrak{M}(E)$. Then, one gets*

1) $\rho^{-1}(h(A)) = h(\phi(A))$

2) $\phi(k(B)) = k(\rho^{-1}(B))$

P r o o f. Concerning the first assertion,

$$\begin{aligned}\rho^{-1}(h(A)) &= \{\overline{f} \in \mathfrak{M}(E/I) : \overline{f} \circ \phi \in h(A)\} \\ &= \{\overline{f} \in \mathfrak{M}(E/I) : \overline{f} \circ \phi(x) = 0, \forall x \in A\} \\ &= \{\overline{f} \in \mathfrak{M}(E/I) : \widehat{\phi(x)}(\overline{f}) = 0, \forall x \in A\} \\ &= \{\overline{f} \in \mathfrak{M}(E/I) : \hat{y}(\overline{f}) = 0, \forall y \in \phi(A)\} = h(\phi(A).\end{aligned}$$

On the other hand, for the second one, we have

$$\phi(x) \in k(\rho^{-1}((B)) \Leftrightarrow \forall \overline{f} \in \rho^{-1}((B), \overline{f}(\phi(x)) \Leftrightarrow \forall \rho(\overline{f}) \in B, \rho(\overline{f}) = 0$$
$$\Leftrightarrow \forall f \in B \cap h(I), f(x) = 0 \Leftrightarrow x \in k(B \cap h(I).$$

Now, the rest of our assertion is to prove that,

$$x \in k(B \cap h(I)) \Leftrightarrow \phi(x) \in \phi(k(B \cap h(I)))$$

Indeed, the direct implication is immediate, it suffice only to prove the converse one. So, by assuming that $\phi(x) \in \phi(k(B \cap h(I)))$, there exists $y \in k(B \cap h(I))$ such that $\phi(x) = \phi(y)$. Otherwise, for any $f \in B \cap h(I)$, there exists $\overline{f} \in \mathfrak{M}(E/I)$ such that $f = \overline{f} \circ \phi$. Then, since $y \in k(B \cap h(I))$, we get

$$f(x) = \overline{f} \circ \phi(x) = \overline{f} \circ \phi(y) = 0, \ \forall f \in B \cap h(I).$$

Hence, $x \in k(B \cap h(I))$, and this finish the proof. □

The forgoing enable us to set the following.

Lemma 2.2. *Let E be a topological algebra whose spectrum is $\mathfrak{M}(E)$, and I is a closed ideal of E. Then,*

$$\rho : \mathfrak{M}(E/I) \to \mathfrak{M}(E).$$
$$\overline{f} \longmapsto \overline{f} \circ \phi$$

is continuous, by endowing the spectrums with hk-topologies.

P r o o f. Let B be a regularly closed set in $h(I) \subseteq \mathfrak{M}(E)$. By applying the above lemma, the inverse image of B, under the map ρ, is

$$\rho^{-1}(B) = \rho^{-1}(hk(B)) = h(\phi(k(B)))$$
$$= h(\phi(k(B \cap h(I)))) = hk(\rho^{-1}(B)).$$

Therefore, B is regularly closed in $\mathfrak{M}(E/I)$, hence ρ is continuous. □

Now, the key to our main result is the following proposition, which holds by combining the previous lemma with the known fact that the map ρ is also continuous by endowing the spectrums with the Gel'fand topologies. Namely, more precisely, the map:

$$\rho : \mathfrak{M}(E/I) \to h(I) \subseteq \mathfrak{M}(E). \qquad (2.4)$$
$$\overline{f} \longmapsto \overline{f} \circ \phi$$

is a homeomorphism, in particular the latter is also an open map (see [10, p. 339, Theorem 4.1]). Thus, we have

Proposition 2.3. *Let E be a topological algebra with the spectrum $\mathfrak{M}(E)$, and I is a closed ideal of E. Then, if E is* combinatorially regular*, the quotient topological algebra E/I is also* combinatorially regular*.*

P r o o f. Let K be a (weak) compact set of $\mathfrak{M}(E/I)$. If a *finite (weak) open cover*, $\{U_1, \ldots, U_n\}$, of K is given; then by (2.4), the family $\{\rho(U_1), \ldots, \rho(U_n)\}$ is also a *finite (weak) open cover* of the compact $\rho(K)$ in $h(I) \subseteq \mathfrak{M}(E)$. So, there exists $\{U_1', \ldots, U_n'\}$ an open sets of $\mathfrak{M}(E)$, such that for each $i \in \{1, \ldots, n\}$, one has $U_i = U_i' \cap h(I)$. Therefore, $\{U_1', \ldots, U_n'\}$ is a *finite (weak) open cover* of the compact $\rho(K)$ in $\mathfrak{M}(E)$. As a byproduct, since E is combinatorially regular, there exists *regularly closed* sets $\{F_1', \ldots, F_n'\}$ which cover $\rho(K)$, and *regularly open* $\{V_1', \ldots, V_n'\}$, such that

$$F_i' \subseteq V_i' \subseteq U_i' \quad , \quad \forall\, i = 1, \ldots, n.$$

Now, by continuity of ρ (Lemma 2.2), the sets $F_i = \rho^{-1}(F_i')$ are regularly closed which, furthermore, cover K, and $V_i = \rho^{-1}(V_i')$ are regularly open sets in $\mathfrak{M}(E/I)$ such that

$$F_i \subseteq V_i \subseteq U_i \quad , \quad \forall\, i = 1, \ldots, n.$$

And that is our desired result. □

We come now to our main result, as promised in the Abstract. However, we still have to recall the following important result, which recently proved in [17, p. 494, Theorem 1], so we have

(2.5) If $(E_\alpha, \pi_{\alpha\beta})_{\alpha \in I}$ is a strictly dense projective system [10: p. 174] of local semi-simple topological algebras, then the projective limit, $E = \varprojlim E_\alpha$, is local.

That is, one gets

Theorem 2.3. *Let E be a* combinatorially regular *uniform complete locally m-convex with identity. Then, E is local.*

P r o o f. By considering the Arenes Micheal decomposition of E, one obtains, as we mentioned it before in Lemma 1.1,

$$E = \varprojlim E_\alpha.$$

Furthermore, the latter is is a strictly dense projective system of Banach algebras (see for instance, [10, p. 176, (7:18)]). On the other hand, it follows from Proposition 2.3, since E is combinatorially regular, that each one of the algebras E_α is combinatorially regular. Moreover, by the semi-simplicity of E, we conclude that E_α is also semi-simple; indeed, the norm of E_α is given by $\|\rho_\alpha(x)\|_\alpha = p_\alpha(x)$, so that one has by hypothesis $p_\alpha(x^2) = (p_\alpha(x))^2$, hence $\|\rho_\alpha(x^2)\|_\alpha = (\|\rho_\alpha(x)\|_\alpha)^2$, for every $x \in E$ (cf., [10: p. 93, (4.6), as well as, p. 274, (5.1)]). As a byproduct, each one of the algebra E_α is semi-simple [10, p. 275, Lemma 5.1]. Finally, the above result (2.5) says that E is local. $\square$

We conclude this paper with an immediate consequences of the preceding, we remark that:

Remark 2.5. 1) According to [10, p. 279, Corollary 5.2], a semi-simple Michael algebra [10, p. 269, Definition 3.4] is a uniform algebra. Thus, by the above theorem we set

A unital semi-simple combinatorially regular complete Michael algebra is local.

2) Moreover, in view of [10, p. 271, Theorem 4.1] a Warner algebra [10, p. 271, Definition 4.1] with continuous Gel'fand map is uniform algebra. On the other hand, a Fréchet algebra is a Warner algebra, having continuous Gel'fand map [10, p. 183, Corollary 1.1], hence a uniform algebra. Therefore, on has the following

A unital semi-simple combinatorially regular Fréchet algebra is local.

In this respect, we finally note that the previous result follows also from [21]. However the prof there is based on the existence of partitions of unity, some thing that we do not apply here.

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