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ON THE RESIDUUM OF CONCAVE UNIVALENT FUNCTIONS

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Abstract. Let D denote the open unit disc and $f : D \to \overline{\mathbb{C}}$ be meromorphic and injective in D. We further assume that f has a simple pole at the point $p \in (0,1)$ and is normalized by $f(0) = 0$ and $f'(0) = 1$. In particular, we are concerned with f that map D onto a domain whose complement with respect to $\overline{\mathbb{C}}$ is convex. Because of the shape of $f(D)$ these functions will be called concave univalent functions with pole p and the family of these functions is denoted by $Co(p)$.
We determine for fixed $p \in (0,1)$ the set of variability of the residuum of $f, f \in Co(p)$.

Let D denote the open unit disc and $f : D \to \overline{\mathbb{C}}$ be meromorphic and injective in D. We further assume that f has a simple pole at the point $p \in (0,1)$ and is normalized by $f(0) = 0$ and $f'(0) = 1$. In the paper [8], S. M. Zemyan denoted this class by S_p. He determined the exact set of variability of

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the residuum of the functions $f \in S_p$ at the point p for fixed p. The union of these sets is the whole plane $\mathbb{C}$ punctured in the origin. In the present paper we consider similar questions for a subclass $Co(p)$ of S_p, which is called class of concave univalent functions and defined as follows.

We say that a function $f : D \to \overline{\mathbb{C}}$ belongs to the family $Co(p)$ if and only if:

(i) f is meromorphic in D and has a simple pole in the point $p \in (0,1)$.

(ii) $f(0) = 0$ and $f'(0) = 1$.

(iii) f maps D conformally onto a set whose complement with respect to $\overline{\mathbb{C}}$ is convex.

Concerning the history of this class, we refer to [1], [3], [4], [5], [6] and [7]. In the extremal problems considered in these references it occurs very often that extremal problems in $Co(p)$ have as extremal functions the conformal maps of D onto the extended plane $\overline{\mathbb{C}}$ slit in a part of a straight line. We shall prove that the same is the case for the set of variability of the residuum of the functions $f \in Co(p)$ at the point p. This is the content of the following theorem.

Theorem. *Let $p \in (0,1)$. For $a \in \mathbb{C}$ there exists a function $f \in Co(p)$ such that $a = \operatorname{res}(f(z), z = p)$ if and only if*

$$\left| a + \frac{p^2}{1-p^4} \right| \le \frac{p^4}{1-p^4}. \tag{1}$$

Let $\theta \in [0, 2\pi)$. A function $f \in Co(p)$ has the residuum

$$a = -\frac{p^2}{1-p^4} + \left(\frac{p^4}{1-p^4} \right) e^{i\theta} \tag{2}$$

if and only if

$$f(z) = f_\theta(z) = \frac{z - \frac{p}{1+p^2}\left(1 + e^{i\theta}\right) z^2}{\left(1 - \frac{z}{p}\right)(1 - zp)}. \tag{3}$$

P r o o f. We first prove that any point of the disc described by (1) occurs as the residuum of a function $f \in Co(p)$. To that end we use the following characterization of the class $Co(p)$ proved in [7]:

A function f belongs to the class $Co(p)$ if and only if $f(0) = 0$, $f'(0) = 1$, and there exists a function ω holomorphic in D such that $\omega(D) \subset \overline{D}$ and

$$\frac{d}{dz} \log \left(f'(z) \left(1 - \frac{z}{p}\right)^2 (1 - zp)^2 \right) = \frac{-\frac{2p}{1+p^2} - \left(2z - \frac{2p}{1+p^2}\right)\omega(z)}{1 - \frac{2p}{1+p^2} z - \left(z^2 - \frac{2p}{1+p^2} z\right)\omega(z)} \tag{4}$$

for $z \in D$.

In the special case $\omega \equiv c$, $c \in \overline{D}$, it is very easy to integrate the differential equation (4) to get

$$f'(z) = \frac{1 - \frac{2p}{1+p^2} z - \left(z^2 - \frac{2p}{1+p^2} z\right) c}{\left(1 - \frac{z}{p}\right)^2 (1 - zp)^2}.$$

Obviously, this function f has the residuum

$$-\frac{p^2}{1 - p^4} - \left(\frac{p^4}{1 - p^4}\right) c.$$

To prove the other part of the first assertion, we use Theorem 4 of [4]:

If $f \in Co(p)$ and $z \in D \setminus \{0, p\}$, then

$$\left| \frac{1}{f(z)} - \frac{1}{z} + \frac{1 + p^2}{p} \right| \leq 1. \tag{5}$$

In fact, the inequality (5) was proved by J. Miller for another class of functions, which later on was seen to be equal to $Co(p)$ (see [3] and [5]).

It is evident that the function

$$w(z) = \frac{1}{f(z)} - \frac{1}{z} + \frac{1 + p^2}{p} \tag{6}$$

has under our circumstances a completion holomorphic in the unit disc that satisfies $w(p) = p$. Since $|w(z)| \leq 1$ for $z \in D$, we get as a consequence of the Schwarz Lemma (see for instances [2], p. 18), that

$$|w'(p)| \leq \frac{1 - |w(p)|^2}{1 - p^2} = 1. \tag{7}$$

In (7) equality is attained if and only if w is a holomorphic automorphism of D with fixed point p. The evaluation of $w'(p)$ using (6) yields

$$w'(p) = \frac{1}{a} + \frac{1}{p^2},$$

where $a = \operatorname{res}(f(z), z = p)$. A little computation using this identity shows that (1) is equivalent to (7).

According to the above, equality in (1) can be attained if and only if there exists $\varphi \in [0, 2\pi)$ such that

$$w(z) = \frac{p + e^{i\varphi}\frac{z-p}{1-zp}}{1 + pe^{i\varphi}\frac{z-p}{1-zp}}, \quad z \in D. \tag{8}$$

By a calculation of f from (6) and (8) we get (3) with

$$e^{i\theta} = \frac{p^2 - e^{i\varphi}}{1 - p^2 e^{i\varphi}}.$$

This completes the proof of the Theorem. □

We want to add two remarks.

Remark 1. From the inequality (1) it is immediately clear that the functions $f \in Co(p)$ have residua with negative real part. In fact, the set of all residua of these functions, where p varies in the interval $(0, 1)$, is a proper subset of the left half plane. A computation of the envelope of the circles described by (2) reveals that $a = x + iy$ is the residuum of a function f in one of the classes $Co(p), p \in (0, 1)$, if and only if $x + iy$ satisfies one of the following conditions.

(i) $|y| \geq \frac{1}{2}$ and $x < -\frac{1}{2}$.

(ii) $|y| \in (0, \frac{1}{2})$ and $x \leq -\sqrt{|y| - y^2}$.

(iii) $y = 0$ and $x < 0$.

Remark 2. Since for any function w holomorphic in D with $w(D) \subset D$ and fixed point $p \in (0, 1)$ there exists a function v holomorphic in D such that $v(D) \subset \overline{D}$ and

$$w(z) = \frac{p + \frac{z-p}{1-zp}v(z)}{1 + p\frac{z-p}{1-zp}v(z)}, \quad z \in D,$$

we get as a consequence of (5) and (6) the following representation formula for concave univalent functions.

Let $p \in (0,1)$. *For any* $f \in Co(p)$, *there exists a function* v *holomorphic in* D *such that* $v(D) \subset \overline{D}$ *and*

$$f(z) = z\frac{1 - zp + p(z-p)v(z)}{\left(1 - \frac{z}{p}\right)(1-zp)(1-p^2v(z))}, \quad z \in D. \tag{9}$$

This formula can be simplified a lot, if we set

$$v(z) = \frac{p^2 - u(z)}{1 - p^2u(z)}, \quad z \in D. \tag{10}$$

The insertion of (10) into (9) yields a second possibility to express concave univalent functions by unimodular bounded functions.

Let $p \in (0,1)$. *For any* $f \in Co(p)$, *there exists a function* u *holomorphic in* D *such that* $u(D) \subset \overline{D}$ *and*

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+u(z))z^2}{\left(1 - \frac{z}{p}\right)(1-zp)}, \quad z \in D.$$

We want to express our hope that this formula will help to solve further extremal problems for $Co(p)$.

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