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SOME GENERALIZATION OF DESARGUES AND VERONESE CONFIGURATIONS

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Abstract. We propose a method of constructing partial Steiner triple system, which generalizes the representation of the Desargues configuration as a suitable completion of three Veblen configurations. Some classification of the resulting configurations is given and the automorphism groups of configurations of several types are determined.

Introduction. Let us start with the classical Desargues configuration 10_3 arising from the Desargues theorem on the perspective of two triangles in a projective space (cf. eg. [5]). This configuration consists of three lines of size 3 through a point p, three Veblen subconfigurations inscribed into every pair of the given lines, and an axis which joins corresponding points of intersection. This description does not characterize the Desargues configuration, actually also the combinatorial Veronese space $\mathbf{V}_3(3)$ (cf. [13]) may be presented in the same way,

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and it is not isomorphic to the Desargues configuration. Besides, these two are the only possible. Constructing the Desargues configuration we join six points on the lines through p so as two triangles appear; constructing a Veronese space we draw a hexagon, which makes the resulting configuration a cousin of the Pappus Configuration. As a generalization of such situation the following question arises. Given a set of *points* $\mathcal{S}$, let p be a point and let $\mathcal{L}_p$ be a set of triples of points of $\mathcal{S}$, called *lines*, all containing $p \in \mathcal{S}$. What configurations can appear when every pair of these lines yields a Veblen figure (such a situation appears eg. when we consider the perspective of two n-simplices in a projective space, cf. [14]). In the paper we give some answers to this problem. It is also worth to point out that our investigations lead us to purely combinatorial problems concerning, in fact, determining partial Steiner triple systems defined on the universe of 2-element subsets of a given set (cf. Representation 3).

The resulting configuration $\mathfrak{M}$ is determined by some graph $\mathcal{P}$ on n vertices and the way of joining points of intersection of "second" pairs of lines in the corresponding Veblen figures. The way to join points in the Veblen figures is characterized by an isomorphism γ determining the type $\mathfrak{H}$ of the configuration which constitute these points. The obtained configuration will be written $\mathfrak{M} = \mathbb{M}^n \rhd^{\gamma}_{\mathcal{P}} \mathfrak{H}$. A natural rule of such a joining is suggested by the construction of the combinatorial Grassmann space $\mathbf{G}_2(n)$. In accordance with this rule every triple of lines through p yields in $\mathfrak{M}$ either the Desargues Configuration or the $\mathbf{V}_3(3)$ space. In most of the considered examples this rule will be adopted (one interesting exception is discussed in Representation 3 and Proposition 17). After that the configuration $\mathfrak{M} = \mathbb{M}^n \rhd_{\mathcal{P}} \mathbf{G}_2(n)$ is determined by a graph $\mathcal{P}$ only. A classification of the investigated configurations follows from some natural classification of graphs, proposed in the paper.

For fixed n all the configurations $\mathbb{M}^n \rhd^{\gamma}_{\mathcal{P}} \mathfrak{H}$ have the same parameters (numbers of points and lines), but, of course, they need not to be isomorphic. Section 4 contains the classification of $\mathbb{M}^n \rhd_{\mathcal{P}} \mathbf{G}_2(n)$, for $n \leq 5$ (Theorems 4 and 5). It turns out that there are exactly three nonisomorphic configurations $\mathbb{M}^4 \rhd_{\mathcal{P}} \mathbf{G}_2(4)$, and exactly seven nonisomorphic configurations $\mathbb{M}^5 \rhd_{\mathcal{P}} \mathbf{G}_2(5)$. In Section 3 we determine automorphism groups $\mathrm{Aut}(\mathbb{M}^n \rhd_{\mathcal{P}} \mathbf{G}_2(n))$ for arbitrary n and some more regular graphs $\mathcal{P}$ (Proposition 10, Corollary 1, Propositions 12, 13, and 14). As a consequence we obtain a characterization of automorphisms of our small configurations classified in Section 4.

Among the structures discussed in the paper two types seem especially interesting, possibly for their own. The first one consitute structures of the form $\mathbb{M}^n \rhd_{K_n} \mathbf{G}_2(n) \cong \mathbf{G}_2(n+2)$, generalizing the Desargues configuration, which are studied in details in [14]. The second type constitute structures $\mathbb{M}^n\rhd_{N_n}\mathbf{G}_2(n)$ determined by an empty graph N_n; these structures generalizing the Veronese space $\mathbf{V}_3(3)$ slightly remind also generalization of (dual minor) Pappus configuration.

1. Generalities, definitions, and basic facts. Let X be a nonempty n-element set. For every nonnegative integer k let $\wp_k(X)$ denote the set of all k-element subsets of X. We begin with recalling some fundamental types of graphs (nonoriented, without loops) defined on X (cf. [18]). We write

K_n – for the complete graph $\langle X, \wp_2(X)\rangle$, and N_n for the empty graph $\langle X, \emptyset\rangle$,

L_n – for the linear graph $\big\langle X, \big\{\{x_i, x_{i+1}\}\colon i = 1, \dots, n-1\big\}\big\rangle$ for some ordering $x_1, \dots, x_n$ of the set X,

C_n – for the (closed) n-gon $\big\langle X, \big\{\{x_i, x_{i+1}\}\colon i = 1, \dots, n-1\big\} \cup \big\{\{x_n, x_1\}\big\}\big\rangle$,

K_{n_1,n_2} – for the complete bipartite graph $\big\langle X, \big\{\{x_i, x_{n_1+j}\}\colon i = 1, \dots, n_1,\ j = 1, \dots, n_2\big\}\big\rangle$, $(n = n_1 + n_2)$; in particular, $M_{n-1} = K_{1,n-1}$ is the pencil with $n-1$ edges;

if $X \subset Y$, $|X| = n$, $|Y| = m$, and T is any of the above types of graphs on X, we write

T^m – for the image of the graph of the type T defined on X under natural embedding of X into Y.

If $\mathcal{P}$ is a graph defined on a set X (ie. $\mathcal{P} \subset \wp_2(X)$) and $A \subset X$, we write $\mathcal{P} \curlywedge A$ for the restriction $\mathcal{P} \cap \wp_2(A)$ of $\mathcal{P}$ to A.

Further, we briefly recall the definitions of some (combinatorial) structures, which will be used in the paper.

Desarguesian closure $\mathbf{D}(\mathfrak{S})$ of a graph $\mathfrak{S}$ (cf. [15], [7]) Let $\mathfrak{S} = \langle S, \mathcal{E}\rangle$, where $\mathcal{E} \subset \wp_2(S)$ is a nonoriented graph without loops. We complete its every edge $e \in \mathcal{E}$ with a new point e^∞ in such a way that distinct edges get distinct improper points. Let $\mathcal{T}$ be the set of all triangles in $\mathfrak{S}$. With every triangle $T \in \mathcal{T}$ we associate a new line T^∞ consisting of the points e^∞, where $e \in \mathcal{E}$, $e \subset T$. The structure $\mathbf{D}(\mathfrak{S})$ is the incidence structure

$$\big\langle S \cup \{e^\infty\colon e \in \mathcal{E}\}, \big\{e \cup \{e^\infty\}\colon e \in \mathcal{E}\big\} \cup \big\{T^\infty\colon T \in \mathcal{T}\big\}\big\rangle.$$

Combinatorial Grassmannian $\mathbf{G}_k(X)$ (cf. [13], [14], [9]) For any positive integer k such that $1 \le k < n$ we put $\mathbf{G}_k(X) = \langle \wp_k(X), \wp_{k+1}(X), \subset\rangle$. We write, shortly, $\mathbf{G}_k(n) \cong \mathbf{G}_k(X)$, where $|X| = n$. (The structure $\mathbf{G}_2(n)$ formalizes the perspective of two $(n-1)$-simplices, cf. [14].)

Combinatorial Veronesian $\mathbf{V}_m(X)$ (cf. [11], [13]) We write $\mathfrak{y}_m(X)$ for the set of m-element multisets with elements from X, coded with the rule

$$x_1^{m_1}\dots x_\nu^{m_\nu} = \{\underbrace{x_1,\dots,x_1}_{m_1\text{times}},\dots,\underbrace{x_\nu,\dots,x_\nu}_{m_\nu\text{times}}\},$$

where the m_i are nonnegative integers, $m = m_1 + \ldots + m_\nu$, and $x_1,\ldots x_\nu \in X$. The structure $\mathbf{V}_m(X)$ is the incidence structure whose points are elements of $\mathfrak{y}_m(X)$, and lines are all the sets of the form $fX^r = \{fx^r : x \in X\}$ with $1 \le r \le m$ and $f \in \mathfrak{y}_{m-r}(X)$. For short, we write $\mathbf{V}_k(n) \cong \mathbf{V}_k(X)$, where $|X| = n$. (For some results on classical projective Veronesians we refer the reader, eg. to [2, 10, 17].)

Let $\alpha \in S_X$ i.e. let α be a permutation of X; we write $\alpha^{(m)}$ for the natural action of α on $\wp_m(X)$. Clearly, $\alpha^{(k)} \in \mathrm{Aut}(\mathbf{G}_k(X))$. In a similar way S_X acts (faithfully) as an automorphism group of $\mathbf{V}_k(X)$.

Example 1. $\mathbf{G}_2(3) \cong \mathbf{V}_1(3)$ is a single 3-element line. $\mathbf{G}_2(4) \cong \mathbf{V}_2(3)$ is the Veblen Configuration. Moreover, $\mathfrak{D}^o := \mathbf{G}_2(5) \cong \mathbf{D}(K_4)$ is simply the Desargues configuration, and $\mathfrak{V}^o := \mathbf{V}_3(3)$ is the 10_3G-configuration of Kantor (cf. [6], see also [3]), presented in Figure 1. ○

Finally, let us recall some standard notations from the theory of partial linear spaces. If $\mathfrak{M}$ is a partial linear space with constant point degree and line size we write $\boldsymbol{\nu}_{\mathfrak{M}}$ for the number of its points, $\boldsymbol{b}_{\mathfrak{M}}$ for the number of its lines, $\boldsymbol{r}_{\mathfrak{M}}$ for the degree of any of its points, and $\boldsymbol{\kappa}_{\mathfrak{M}}$ for the size of any of its lines; $\mathfrak{M}$ is also called a $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_{\boldsymbol{\kappa}})$-configuration, where $\boldsymbol{\nu} = \boldsymbol{\nu}_{\mathfrak{M}}$, $\boldsymbol{r} = \boldsymbol{r}_{\mathfrak{M}}$, $\boldsymbol{\kappa} = \boldsymbol{\kappa}_{\mathfrak{M}}$, and $\boldsymbol{b} = \boldsymbol{b}_{\mathfrak{M}}$. A partial Steiner triple system is a partial linear space whose lines have size 3; consequently, every $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_3)$-configuration is a partial Steiner triple system.

Proposition 1. *Let $\mathfrak{G} = \mathbf{G}_2(n+2)$ and $\mathfrak{V} = \mathbf{V}_n(3)$. Then $\boldsymbol{\nu}_{\mathfrak{G}} = \boldsymbol{\nu}_{\mathfrak{V}} = \binom{n+2}{2}$, $\boldsymbol{b}_{\mathfrak{G}} = \boldsymbol{b}_{\mathfrak{V}} = \binom{n+2}{3}$, $\boldsymbol{\kappa}_{\mathfrak{G}} = \boldsymbol{\kappa}_{\mathfrak{V}} = 3$, and $\boldsymbol{r}_{\mathfrak{G}} = \boldsymbol{r}_{\mathfrak{V}} = n$.*

This means that $\mathfrak{G}$ and $\mathfrak{V}$ both are $\left(\binom{n+2}{2}_n, \binom{n+2}{3}_3\right)$-configurations.

In this paper we are going to construct and investigate a class of $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_3)$ configurations. In particular, we are interested how do they look like, and what are their automorphisms. We end this section by recalling some classical results on $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_{\boldsymbol{\kappa}})$ configurations, and the definition of subspace of a partial linear space.

Proposition 2 (Kirkmann)**.** *A Steiner triple system can be defined on an $\boldsymbol{\nu}$-element set if and only if $\boldsymbol{\nu} \equiv 1 \mod 6$ or $\boldsymbol{\nu} \equiv 3 \mod 6$.*

Proposition 3 [1]**.** *If $\mathfrak{M}$ is a $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_{\boldsymbol{\kappa}})$-configuration, then $\boldsymbol{\nu}\boldsymbol{r} = \boldsymbol{b}\boldsymbol{\kappa}$. A $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_{\boldsymbol{\kappa}})$-configuration is a linear space if and only if $\binom{\boldsymbol{\nu}}{2} = \boldsymbol{b}\binom{\boldsymbol{\kappa}}{2}$.*

Theorem 1 [4]**.** *There is a* $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_3)$*-configuration if and only if* $\boldsymbol{\nu} \geq 2\boldsymbol{r}+1$ *and* $\boldsymbol{\nu}\boldsymbol{r} = 3\boldsymbol{b}$.

A subset Z of the point set of a partial linear space $\mathfrak{M}$ is a *subspace* of $\mathfrak{M}$ if every line of $\mathfrak{M}$ which crosses Z in at least two points is entirely contained in Z.

2. Construction of some $\left(\binom{n+2}{2}_{n}, \binom{n+2}{3}_{3}\right)$-configurations. In this section we are going to give our constructions. Let us start with a representation of $\mathfrak{V}^o = \mathbf{V}_3(\{a,b,c\})$, which consists in suitable modification of the construction of $\mathbf{D}(K_4)$.

Representation 1. It is seen that the set $\mathfrak{y}_3(\{a,b\})$ yields in $\mathfrak{V}^o$ the complete graph K_4 with vertices

$$(1) = a^2b,\ (2) = b^3,\ (3) = ab^2,\ \text{and}\ \ (4) = a^3.$$

We have a new point $(i,j)^\infty$ added on the edge $\overline{(i),(j)}$ for every pair i,j with $1 \leq i < j \leq 4$.

$$(1,2)^\infty = bc^2, (1,3)^\infty = abc, (1,4)^\infty = a^2c, (2,3)^\infty = b^2c,$$
$$(2,4)^\infty = c^3, (3,4)^\infty = ac^2.$$

For two triangles (1)(2)(3) and (1)(4)(3) of K_4 (with the common side $\overline{(1),(3)}$) we add two lines which join their improper points:

$$(1,2)^\infty, (1,3)^\infty, (2,3)^\infty \mid bcX \ \text{and}\ (1,3)^\infty, (1,4)^\infty, (3,4)^\infty \mid acX;$$

the other two new lines join improper points of three edges which complete $\overline{(1),(3)}$ to a quadrangle in K_4:

$$(1,4)^\infty, (2,4)^\infty, (2,3)^\infty \mid cX^2 \ \text{and}\ (1,2)^\infty, (2,4)^\infty, (3,4)^\infty \mid c^2X.$$

Recall, that to obtain the Desargues configuration $\mathfrak{D}^o$ we need to add every of the four new lines as joining improper points of edges of a triangle in K_4 (cf. [7]). ○

On the other hand, the configuration $\mathfrak{V}^o$ can be, more intuitively presented in the following way:

Representation 2. The three lines $L_1 = abX$, $L_2 = acX$, and $L_3 = bcX$ of $\mathfrak{V}^o$ pass through the point $p = abc$ (the *center*). The other two points a_i, b_i

on the corresponding L_i are: $a_1 = a^2b$, $b_1 = ab^2$, $a_2 = ac^2$, $b_2 = a^2c$, $a_3 = b^2c$, $b_3 = bc^2$. The structure $\mathfrak{V}^o$ contains also the lines $G_{i,j} = \overline{a_i, b_j}$ $(i \neq j)$; namely $G_{1,2} = a^2X$, $G_{2,1} = aX^2$, $G_{1,3} = bX^2$, $G_{3,1} = b^2X$, $G_{2,3} = c^2X$, $G_{3,2} = cX^2$. After that the *diagonal point* c_l is placed on $G_{i,j}, G_{j,i}$, where $\{i,j,l\} = \{1,2,3\}$ $(c_1 = c^3$, $c_2 = b^3$, $c_3 = a^3)$. Finally, $\mathfrak{V}^o$ is obtained by adding one new line X^3 (the *axis*) which joins the diagonal points of the corresponding three quadrangles (see Figure 1).

Now, we can immediately recognize a similarity between $\mathfrak{V}^o$ and other classical configurations, in particular, the Pappus (and Pascal-Brianchon) configuration (cf. [5, 8, 12]).

It is worth to note that if we introduce the lines $A_{i,j} = \overline{a_i, a_j}$ and $B_{i,j} = \overline{b_i, b_j}$ and require that the points c_l on $A_{i,j}$, $B_{i,j}$ $(\{i,j,l\} = \{1,2,3\})$ are on one axis, then simply the Desargues Configuration will arise (cf. [5]). ○

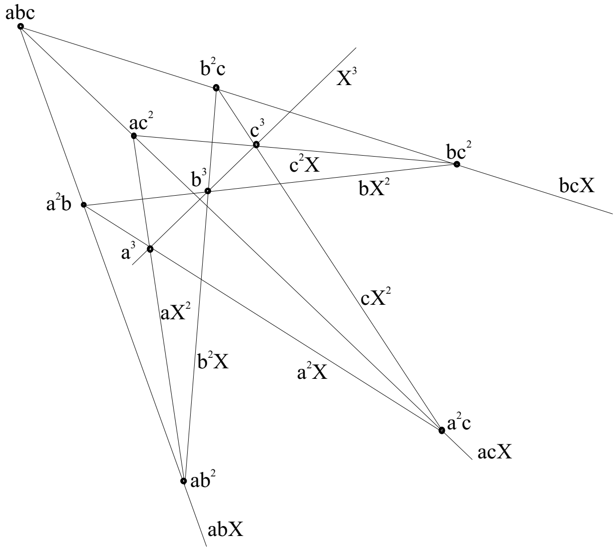


Fig. 1. Configuration $\mathfrak{V}^o$

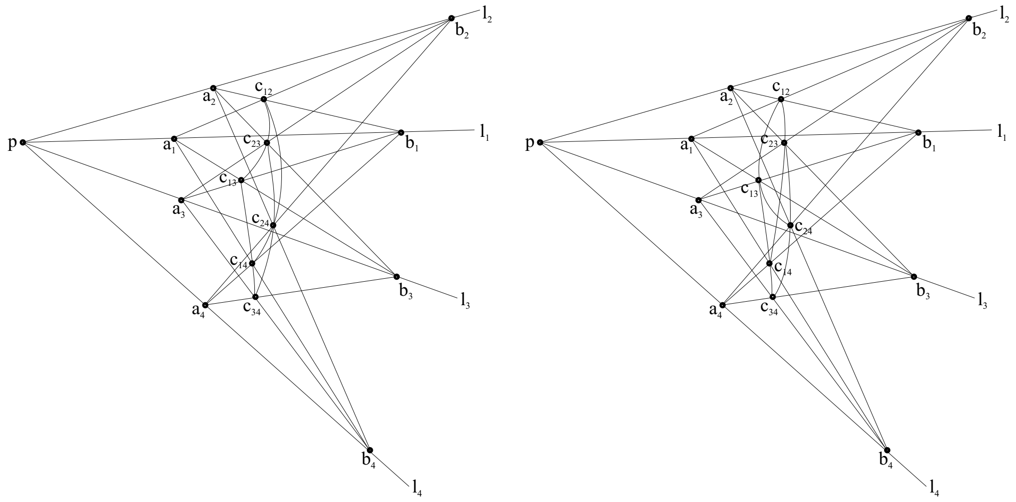
Both Representation 1 and Representation 2 can be generalized.

Construction 1. We define the closure $\widetilde{\mathbf{D}}(K_n)$ of the complete graph K_n as follows. First, we complete every edge of K_n by an improper point, like in the case of defininig $\mathbf{D}(K_n)$. The obtained triples constitute one class of lines of $\widetilde{\mathbf{D}}(K_n)$. Let e be a fixed edge of K_n. The second class of lines of $\widetilde{\mathbf{D}}(K_n)$ consists of the sets of the form $\{e_1^\infty, e_2^\infty, e_3^\infty\}$, where the e_i are edges of K_n such that one of the following holds:

- e_1, e_2, e_3 is a triangle in K_n which either misses e or has e as one of its sides;
- e, e_1, e_2, e_3 is a quadrangle in K_n.

Since the automorphism group of K_n is transitive on its edges, the isomorphism type of $\widetilde{\mathbf{D}}(K_n)$ does not depend on the choice of a particular edge e. $\bigcirc$

Construction 2. Let us fix a natural number n and let us write $X = \{1, \ldots, n\}$. Let p be a point, and let $L_1, \ldots, L_n$ be distinct lines (*rays*) through p. On every line L_i we consider two other points a_i, b_i, and then we have lines $G_{i,j} = \overline{a_i, b_j}$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. After that we complete every system of points on L_i, L_j to the Veblen figure adding a point $c_{i,j}$ on $G_{i,j}, G_{j,i}$ (note: we can write, in fact, $c_{i,j} = c_{\{i,j\}}$, i.e. we can consider the points c's as labelled with elements of $\wp_2(X)$). Finally, for every $T \in \wp_3(X)$ we consider the line $C_T = \big\{c_z : z \in \wp_2(T)\big\}$.



The configuration $\mathbb{B}(4)$ The configuration $\mathbb{NV}^4 \rhd_{N_4} \mathbb{B}(2)$

Fig. 2. The configuration $\mathbb{B}(4)$ and its cousin (cf. Example 2)

The obtained system of points and lines will be denoted by $\mathbb{B}(n)$. $\bigcirc$

As an example we present Figure 2 which illustrates the structure of $\mathbb{B}(4)$. In view of Representation 2 the following is immediate:

Proposition 4. $\mathbb{B}(3) = \mathfrak{V}^o$.

It is slightly more difficult to prove

Proposition 5. $\widetilde{\mathbf{D}}(K_{n+1}) \cong \mathbb{B}(n)$.

Proof. Let $X = \{1, \dots, n+1\}$ be the set of vertices of K_{n+1} and let the edge $\{1, 2\}$ be fixed. We label the points and lines of $\widetilde{\mathbf{D}}(K_{n+1})$ in the following way:
$p = \{1, 2\}^\infty$, $\quad a_0 = 2$, $b_0 = 1$, $\quad L_0 = \overline{\{1, 2\}}$,
$a_i = \{1, i\}^\infty$, $b_i = \{2, i\}^\infty$, $L_i = \{1, 2, i\}^\infty$, $G_{0,i} = \overline{\{2, i\}}$, $G_{i,0} = \overline{\{1, i\}}$, $c_{\{0,i\}} = i$, for $i = 3, \dots, n+1$,
$c_{\{i,j\}} = \{i, j\}^\infty$, $G_{i,j}$ joins improper points of the quadrangle $(1, i, j, 2)$ in K_{n+1} (with p omitted), $C_{\{0,i,j\}} = \overline{\{i, j\}}$, for $\{i, j\} \in \wp_2(\{3, \dots, n+1\})$,
$C_T = T^\infty$ for $T \in \wp_3(\{3, \dots, n+1\})$.

It is seen that the above labelling establishes an isomorphism of $\widetilde{\mathbf{D}}(K_{n+1})$ on $\mathbb{B}(n)$ with its rays numbered by the integers $0, 3, 4, \dots, n+1$. $\square$

Proposition 6. *The incidence structure* $\mathbb{B}(n)$ *is isomorphic to the dual of* $\mathbf{V}_3(n)$.

Proof. Let $X = \{t_1, \dots, t_n\}$, $\mathfrak{V} = \mathbf{V}_3(X)$, and $\mathfrak{M}$ be the dual of $\mathfrak{V}$. Set $L_i = t_i^3$ for $i = 1, \dots, n$; these lines of $\mathfrak{M}$ meet in the point $p = X^3$. Next, we define $a_i = t_i X^2$, $b_i = t_i^2 X$ for $i = 1, \dots, n$, and $c_{\{i,j\}} = t_i t_j X$ for distinct $i, j \in \{1, \dots, n\}$. A straightforward verification shows (see Section 3 of [13] for details) that the above yields a required isomorphism. $\square$

The following is evident.

Proposition 7. *For every natural* $n \geq 3$ *the structure* $\mathfrak{B} = \mathbb{B}(n)$ *is a partial linear space (a partial Steiner triple system, cf. [16]), with parameters:*

$$\boldsymbol{\nu}_{\mathfrak{B}} = \binom{n+2}{2}, \quad \boldsymbol{b}_{\mathfrak{B}} = \binom{n+2}{3}, \quad \boldsymbol{r}_{\mathfrak{B}} = n, \quad \boldsymbol{\kappa}_{\mathfrak{B}} = 3. \tag{1}$$

Remark 1. Let us modify the construction of $\mathbb{B}(n)$ so as we draw lines $A_{i,j} = \overline{a_i, a_j}$, $B_{i,j} = \overline{b_i, b_j}$, and after that $c_{i,j}$ is on $A_{i,j}$, $B_{i,j}$. It is seen that we obtain simply $\mathbf{G}_2(n+2) \cong \mathbf{D}(K_{n+1})$.

The way in which the points c_z are grouped into lines is, from some point of view, natural. Following this way we obtain, in particular, that the subconfiguration of $\mathbb{B}(n)$ spanned by the points c_z is isomorphic to $\mathbf{G}_2(n)$. But it is not the unique one. In what follows we shall generalize our construction.

Construction 3. Let n be a fixed natural number and $X = \{1, \dots, n\}$. The construction goes in several steps.

Step A Let p be an arbitrary "point".

Step B Through p we have lines L_i, and new points a_i, b_i on L_i, for every $i \in X$.

Step C We choose a subset $\mathcal{P}$ of $\wp_2(X)$, and after that

if $\{i, j\} \in \mathcal{P}$**:** we draw lines $A_{i,j} = \overline{a_i, a_j}$ and $B_{i,j} = \overline{b_i, b_j}$; the point $c_{\{i,j\}}$ is common for $A_{i,j}$ and $B_{i,j}$,

if $\{i, j\} \in \wp_2(X) \setminus \mathcal{P}$**:** we draw lines $G_{i,j} = \overline{a_i, b_j}$; the point $c_{\{i,j\}}$ is common for $G_{i,j}$ and $G_{j,i}$,

for every $\{i, j\} \in \wp_2(X)$. It is seen that the point p and the points a_i, b_i $(i \in X)$ have degree n, while (up to now) c_z with $z \in \wp_2(X)$ has degree 2. Moreover, the number of the points c_z is $\binom{n}{2}$.

Step D Let $\mathfrak{H}$ be any $\left(\binom{n}{2}_{n-2}, \binom{n}{3}_3\right)$-configuration. Finally, we identify the points c_z constructed above with points of $\mathfrak{H}$ (under some bijection γ) and, consequently, we group the points c_z into new $\binom{n}{3}$ lines obtained as coimages of the lines of $\mathfrak{H}$ under γ.

The resulting configuration will be written as $\mathbf{M}^n \triangleright^{\gamma}_{\mathcal{P}} \mathfrak{H}$. ○

We write

$$\mathcal{C} = \left\{c_z \colon z \in \wp_2(X)\right\}.$$

If a bijection γ is fixed (or evident), we write simply $\mathbf{M}^n \triangleright_{\mathcal{P}} \mathfrak{H}$.
In particular, if $\mathfrak{H} = \mathbf{G}_2(n)$, it is natural to put $\gamma \colon c_{\{i,j\}} \longmapsto \{i, j\}$. It is evident now that

$$\mathbf{M}^n \triangleright_{N_4} \mathbf{G}_2(n) \cong \mathbf{B}(n); \text{ moreover, } \mathbf{M}^n \triangleright_{K_n} \mathbf{G}_2(n) \cong \mathbf{G}_2(n+2).$$

From the definitions the following generalization of Proposition 7 follows

Theorem 2. *Let n be a natural number, $\mathcal{P}$ be a subset of $\wp_2(\{1, \dots, n\})$, and $\mathfrak{H}$ be any $\left(\binom{n}{2}_{n-2}, \binom{n}{3}_3\right)$-configuration. Then $\mathbf{M}^n \triangleright^{\gamma}_{\mathcal{P}} \mathfrak{H}$ is a $\left(\binom{n+2}{2}_n, \binom{n+2}{3}_3\right)$-configuration, for every bijection γ, as in Step D of Construction* 3.

Now we see (cf. Propositions 1 and 7, and Theorem 2) that the construction can be iterated: it makes sense to consider structures of the form

$$\mathbf{M}^n \triangleright^{\gamma_1}_{\mathcal{P}_1} (\mathbf{M}^{n-2} \triangleright^{\gamma_2}_{\mathcal{P}_2} \dots (\mathbf{M}^{n-2k} \triangleright^{\gamma_{k-1}}_{\mathcal{P}_{k-1}} \mathfrak{H})). \tag{2}$$

But note that now the choice of particular bijections γ_1, γ_2, γ_{k-1} may be essential (even if we fix, e.g. $\mathcal{P}_1 = \mathcal{P}_2 = \ldots \mathcal{P}_{k-1} = \emptyset$). Such a general approach seems too complex, and in the paper we shall restrict ourselves to some particular cases of the definition (2).

Still, one "standard" way of handling with structures of the form (2) seems natural, which (though simple) may be also of some interest from the point of view of combinatorics.

Representation 3. Let $X = \{1, \ldots, n\}$ and $\mathfrak{B} = \mathbf{M}^n \triangleright^{\gamma}_{\mathcal{P}} \mathfrak{H}$ be the configuration obtained from Construction 3. Clearly, γ^{-1} defines the structure of a partial linear space on $\mathcal{C}$ and thus, under the identification $c_z \mapsto z$, on the set $\wp_2(X)$ as well; let us write $\mathcal{L}$ for the obtained set of lines. Let $X' = X \cup \{n+1, n+2\}$. Consider the following families of blocks:

$$\begin{aligned}
\mathcal{L}_1 &= \Big\{\{\{n+1,n+2\},\{n+1,i\},\{n+2,i\}\} : i \in X\Big\},\\
\mathcal{L}_2 &= \Big\{\{\{i,j\},\{n+1,i\},\{n+2,j\}\}, : i,j \in X,\ i \neq j,\ \{i,j\} \notin \mathcal{P}\Big\},\\
\mathcal{L}_3 &= \Big\{\{\{i,j\},\{n+1,i\},\{n+1,j\}\},\{\{i,j\},\{n+2,i\},\{n+2,j\}\} : i,j \in X,\ \{i,j\} \in \mathcal{P}\Big\}.
\end{aligned}$$

Then under the identification $p = \{n+1, n+2\}$, $a_i = \{n+1, i\}$, $b_i = \{n+2, i\}$, and $c_z = z$ the structure $\langle \wp_2(X'), \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \rangle$ is isomorphic to $\mathfrak{B}$. ○

A representation of the structure $\mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ as a closure of the complete graph K_{n+1} is also available (cf. Representation 1 and Construction 1). We shall mention this representation below, however, it will not be used in the next parts of the paper.

Representation 4. Let $X = \{1, \ldots, n\}$ and $\mathfrak{B} = \mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(X)$. Next, let $\mathcal{C}' = \{c_z \in \mathcal{C} \colon n \in z\}$, $e = \{a_n, b_n\}$, and $\mathcal{K} = \mathcal{C}' \cup e$. Then any two points in $\mathcal{K}$ are collinear in $\mathfrak{B}$ i.e. $\mathcal{K}$ is the complete graph K_{n+1}. With every edge $q = \{x, y\}$ of $\mathcal{K}$ we can associate the unique third point point q^∞ on the line $\overline{x, y}$ of $\mathfrak{B}$; in this way all the points of $\mathfrak{B}$ are exhausted. The elements of $\mathcal{C}'$ can be identified with the numbers in $X' = X \setminus \{n\}$ under the map $i \longmapsto c_{\{i,n\}}$, let the subgraph $\mathcal{P}'$ of $\mathcal{K}$ be the image of $\mathcal{P} \curlywedge X'$ under this correspondence. The class of lines of $\mathfrak{B}$ is the union of the family of the sets $\{x, y, \{x, y\}^\infty\}$ with $x, y \in \mathcal{K}$, $x \neq y$, and the family of all the sets of the form $\{e_1^\infty, e_2^\infty, e_3^\infty\}$ where one of the following holds:

- e_1, e_2, e_3 are the sides of a triangle in $\mathcal{K}$ which misses e or has e as a side;
- e_1, e_2, e_3 are the sides of a triangle in $\mathcal{K}$ with (exactly) one vertex in e and the side opposite to this edge in $\mathcal{P}'$;
- e, e_1, e_2, e_3 are the sides of a quadrangle in $\mathcal{K}$ in which the side opposite to e does not belong to $\mathcal{P}'$. ○

3. Automorphisms. Let us try to establish the automorphism group of the structures of the form $\mathfrak{B} = \mathbf{M}^n \triangleright^{\gamma}_{\mathcal{P}} \mathfrak{H}$ defined by Construction 3 (in what follows the notation is taken from Construction 3 as well). Let $X = \{1, \dots, n\}$. The following three lemmas are immediate.

Lemma 1. *Let σ be the bijection of the points of $\mathfrak{B}$ defined by*

$$\begin{array}{lll} & \sigma(p) = p, & \\ (3) & \sigma(c_z) = c_z & \text{for every } z \in \wp_2(X), \\ & \sigma(a_i) = b_i, \sigma(b_i) = a_i & \text{for every } i \in X. \end{array}$$

Then σ is an involutory automorphism of $\mathfrak{B}$.

Lemma 2. *Let $\alpha \in S_X$. Assume that*

(i) *α is an automorphism of the graph $\langle X, \mathcal{P} \rangle$, and*

(ii) *$\alpha^{(2)}$ is (up to the bijection γ) an automorphism of $\mathfrak{H}$.*

Then the map F_α defined by

$$(4) \qquad F_\alpha(p) = p, \quad F_\alpha(a_i) = a_{\alpha(i)}, \quad F_\alpha(b_i) = b_{\alpha(i)}, \quad F_\alpha(c_{i,j}) = c_{\alpha(i),\alpha(j)}$$

for $i, j \in X$ is an automorphism of $\mathfrak{B}$, and $F_\alpha \circ \sigma = \sigma \circ F_\alpha$.

Under the isomorphism defined in Representation 3 the map σ given in Lemma 1 corresponds to $\beta^{(2)}$, where $\beta \in S_{X \cup \{n+1, n+2\}}$ is the transposition $(n+1, n+2)$. The map F_α of Lemma 2 corresponds to $\beta^{(2)}$, where $\beta \in S_{X \cup \{n+1, n+2\}}$ is the extension of α by the identity on $\{n+1, n+2\}$.

Lemma 3. *Let $\mathfrak{B} = \mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ and $f, g \in \mathrm{Aut}(\mathfrak{B})$ fix the point p. Then*

(i) *f leaves the set $\mathcal{C}$ invariant.*

(ii) *f determines a permutation $\alpha_f = \alpha \in S_X$ by the rule $f(L_i) = L_{\alpha(i)}$, and then $f(c_z) = c_{\alpha^{(2)}(z)}$.*

(iii) *Assume that $\alpha_f = \alpha_g$ and $f(a_i) = g(a_i)$ for some $i \in X$. Then $f = g$.*

Moreover, if $f \in \mathrm{Aut}(\mathfrak{B})$ leaves the set $\mathcal{C} \cup \{p\}$ invariant, then $f(p) = p$.

Proof. (i) and (ii) are easy to verify. To prove (iii) it suffices to note that for every $i, j \in X$, $j \neq i$, there is exactly one point $s_{i,j}$ on L_j collinear with a_i, and thus both f and g must map $s_{i,j}$ onto $s_{\alpha(i),\alpha(j)}$.

Finally we observe that in the substructure of $\mathfrak{B}$ determined by $\mathcal{X} = \mathcal{C} \cup \{p\}$ the point p is the only one isolated, and thus it must be fixed by f, provided $\mathcal{X}$ is preserved. □

As a consequence of Lemmas 1 and 2 we can infer, eg., that the groups $\mathrm{Aut}(\mathbf{G}_2(n+2)) \cong \mathrm{Aut}(\mathbf{M}^n \triangleright_{K_n} \mathbf{G}_2(n))$ and $\mathrm{Aut}(\mathbf{B}(n)) \cong \mathrm{Aut}(\mathbf{M}^n \triangleright_{N_n} \mathbf{G}_2(n))$ both contain $C_2 \oplus S_n$. Partly, it is a trivial result since we know that $\mathrm{Aut}(\mathbf{G}_2(n+2)) \cong S_{n+2}$ (comp. [14]). However, as we shall see in Proposition 10, $\mathbf{B}(n)$ has no other automorphisms.

As a convenient tool for distinguishing the types of points of $\mathfrak{B}$ we use the notion of *the neighborhood* $\mathcal{N}^+(q)$ and *the antineighborhood* $\mathcal{N}^-(q)$ of a point q. We write $\mathcal{N}^-(q)$ for the substructure of $\mathfrak{V}$ whose points are points of $\mathfrak{V}$ *not* collinear with q, and whose lines are at least two element sections of lines of $\mathfrak{V}$ with points in $\mathcal{N}^-(q)$. Similarly, $\mathcal{N}^+(q)$ is build from points of $\mathfrak{B}$ collinear with q. The following is just an easy though useful observation and thus we write it down explicitly.

Lemma 4. *Let q be a point of $\mathfrak{B}$, $\mathcal{N}^+ = \mathcal{N}^+(q)$, and $\mathcal{N}^- = \mathcal{N}^-(q)$. Then*

$$q = p : \begin{cases} \mathcal{N}^+ = a_i, b_i \colon i \in X, \\ \mathcal{N}^- = c_z \colon z \in \wp_2(X), \end{cases}$$

$$q = a_i : \begin{cases} \mathcal{N}^+ = b_i, p, \quad a_j \colon \{i,j\} \in \mathcal{P}, \quad b_j \colon \{i,j\} \notin \mathcal{P}, \quad c_z \colon i \in z \in \wp_2(X), \\ \mathcal{N}^- = a_j \colon \{i,j\} \notin \mathcal{P},\ j \neq i, \quad b_j \colon \{i,j\} \in \mathcal{P}, \quad c_z \colon i \notin z \in \wp_2(X), \end{cases}$$

$$q = c_z : \begin{cases} \mathcal{N}^+ = a_i, b_i \colon i \in z, \quad c_w \colon c_w \textit{ is collinear with } c_z \textit{ in } \mathfrak{B},\ w \in \wp_2(X), \\ \mathcal{N}^- = p, \quad a_i, b_i \colon i \in X \setminus z, \quad c_w \colon c_w \textit{ is not collinear in } \mathfrak{B} \textit{ with } c_z. \end{cases}$$

Lemma 5. *$\mathcal{N}^-(p)$ is isomorphic to $\mathfrak{H}$. Moreover, it is a subspace of $\mathfrak{B}$.*

Lemma 6. *Let $z \in \wp_2(X)$; set $X' = X \setminus z$. Then $\mathcal{N}^-(c_z)$ is contained in $\mathbf{M}^{n-2} \triangleright^{\gamma'}_{\mathcal{P}'} \mathfrak{H}'$, where $\mathcal{P}' = \mathcal{P} \curlywedge X'$, γ' is the restriction of γ to $\wp_2(X')$, and $\mathfrak{H}'$ is the suitable restriction of $\mathfrak{H}$.*

If (up to γ) points of $\mathfrak{H}$ noncollinear with c_z are exactly all the c_w with $z \cap w = \emptyset$, then $\mathcal{N}^-(c_z)$ is isomorphic to $\mathbf{M}^{n-2} \triangleright^{\gamma'}_{\mathcal{P}'} \mathfrak{H}'$.

Lemma 7. *Let $z \in \wp_2(X)$. The set $\mathcal{N}^-(c_z)$ yields a subspace of $\mathfrak{B}$ if and only if the following two conditions hold:*

(a) *for every $w \in \wp_2(X)$ the points c_w and c_z are collinear in $\mathfrak{H}$ if and only if $w \cap z \neq \emptyset$, and*
(b) *the set $\{c_w \colon w \in \wp_2(X \setminus z)\}$ yields a subspace of $\mathfrak{H}$.*

P r o o f. Two observations are sufficient:

(i) *Let $w \cap z = \emptyset$ and c_w, c_z be collinear.* Write $w = \{i, j\}$. Then either $\{a_i, c_w, a_j\}$ or $\{a_i, c_w, b_j\}$ is a line of $\mathfrak{B}$. Since $a_i, a_j, b_j \in \mathcal{N}^-(c_z)$ and $c_w \notin \mathcal{N}^-(c_z)$, *the set $\mathcal{N}^-(c_z)$ is not a subspace of $\mathfrak{B}$.*

(ii) *Let $w \cap z \neq \emptyset$ and c_w, c_z be not collinear.* Write $w = \{i, j\}$, where $i \in z$. Then, again, we consider the line through b_j and c_w (b_j and c_z are in $\mathcal{N}^-(c_z)$!). Its third point is a_i or b_i and they both are not in $\mathcal{N}^-(c_z)$ and thus $\mathcal{N}^-(c_z)$ *is not a subspace of $\mathfrak{B}$.*

The proof of the converse implication consists in direct verification. □

Lemma 8. *Let $i \in X$. The set $\mathcal{N}^-(a_i)$ yields a subspace of $\mathfrak{B}$ if and only if the following holds*

(a) *there are no $j_1, j_2 \in X$ such that $\mathcal{P} \curlywedge \{i, j_1, j_2\} \cong N_3$ or $\mathcal{P} \curlywedge \{i, j_1, j_2\} \cong L_3$, and*

(b) *the set $\big\{c_w \colon w \in \wp_2(X \setminus \{i\})\big\}$ yields a subspace of $\mathfrak{H}$.*

P r o o f. Clearly, we see that if $\mathcal{N}^-(a_i)$ a subspace then (b) follows from Lemma 4. To prove (a) we take arbitrary $j_1 \in X \setminus \{i\}$. Assume that $\{i, j_1\} \notin \mathcal{P}$ so, from Lemma 4 we have $a_{j_1} \in \mathcal{N}^-(a_i)$. For arbitrary $j_2 \in X \setminus \{i, j_1\}$ there is a line L of $\mathfrak{B}$ through a_{j_1} and $c_{\{j_1, j_2\}}$, and from Lemma 4, $c_{\{j_1, j_2\}} \in \mathcal{N}^-(a_i)$; let q be the third point on L. If $\{j_1, j_2\} \in \mathcal{P}$, then $q = a_{j_2}$ and to prove that $\mathcal{N}^-(a_i)$ is a subspace we need $\{i, j_2\} \notin \mathcal{P}$. Similarly, if $\{j_1, j_2\} \notin \mathcal{P}$, then $q = b_{j_2}$ and we need $\{i, j_2\} \in \mathcal{P}$. The case $\{i, j_1\} \in \mathcal{P}$ is considered analogously.

The converse implication is verified directly. □

In the particular case $\mathfrak{B} = \mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ as a direct consequence of Lemmas 5, 6, 7, and 8 we obtain a more explicit classification of antineighborhoods.

Proposition 8. *Let $\mathcal{P}$ be a graph on the set $X = \{1, \ldots, n\}$ and $\mathfrak{B} = \mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$.*

(i) *The set $\mathcal{N}^-(p)$ yields a subspace of $\mathfrak{B}$ isomorphic to $\mathbf{G}_2(X)$.*

(ii) *Let $z \in \wp_2(X)$. Then $\mathcal{N}^-(c_z)$ yields a subspace of $\mathfrak{B}$, which is isomorphic to $\mathbf{M}^{n-2} \triangleright_{\mathcal{P} \curlywedge (X \setminus z)} \mathbf{G}_2(X \setminus z)$.*

(iii) *Let $i \in X$. The set $\mathcal{N}^-(a_i)$ yields a subspace of $\mathfrak{B}$ if and only if* (a) *of Lemma* 8 *holds.*

Remark 2. Some other cases are also easy to determine. Let $\mathfrak{B} = \mathbf{M}^n \triangleright_{\mathcal{P}} \mathfrak{H}$ and $i \in X$.

(i) *If $\mathcal{P} = \emptyset$, then $\mathcal{N}^-(a_i)$ is not a subspace of $\mathfrak{B}$.*

(ii) *If* $\mathcal{P} = K_n$, *then* $\mathcal{N}^-(a_i)$ *is a subspace of* $\mathfrak{B}$ *if and only if* (b) *of Lemma* 8 *holds.*

Let us come back to the case $\mathfrak{B} = \mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$. We write

$\mathcal{Z}$ = the set of the points q such that $\mathcal{N}^-(q)$ is a subspace of $\mathfrak{B}$.

From Proposition 8, $\mathcal{C} \cup \{p\} \subseteq \mathcal{Z}$, and clearly, every automorphism of $\mathfrak{B}$ leaves the set $\mathcal{Z}$ invariant.

To classify all the structures $\mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ the following criterion is useful.

Let $X \neq \emptyset$ and $\mathcal{X} = \wp(\wp_2(X))$ be the family of all graphs defined on X. For every $x \in X$ we define the transformation μ_x of the family $\mathcal{X}$ by the formula

$$\begin{array}{rcccc} x \neq y, z & \Longrightarrow & \big(\{y,z\} \in \mu_x(\mathcal{P}) & \iff & \{y,z\} \in \mathcal{P}\big); \\ x \neq y & \Longrightarrow & \big(\{x,y\} \in \mu_x(\mathcal{P}) & \iff & \{x,y\} \notin \mathcal{P}\big). \end{array}$$

We write $\mu(\mathcal{P}) = \wp_2(X) \setminus \mathcal{P}$ for the boolean complementation of the graph $\mathcal{P} \in \mathcal{X}$. Note that if $x_1, x_2 \in X$ then $\mu_{x_1}\mu_{x_2} = \mu_{x_2}\mu_{x_1}$ and $\mu_{x_1}\mu = \mu\mu_{x_1}$.

Two graphs $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{X}$ are said to be equivalent iff there exists a sequence $x_1, \ldots, x_n \in X$ such that $\mu_{x_n} \ldots \mu_{x_1}(\mathcal{P}_1) = \mathcal{P}_2$; then we write $\mathcal{P}_1 \approx \mathcal{P}_2$. Clearly, $\mathcal{P}_1 \approx \mathcal{P}_2$ if and only if $\mu(\mathcal{P}_1) \approx \mu(\mathcal{P}_2)$.

Proposition 9. $\mathbb{M}^n \triangleright_{\mathcal{P}_1} \mathbf{G}_2(n) \cong \mathbb{M}^n \triangleright_{\mathcal{P}_2} \mathbf{G}_2(n)$ *whenever* $\mathcal{P}_1 \approx \mathcal{P}_2$.

Proof. It suffices to prove the claim for $\mathcal{P}_2 = \mu_m(\mathcal{P}_1)$, where $m \in X$. In this case we define an isomorphism F by the requirements: all points of $\mathbb{M}^n \triangleright_{\mathcal{P}_1} \mathbf{G}_2(n)$ remain unchanged except a_m, b_m, and these two are interchanged. □

It is relatively easy to determine the automorphism groups of the structures of the form $\mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ for some simple graphs $\mathcal{P}$. The following observation is essential here:

Lemma 9. *Let* $\mathcal{P}$ *be a graph defined on a set* X *with* $|X| = n$, $i_1, i_2, i_3 \in X$, *and* $L_{i_1}, L_{i_2}, L_{i_3}$ *be three lines of* $\mathfrak{B} = \mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ *through* p. *Then the* L_{i_j} $(j = 1, 2, 3)$ *determine in* $\mathfrak{B}$ *the subconfiguration with points* p, $a_{i_j}, b_{i_j}, c_{i_{j_1}, i_{j_2}}$, *isomorphic to* $\mathbb{M}^3 \triangleright_{\mathcal{P}'} \mathbf{G}_2(3)$, *where* $\mathcal{P}' = \mathcal{P} \curlywedge \{i_1, i_2, i_3\})$, *and thus isomorphic either to the Desargues configuration, when* $\mathcal{P}' \approx K_3$, *or to* $\mathbf{V}_3(3)$, *when* $\mathcal{P}' \approx N_3$. *Clearly, every* $f \in \mathrm{Aut}(\mathfrak{B})_p$ *preserves these two types of* 3*-subsets of* X.

Proposition 10. *Let* $n \geq 4$. *Then* $\mathrm{Aut}(\mathbb{B}(n)) \cong C_2 \oplus S_n$.

Proof. Let $X = \{1, \ldots, n\}$ and $\mathfrak{V} = \mathbb{B}(n)$. Take $i \in X$; from (i) of Remark 2 we find that $a_i, b_i \notin \mathcal{Z}$. From the above and Lemma 3 we get that every automorphism f of $\mathfrak{V}$ fixes p and it determines the permutation $\alpha = \alpha_f$ of the set X such that $f(L_i) = L_{\alpha(i)}$. Evidently, every $\alpha \in S_n$ yields the automorphism $\alpha^{(2)}$ of $\mathbf{G}_2(X)$, and $S_n \subset \mathrm{Aut}(\langle X, \emptyset\rangle)$. In view of the above and Lemmas 1, 2, and (iii) of Lemma 3, the group $\mathrm{Aut}(\mathfrak{V})$ consists exactly of the maps $F_\alpha \circ \sigma^\varepsilon$ $(\varepsilon = 0, 1)$, which proves our claim. $\square$

Proposition 11. *Let* $4 \leq n_1 + 2 \leq n$ *and* $\mathcal{P} = K^n_{n_1}$ *or* $\mathcal{P} = \mu(K^n_{n_1})$. *Set* $\mathfrak{B} = \mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$. *Then*

(i) $\mathrm{Aut}(\mathfrak{B}) \cong C_2 \oplus (S_{n_1} \oplus S_{n-n_1})$.

(ii) *If* $n - n_1 > 2$, *then* $\mathbb{M}^n \triangleright_{K^n_{n_1}} \mathbf{G}_2(n) \ncong \mathbb{M}^n \triangleright_{\mu(K^n_{n_1})} \mathbf{G}_2(n)$.

P r o o f. Let $X = \{1, \dots, n\}$. Set $X_1 = \{1, \dots, n_1\}$ and $X_2 = X \setminus X_1$, and let $K^n_{n_1} = \wp_2(X_1)$. Clearly, for every $\alpha_1 \in S_{X_1} = S_{n_1}$ and $\alpha_2 \in S_{X_2} = S_{n-n_1}$ the permutation $\alpha = \alpha_1 \cup \alpha_2$ of X is an automorphism of $K^n_{n_1}$ and of $\mu(K^n_{n_1})$. Consequently, from Lemma 2, we obtain the induced automorphism F_α of $\mathfrak{B}$.

Let $f \in \mathrm{Aut}(\mathfrak{B})_p$ and let $\alpha = \alpha_f$ be the induced permutation of X. Assume, first, that $\mathcal{P} = K^n_{n_1}$. Then the only triples $a \in \wp_3(X)$ such that $\mathcal{P} \curlywedge a \approx K_3$ (cf. Lemma 9) are those, which meet X_1 in at least 2 elements. What is more, if $w \in \wp_2(X)$ then $w \subset X_1$ is equivalent to $\mathcal{P} \curlywedge a \approx K_3$ for every $a \in \wp_3(X)$ such that $w \subset a$.

Consequently, there are $\binom{n_1}{2}(n - n_1)$ Desargues subconfigurations of $\mathfrak{B}$ spanned by lines through p. The remaining 3-subsets of X determine $\mathbf{V}_3(3)$ subconfigurations and thus $\mathfrak{B}$ contains $\binom{n}{3} - \binom{n_1}{2}(n - n_1)$ such subconfigurations. Note that if $n - n_1 > 2$ then $2\binom{n_1}{2}(n - n_1) < \binom{n}{3}$. Since replacing $\mathcal{P} = K^n_{n_1}$ by $\mathcal{P} = \mu(K^n_{n_1})$ results in interchanging Desargues subconfigurations with $\mathbf{V}_3(3)$-subconfigurations we conclude that for $n - n_1 > 2$ there is no isomorphism of $\mathbb{M}^n \triangleright_{K^n_{n_1}} \mathbf{G}_2(n)$ and $\mathbb{M}^n \triangleright_{\mu(K^n_{n_1})} \mathbf{G}_2(n)$ that preserves p.

Continuing, we note that α preserves X_1 and, consequently, α preserves X_2. Therefore, α can be written in the form $\alpha = \alpha_1 \cup \alpha_2$, where $\alpha_j \in S_{X_j}$ which, together with Lemmas 1 and 3 proves that $\mathrm{Aut}(\mathfrak{B})_p \cong C_2 \oplus (S_{n_1} \oplus S_{n-n_1})$.

To close the proof let us determine the set $\mathcal{Z}$. For any $j_1 \in X_2$ and $i \in X_1$ one can find $j_2 \in X_2$ with $j_1 \neq j_2$ and then $\mathcal{P} \curlywedge \{i, j_1, j_2\} \cong N_3$. Moreover, for every $j \in X_2$ and distinct $i_1, i_2 \in X_1$ we have $\mu(\mathcal{P}) \curlywedge \{i_1, i_2, j\} \cong L_3$. In view of Proposition 8, the set $\mathcal{Z}$ is the union $\mathcal{C} \cup \{p\}$; from Lemma 3 we obtain $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}(\mathfrak{B})_p$. $\square$

Corollary 1. *Let* $n \geq 4$ *and* $\mathcal{P} = L^n_2$ *or* $\mathcal{P} = \mu(L^n_2)$. *Set* $\mathfrak{B} = \mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$. *Then* $\mathrm{Aut}(\mathfrak{B}) \cong C_2 \oplus (C_2 \oplus S_{n-2})$.

Moreover, if $n > 4$, *then* $\mathbb{M}^n \triangleright_{L^n_2} \mathbf{G}_2(n) \ncong \mathbb{M}^n \triangleright_{\mu(L^n_2)} \mathbf{G}_2(n)$.

Proposition 12. *Let $n_1 + n_2 + 1 = n$ with $n_1, n_2 \geq 2$, and $\mathcal{P} = M^n_{n_1}$ or $\mathcal{P} = \mu(M^n_{n_1})$. Set $\mathfrak{B} = \mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$. Then*

(i) $\mathrm{Aut}(\mathfrak{B}) \cong C_2 \oplus \mathrm{Aut}(K_{n_1,n_2})$. *In particular, if $n_1 \neq n_2$, then* $\mathrm{Aut}(\mathfrak{B}) \cong C_2 \oplus (S_{n_1} \oplus S_{n_2})$.

(ii) $\mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n) \ncong \mathbf{M}^n \triangleright_{\mu(\mathcal{P})} \mathbf{G}_2(n)$.

Proof. Without loss of generality we can consider $X = \{1, \ldots, n\}$. Let $X_1 = \{i \colon i = 1, \ldots, n_1\}$, $X_2 = \{n_1 + i \colon i = 1, \ldots, n_2\}$. Let us take $\mathcal{P}_j = \big\{\{n, i\} \colon i \in X_j\big\}$ for $j = 1, 2$. It is seen that $\mu_n(\mathcal{P}_1) = \mathcal{P}_2$ so, without loss of generality we can assume that $n_1 \leq n_2$ and $\mathcal{P} = \mathcal{P}_1$.

Let $\alpha_j \in S_{X_j} = S_{n_j}$ for $j = 1, 2$, we take $\alpha = \alpha_1 \cup \alpha_2 \cup \{(n, n)\}$. Clearly, $\alpha \in \mathrm{Aut}(\mathcal{P})$ so, as a consequence of Lemma 2 we obtain an automorphism $F_\alpha \in \mathrm{Aut}(\mathfrak{B})$.

Next, assume that $n_1 = n_2$ and consider any bijection $\beta_0 \colon X_1 \longrightarrow X_2$ (e.g. defined by $\beta(i) = n_1 + i$ for $i \in X_1$); let the map $\beta \colon X \longrightarrow X$ be defined as $\beta = \beta_0 \cup \beta_0^{-1} \cup \{(n, n)\}$. Then we consider the map G_β defined as follows:

$$(5) \quad G_\beta(p) = p, \quad G_\beta(a_n) = a_n,\ G_\beta(b_n) = b_n, \quad G_\beta(c_{i,j}) = c_{\beta(i),\beta(j)}\ (i, j \in X), \\ G_\beta(a_i) = b_{\beta(i)} = G_\beta^{-1}(a_i),\ \text{ for } i \in X_1 \cup X_2.$$

Clearly, $\beta \notin \mathrm{Aut}(\mathcal{P})$, but $\beta \restriction (X_1 \cup X_2) \in \mathrm{Aut}(K_{n_1,n_2})$ and a straightforward computation gives that $G_\beta \in \mathrm{Aut}(\mathfrak{B})$. This proves that every automorphism of the graph K_{n_1,n_2} determines an automorphism of $\mathfrak{B}$.

Conversely, let us first determine the stabilizer of the point p in the group $\mathrm{Aut}(\mathfrak{B})$. Let $f \in \mathrm{Aut}(\mathfrak{B})_p$. From Lemma 3, f leaves the set $\mathcal{C}$ invariant and determines the permutation $\alpha = \alpha_f$ of the set of lines through p; clearly, α can be considered as a permutation in S_n.

Let $a \in \wp_3(X)$. Note that if $a \subset X_1$ or $a \subset X_2$, or $a = \{n, i_1, i_2\}$ where $i_1, i_2 \in X_1$ or $i_1, i_2 \in X_2$, then $\mathcal{P} \curlywedge a \approx N_3$. But if $a = \{n, i_1, i_2\}$, where $i_1 \in X_1$, $i_2 \in X_2$, then $\mathcal{P} \curlywedge a \approx K_3$. Consequently, α leaves the family $\big\{\{n, i_1, i_2\} \colon i_1 \in X_1,\ i_2 \in X_2\big\}$ invariant so, $\alpha(n) = n$. With Lemma 1 we can assume that $f(a_n) = a_n$ and $f(b_n) = b_n$. Now, we see that either α preserves X_1 and X_2 or interchanges these two sets (which can happen if $n_1 = n_2$ only). In the first case f can be identified with a pair of permutations $\alpha_j \in S_{X_j}$, $j = 1, 2$. Thus $\mathrm{Aut}(\mathfrak{B})_p \cong C_2 \oplus \mathrm{Aut}(K_{n_1,n_2})$.

To close this part of proof we determine the set $\mathcal{Z}$. For every $i_1, i_2 \in X_1$ such that $i_1 \neq i_2$ and every $j \in X_2$ the set $\{i_1, i_2, j\}$ is an empty subgraph of $\mathcal{P}$, and $\{n, i_1, i_2\}$ is a L_3-subgraph of $\mathcal{P}$. From Proposition 8 we infer that $\mathcal{Z} = \mathcal{C} \cup \{p\}$ which, in view of Lemma 3, proves (i) for $\mathcal{P} = \mathcal{P}_1$.

Let us adopt $\mathcal{P} = \mu(\mathcal{P}_1)$. It is evident that G_β defined by (5) is an automorphism of $\mathfrak{B}$. Let $f \in \mathrm{Aut}(\mathfrak{B})_p$ and let $\alpha = \alpha_f$ be the induced permutation of X.

Let $a \in \wp_3(X_1) \cup \wp_3(X_2)$ or $a = \{n\} \cup w$ with $w \in \wp_2(X_1) \cup \wp_2(X_2)$; clearly we have $\mathcal{P} \curlywedge a \approx K_3$. The only triples a such that $\mathcal{P} \curlywedge a \approx N_3$ have form $\{n, i, j\}$ with $i \in X_1$ and $j \in X_2$. Their intersection is the point n, and thus $\alpha(n) = n$. The rest of reasoning goes as in the case $\mathcal{P} = \mathcal{P}_1$ ending with the (required) form of $\mathrm{Aut}(\mathfrak{B})_p$.

Finally, we note that for every $i \in X_1$ and $j \in X_2$ the set $\{1, i, j\}$ is a L_3-subgraph of $\mathcal{P}$ and thus, again from Proposition 8 we get $\mathcal{Z} = \mathcal{C} \cup \{p\}$, which, together with Lemma 3 yields $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}(\mathfrak{B})_p$.

Note that the number of $a \in \wp_3(X)$ such that $\mathcal{P}_1 \curlywedge a \approx K_3$ is $n_1 n_2$, and the number of $a \in \wp_3(X)$ such that $\mu(\mathcal{P}_1) \curlywedge a \approx K_3$ is $\binom{n}{3} - n_1 n_2$. If there were an isomorphism F of $\mathbf{M}^n \triangleright_{\mathcal{P}_1} \mathbf{G}_2(n)$ and $\mathbf{M}^n \triangleright_{\mu(\mathcal{P}_1)} \mathbf{G}_2(n)$, then $F(p) = p$ and we would have $2n_1 n_2 = \binom{n}{3}$. Clearly, $\binom{n}{2} > 2n_1 n_2$ and $\frac{n_1+n_2-1}{3} \geq 1$ so, $\binom{n}{3} > 2n_1 n_2$ and a contradiction arises. This proves (ii). $\square$

With similar techniques we can prove

Proposition 13. *Let $n > 4$, $n \neq 6, 8$, and $\mathcal{P} = C_n$ or $\mathcal{P} = \mu(C_n)$. Then* $\mathrm{Aut}(\mathbf{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n) \cong C_2 \oplus D_n$.

Proof. Let $X = \{1, \ldots, n\}$ and edges of C_n join consecutive points $(\mod n)$. Evidently, every $\alpha \in D_n$ is an automorphism of C_n, and of $\mu(C_n)$ as well; as a consequence of Lemma 2, α determines an automorphism of $\mathfrak{B} = \mathrm{Aut}(\mathbf{M}^n \triangleright_{C_n} \mathbf{G}_2(n)$.

Now, let $\mathcal{P} = C_n$. For every fixed pair $\{i_1, i_2\} \in \wp_2(X)$ with $|i_2 - i_1| \leq \frac{n}{2}$ we determine the number $\nu = \nu_{i_1, i_2}$ of triples $a = \{i_1, i_2, j\}$ such that $\mathcal{P} \curlywedge a \approx K_3$:

$i_2 = i_1 + 1$: $\nu = (n-4)$, triples have form $\{i_1, i_1+1, j\}$ with $j \neq i_1, i_1+1, i_1+2, i_1-1)$;

$i_2 = i_1 + 2$: $\nu = 2$, triples are $\{i_1, i_1+2, j\}$ with $j = i_1+3, i_1-1$;

$i_2 = i_1 + m$ $(m > 2)$: $\nu = 4$, corresponding triples are $\{i_1, i_1+m, j\}$, where $j = i_1+m+1, i_1+m-1, i_1+1, i_1-1$.

Let $f \in \mathrm{Aut}(\mathfrak{B})_p$ and $\alpha = \alpha_f$ be the induced permutation of X (cf. Lemma 3). In view of the the above, α preserves the distance 1 between points of X and thus it is an element of D_n. With (iii) of Lemma 3 we conclude that either $f = F_\alpha$, or $f = \sigma \circ F_\alpha$. We end the proof with the observation that for every $i \in X$ the set

$\{i-1, i, i+1\}$ (taken $\mod n$) yields a L_3-subgraph of $\mathcal{P}$. From Proposition 8 we infer that the set $\mathcal{Z}$ is the union of $\mathcal{C}$ and the point p so, $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}(\mathfrak{B})_p$.

If $\mathcal{P} = \mu(C_n)$, then the numbers $\nu(i_1, i_2)$ established above give the number of triples $\{i_1, i_2, j\}$ which yield $\mathbf{V}_3(3)$ subconfigurations of $\mathfrak{B}$. The rest of reasoning runs analogously. We only note now that $\mu(C_n) \curlywedge \{i-1, i, i+2\} \cong L_3$. $\square$

Proposition 14. *Let $n > 8$ and $\mathcal{P} = L_n$ or $\mathcal{P} = \mu(L_n)$. Then*

$$\mathrm{Aut}(\mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)) \cong C_2 \oplus C_2.$$

P r o o f. Let $X = \{1, \ldots, n\}$ and edges of L_n join consecutive points $\{i, i+1\}$, $i = 1, \ldots, n-1$. Clearly, the permutation α_0 of X given by $\alpha_0(i) = (n+1) - i$ is an involutory automorphism of L_n (and thus of $\mu(L_n)$ as well), therefore (cf. Lemma 2) it determines an automorphism F_{α_0} of $\mathfrak{B} = \mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$, where $\mathcal{P} = L_n$, or $\mathcal{P} = \mu(L_n)$.

Let $\mathcal{P} = L_n$. Similarly as in the proof of Proposition 13 for $z = \{i_1, i_2\} \in \wp_2(X)$ we determine the number $\nu = \nu_{i_i, i_2}$ of indices $j \in X$ such that $\mathcal{P} \curlywedge \{i_1, i_2, j\} \approx K_3$. Here are the corresponding values:

$z = \{1, n\}$**:** $\nu = 2$ $(j = 2, n-1)$;
$z = \{1, 2\}$**:** $\nu = n-3$ $(j \neq 1, 2, 3)$;
$z = \{1, 3\}$**:** $\nu = 1$ $(j = 4)$;
$z = \{1, i_2\}$ **(**$3 < i_2 < n$**):** $\nu = 3$ $(j = i_2 - 1, i_2 + 1, 2)$;
$z = \{i_1, i_1 + 1\}$ **(**$1 < i_1 < n-1$**):** $\nu = n-4$ $(j \neq i_1, i_1+1, i_1+2, i_1-1)$;
$z = \{i_1, i_1 + 2\}$ **(**$1 < i_1 < n-2$**):** $\nu = 2$ $(j = i_1 - 1, i_1 + 2)$;
$z = \{i_1, i_1 + m\}$ **(**$1 < i_1 < n-m$**,** $m > 2$**):** $\nu = 4$ $(j = i_1 - 1, i_1 + 1, i_1 + m - 1, i_1 + m + 1)$.

Let $f \in \mathrm{Aut}(\mathfrak{B})_p$ and let $\alpha = \alpha_f$ be the permutation of X determined by f. In view of the above, under assumptions of our theorem, α preserves the families $\big\{\{1,3\}, \{n, n-2\}\big\}$ and $\big\{\{1,2\}, \{n, n-1\}\big\}$, and thus it preserves $\{1, n\}$ as well. Without loss of generality (composing f with F_{α_0}, if necessary) we can assume that $\alpha(1) = 1$ and $\alpha(n) = n$ and then we obtain $\alpha(2) = 2$, $\alpha(3) = 3$. Considering $\nu_{2,4} = \nu_{2,\alpha(4)}$ we get $\alpha(4) = 4$ and, inductively, we come to $\alpha = \mathrm{id}$. Finally, we examine the set $\mathcal{Z}$. It is seen that every element of X is in one of the sets $\{i, i+1, i+2\}$ with $i = 1, \ldots, n-2$, and every such a set is a L_3-subgraph of $\mathcal{P}$. From Proposition 8 we get that $\mathcal{Z} = \mathcal{C} \cup \{p\}$ so (cf. Lemma 3), every automorphism of $\mathfrak{B}$ fixes p. This proves the statement in the first case.

If $\mathcal{P} = \mu(L_n)$, we search for $\mathbf{V}_3(3)$ subconfigurations of $\mathfrak{B}$; the rest of reasoning determining $\mathrm{Aut}(\mathfrak{B})_p$ remains unchanged. Then we observe that every element of X is in one of the sets $\{1, i, i+1\}$ $(i = 2, \dots, n-2)$, $\{2, n, n-1\}$ which are L_3-subgraphs of $\mathcal{P}$. With standard arguments we close up with $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}(\mathfrak{B})_p$. $\square$

4. Classification. Let us start the section by recalling the following results.

Proposition 15 ([11], [14]). $\mathrm{Aut}(\mathbf{G}_2(n+2)) \cong S_{n+2}$ *and* $\mathrm{Aut}(\mathbf{V}_n(3)) \cong S_3$ *for* $n > 3$.

As a consequence of this and of Proposition 10 we infer immediately

Theorem 3. *Let* $n > 3$. *The following three* $\left(\binom{n+2}{2}_n, \binom{n+2}{3}_3\right)$*-configurations:* $\mathbf{B}(n) = \mathbf{M}^n \triangleright_{N_n} \mathbf{G}_2(n)$, $\mathbf{G}_2(n+2) = \mathbf{M}^n \triangleright_{K_n} \mathbf{G}_2(n)$, *and* $\mathbf{V}_n(3)$ *are pairwise nonisomorphic.*

It is trivial that $N_3 \approx L_3$ and $K_3 \approx L_2^3$. A carefull (though tedious) analysis of all graphs on 4 vertices shows that each of them is equivalent to one of the following three:

$\mathcal{P} = K_4$,
$\mathcal{P} = N_4$, and
$\mathcal{P} = L_4 := \big\{\{1,2\}, \{2,3\}\{3,4\}\big\}$, equivalent to $L_2^4 = \big\{\{1,4\}\big\}$.

Proposition 16. $\mathrm{Aut}(\mathbf{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)) \cong C_2 \oplus (C_2 \oplus C_2)$.

Proof. It suffices to recall that $L_4 \approx L_2^4$ and use Corollary 1 and Proposition 9. $\square$

From the classification of graphs on 4 vertices and Propositions 9, 15, and 16 we conclude with the classification of all $(15_4, 20_3)$-configurations of the form $\mathbf{M}^4 \triangleright_{\mathcal{P}} \mathbf{G}_2(4)$:

Theorem 4. *The following four* $(15_4, 20_3)$*-configurations:* $\mathbf{B}(4)$, $\mathbf{G}_2(6)$, $\mathbf{V}_4(3)$, *and* $\mathbf{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$ *are pairwise nonisomorphic.*

Let $\mathfrak{M} = \mathbf{M}^4 \triangleright_{\mathcal{P}} \mathbf{G}_2(4)$ *for some graph* $\mathcal{P}$ *on* 4 *vertices. Then either* $\mathfrak{M} \cong \mathbf{G}_2(6)$ *or* $\mathfrak{M} \cong \mathbf{B}(4)$, *or* $\mathfrak{M} \cong \mathbf{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$.

Analyzing all the possible graphs on 5 vertices we come to the conclusion, that every of them is equivalent to one of the following

$\mathcal{P} = K_5$;

$\mathcal{P} = N_5$;

$\mathcal{P} = C_5$, equivalent to a triangle K_3 with two extra edges added to two of its vertices;

$\mathcal{P} = L^5_2$, equivalent to C_4 with one edge added to one of its vertex;

$\mathcal{P} = \mu(L^5_2)$, equivalent to a triangle K_3 with a path L_3 connected to one of its vertices;

$\mathcal{P} = L^5_3$ (i.e. M^5_2), equivalent to L_4 with one edge added to an intermediate vertex (i.e. a M_3 with one edge added to the degree 1 point of M_3);

$\mathcal{P} = \mu(L^5_3)$, equivalent to L_5.

Now we are in a position to determine the automorphism group $\mathrm{Aut}(\mathbb{W}^5 \rhd_{\mathcal{P}} \mathbf{G}_2(5))$ for arbitrary graph $\mathcal{P}$ on 5 vertices. Recall that $\mathrm{Aut}(\mathbb{W}^5 \rhd_{K_5} \mathbf{G}_2(5)) \cong S_7$ and $\mathrm{Aut}(\mathbb{W}^5 \rhd_{N_5} \mathbf{G}_2(5)) \cong C_2 \oplus S_5$.

Taking into account the fact that $K_{2,2} \cong C_4$ and $\mathrm{Aut}(C_4) = D_4$ we obtain immediately

Corollary 2.

$$\mathrm{Aut}(\mathbb{W}^5 \rhd_{L^5_2} \mathbf{G}_2(5)) \cong C_2 \oplus (C_2 \oplus S_3) \cong \mathrm{Aut}(\mathbb{W}^5 \rhd_{\mu(L^5_2)} \mathbf{G}_2(5));$$
$$\mathrm{Aut}(\mathbb{W}^5 \rhd_{L^5_3} \mathbf{G}_2(5)) \cong C_2 \oplus D_4 \cong \mathrm{Aut}(\mathbb{W}^5 \rhd_{\mu(L^5_3)} \mathbf{G}_2(5));$$
$$\mathrm{Aut}(\mathbb{W}^5 \rhd_{C_5} \mathbf{G}_2(5)) \cong C_2 \oplus D_5.$$

The three isomorphisms follow from Corollary 1, Proposition 12, and Proposition 13, respectively.

As a direct consequence of the above and of Propositions 10 and 15 we obtain

Theorem 5. *Let $\mathfrak{M} = \mathbb{W}^5 \rhd_{\mathcal{P}} \mathbf{G}_2(5)$ for some graph $\mathcal{P}$ on 5 vertices. Then $\mathfrak{M}$ is isomorphic to (exactly) one from the following seven configurations: $\mathbf{G}_2(7)$, $\mathbb{B}(5)$, $\mathbb{W}^5 \rhd_{L^5_2} \mathbf{G}_2(5)$, $\mathbb{W}^5 \rhd_{\mu(L^5_2)} \mathbf{G}_2(5)$, $\mathbb{W}^5 \rhd_{L^5_3} \mathbf{G}_2(5)$, $\mathbb{W}^5 \rhd_{\mu(L^5_3)} \mathbf{G}_2(5)$, $\mathbb{W}^5 \rhd_{C_5} \mathbf{G}_2(5)$.*

Let us close this section with a more general characterization theorem.

Theorem 6. *The following conditions are equivalent for every $n \geq 3$:*

(i) $\mathbb{W}^n \rhd_{\mathcal{P}} \mathbf{G}_2(n) \cong \mathbf{G}_2(n+2)$,

(ii) $\mathcal{P} \approx K_n$,

(iii) $\mathcal{C} \cup \{p\} \subsetneq \mathcal{Z}$.

P r o o f. The implication (ii) $\Longrightarrow$ (i) follows directly from Proposition 9 and the implication (i) $\Longrightarrow$ (iii) is evident.

Let us denote $X = \{1, \dots, n\}$. If (iii) holds then $a_i \in \mathcal{Z}$ for some $i \in X$. We set $X^+ := \big\{j \in X \colon \{i, j\} \in \mathcal{P}\big\} \cup \big\{i\big\}$ and $X^- := \big\{j \in X \colon \{i, j\} \notin \mathcal{P}, j \neq i\big\}$. In view of Proposition 8, (a) of Lemma 8 holds, which implies that $\wp_2(X^+) \subset \mathcal{P}$, $\wp_2(X^-) \subset \mathcal{P}$, and $\{j_1, j_2\} \notin \mathcal{P}$ for $j_1 \in X^+$, $j_2 \in X^-$. If $X^- = \emptyset$ or $X^+ = \emptyset$, then $\mathcal{P}$ is the complete graph. Assume that both X^+ and X^- are nonempty, then $\mathcal{P}$ is the disjoint union of two complete graphs. It is seen that the composition of all the μ_x with $x \in X^-$ transforms $\mathcal{P}$ onto $\wp_2(X)$. Consequently, (ii) holds. $\square$

Theorem 6 has various interesting consequences. Let us quote one:

Corollary 3. *Let* $\mathfrak{B} = \mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ *and* $n > 2$. *Then either* $\mathrm{Aut}(\mathfrak{B}) = S_{n+2}$ *is transitive on the points of* $\mathfrak{B}$, *or* $\mathrm{Aut}(\mathfrak{B}) = \mathrm{Aut}(\mathfrak{B})_p$ *is a subgroup of* $C_2 \oplus S_n$.

5. Final remarks. At the very end we shall characterize automorphism groups of the structures of the form (2) in one of the most regular cases. The result solves only a particular case, but it indicates the way in which more complex cases can be handled.

Proposition 17. *Let*

$$\mathfrak{M} = \mathbb{M}^{m+2k-2} \triangleright^{\gamma_k}_{N_{m+2k-2}} \dots (\mathbb{M}^{m+2} \triangleright^{\gamma_2}_{N_{m+2}} (\mathbb{M}^m \triangleright^{\gamma_1}_{N_m} \mathbf{G}_2(m))),$$

where the bijections γ_j $(j \leq k)$ *are defined in accordance with Representation* 3. *Then* $\mathrm{Aut}(\mathfrak{M}) \cong C_2^k \oplus S_m$.

P r o o f. Let us write $Y = \{1, \dots, m\} =: Y_0$, $p^j = \{m + 2j, m + 2j - 1\}$ and $Y_j = Y_{j-1} \cup p^j$ for $j = 1, \dots, k$. Next, we define inductively $\mathfrak{M}_0 = \mathbf{G}_2(Y)$, $\mathfrak{M}_j = \mathbb{M}^{m+2(j-1)} \triangleright^{\gamma_j}_{N_{m+2(j-1)}} \mathfrak{M}_{j-1}$; clearly, $\mathfrak{M} = \mathfrak{M}_k$. Then, in accordance with Representation 3, the structure $\mathfrak{M}_j$ is defined on $\wp_2(Y_j)$. Let us write $X := Y_k$.

Let $\mathcal{Z}$ be the set of points w of $\mathfrak{M}$ such that $\mathcal{N}^-(w)$ is a subspace of $\mathfrak{M}$. From (i) of Remark 2 and Lemma 7 we find inductively that if $w \in \wp_2(X)$ then $w \in \mathcal{Z}$ iff $w = p^j$ for some j or $w \in \wp_2(Y)$ (p^j is the "centre" of $\mathfrak{M}_j$, and an "intersection point" of $\mathfrak{M}_{j'}$ for $j < j'$, cf. Representation 3 and Construction 3). Moreover, the p^j are pairwise noncollinear. One can see that p^k is the only one from among the p^j such that any two lines through it yield a Veblen figure. Next,

p^{k-1} is the only one such that in $\mathcal{N}^-(p^k) \cong \mathfrak{M}_{k-1}$ any two lines through it yield a Veblen figure, and so on. Therefore an arbitrary automorphism F of $\mathfrak{M}$ must preserve each one of the points p^j. Moreover, F must preserve the set $\wp_2(Y)$, as only the p^j are isolated in the set $\mathcal{Z}$. Finally, F is determined by a permutation α of X which preserves the sets p^j and the set Y (use Lemmas 2 and 1). $\square$

One particular case seems to be especially interesting, though its proof consists in simple direct computation.

Proposition 18. *Let* $X_0 = \{1, \ldots, n-2\}$, $q = \{n-1, n\}$, $X = X_0 \cup q$, $p = \{n+1, n+2\}$, *and* $Y = X \cup p$. *In accordance with Representation* 3 *we define the structure* $\mathbb{B}(n-2)$ *on the set* $\wp_2(X)$, *with "centre"* $= q$. *Next, we consider the structure* $\mathbb{M}^n \triangleright_{K_n} \mathbb{B}(n-2)$ *and represent it on the set* $\wp_2(Y)$, *with "centre"* $= p$. *Finally, we take* $\mathcal{P} = \wp_2(X) \setminus \wp_2(X_0)$ *and represent* $\mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$ *on the set* $\wp_2(Y)$, *with "centre"* $= p$. *Let* $\alpha = (n-1, n+1)(n, n+2) \in S_Y$. *Then* $\alpha^{(n+2)}$ *is an isomorphism of* $\mathbb{M}^n \triangleright_{K_n} \mathbb{B}(n-2)$ *and* $\mathbb{M}^n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$.

Proposition 18 shows that the representation of a structure $\mathfrak{B}$ in the form $\mathbb{M}^n \triangleright_{\mathcal{P}} \mathfrak{H}$ does not determine neither $\mathcal{P}$ nor $\mathfrak{H}$ (consider $n > 2$).

Example 2. Slightly extending our definitions we can introduce also $\mathbb{B}(2) = \mathbb{M}^2 \triangleright_{N_2} \mathsf{T}$, where T is a single point, and then $\mathbb{B}(2)$ is simply the Veblen configuration defined on $\wp_2(\{1, \ldots, 4\})$ with lines (cf. Representation 3): $\wp_2(\{1,3,4\})$, $\wp_2(\{2,3,4\})$, $\big\{\{1,2\},\{1,3\},\{2,4\}\big\}$, $\big\{\{1,2\},\{1,4\},\{2,3\}\big\}$.

First, let us consider the structure $\mathbb{M}^4 \triangleright_{K_4} \mathbb{B}(2)$. From Proposition 18 we get that $\mathbb{M}^4 \triangleright_{K_4} \mathbb{B}(2)$ and $\mathbb{M}^4 \triangleright_{\mathcal{P}} \mathbf{G}_2(4)$ are isomorphic, where $\mathcal{P} = \wp_2(X) \setminus \{1,2\} \approx L_4$. In this case the isomorphism defined in Proposition 18 corresponds to the following relabelling F of the points of $\mathbb{M}^4 \triangleright_{K_4} \mathbb{B}(2)$:

$x = F(y)$	a_1	b_1	a_2	b_2	a_3	b_3	b_4	p	$c_{\{1,2\}}$
$y = F(x)$	$c_{\{1,3\}}$	$c_{\{1,4\}}$	$c_{\{2,3\}}$	$c_{\{2,4\}}$	a_3	a_4	b_4	$c_{\{3,4\}}$	$c_{\{1,2\}}$

Next, we consider $\mathfrak{B} = \mathbb{M}^4 \triangleright_{N_4} \mathbb{B}(2)$ represented on the set $\wp_2(\{1, \ldots, 6\})$ in accordance with Representation 3. From Remark 2 and Lemma 7 we get that $\mathcal{Z} = \big\{\{1,2\},\{3,4\},\{5,6\}\big\}$. Let $f \in \mathrm{Aut}(\mathfrak{B})$. The unique $q \in \mathcal{Z}$ such that any two lines through q yield in $\mathfrak{B}$ a Veblen figure is $q = \{5,6\}$ so, f fixes q and, consequently, f determines the permutation $\alpha = \alpha_f$ of $\{1,2,3,4\}$ (cf. Lemma 3). Direct verification shows that $\alpha^{(2)} \in \mathrm{Aut}(\mathbb{B}(2))$ if and only if α leaves $\{3,4\}$ invariant. This proves that $\mathrm{Aut}(\mathfrak{B}) = S_{\{1,2\}} \oplus S_{\{3,4\}} \oplus S_{\{5,6\}} \cong C_2^3$.

However, the two structures $\mathbb{M}^4 \triangleright_{N_4} \mathbb{B}(2)$ and $\mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$ are not isomorphic, because their $\mathcal{Z}$-sets have different cardinality: the first has 3, and the second has 7 elements. $\bigcirc$

Example 2 justifies that the choice of γ in Construction 3 may be essential: there are γ_1, γ_2 such that $\mathbb{W}^4 \rhd^{\gamma_1}_{N_4} \mathbf{G}_2(4) \not\cong \mathbb{W}^4 \rhd^{\gamma_2}_{N_4} \mathbf{G}_2(4)$.

In view of Theorem 4, Example 2 shows also that in the paper we have not exhausted all $(15_4, 20_3)$-configurations which can be presented in the form (2). We have not exhausted also all $(21_5, 35_3)$-configurations of this form (e.g. the series $\mathbb{W}^5 \rhd_{\mathcal{P}} \mathbb{B}(3)$, where $\mathcal{P}$ is a graph on 5 vertices was only mentioned). Another problem which was left is to determine how our configurations can be completed to Steiner triple systems. All these questions are addressed in some future papers.

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