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KADEC NORMS ON SPACES OF CONTINUOUS FUNCTIONS

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We study the existence of pointwise Kadec renormings for Banach spaces of the form $C(K)$. We show in particular that such a renorming exists when K is any product of compact linearly ordered spaces, extending the result for a single factor due to Haydon, Jayne, Namioka and Rogers. We show that if $C(K_1)$ has a pointwise Kadec renorming and K_2 belongs to the class of spaces obtained by closing the class of compact metrizable spaces under inverse limits of transfinite continuous sequences of retractions, then $C(K_1 \times K_2)$ has a pointwise Kadec renorming. We also prove a version of the three-space property for such renormings.

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1. Introduction. Let X be a Banach space. Let τ be a tvs topology on X weaker than the norm topology. The norm on X is called *τ-Kadec* if the norm topology coincides with τ on the unit sphere. When τ is the weak topology, the norm is simply said to be Kadec. In our setting we consider mainly spaces of the form $X = C(K)$ for some compact space K. We shall be interested primarily in the question of when there is a norm on X equivalent to the supremum norm which is τ_p-Kadec where τ_p stands for the topology of pointwise convergence, referred to henceforth as the pointwise topology.

Raja has shown in [20] that the existence of a τ-Kadec renorming for X is equivalent to the existence of a countable collection $\{A_n : n \in \mathbb{N}\}$ of convex subsets of X such that the collection of sets of the form $U \cap A_n$, where $U \in \tau$, forms a network for the norm topology. (A collection C of sets in a topological space is a *network* for the topology if every open set is the union of a subcollection of C. In other words, C is like a base except that its members do not have to be open.) It is not known whether the word "convex" can be omitted in this characterization. The notion obtained by deleting convexity goes by several names in the literature. Following [11] (where the notion was introduced), we say that (X, τ) *has a countable cover by sets of small local norm-diameter*, or more briefly (X, τ) *is norm-SLD*, if there is a countable collection $\{A_n : n \in \mathbb{N}\}$ of subsets of X such that the sets $U \cap A_n$, where $n \in \mathbb{N}$ and $U \in \tau$, form a network for the norm topology. The notion of norm-SLD is equivalent to the notion of a *descriptive Banach space* introduced by R. Hansell in [9], as it is shown in [19]. It is shown in [12] that when K is an infinite compact F-space, then $C(K)$ is not σ-fragmentable, in particular $C(K)$ has no Kadec renorming.

In the paper [14], it is shown that for every compact totally ordered space K, $C(K)$ has a τ_p-Kadec renorming. We shall show that the conclusion remains true if K is an arbitrary product of compact linearly ordered spaces. This improves the result in [3, Theorem 5.21(b)] (due to Jayne, Namioka and Rogers for countable products, see [13, Remark (1), p. 329]) that for such a product K, $C(K)$ is norm-SLD in the pointwise topology. It is unknown whether the existence of a τ_p-Kadec renorming for each of $C(K_1)$ and $C(K_2)$ implies the existence of such a renorming for $C(K_1 \times K_2)$. Ribarska has shown in [22] that if $C(K_1)$ has a τ_p-Kadec renorming and $C(K_2)$ is norm-SLD in the pointwise topology, then $C(K_1 \times K_2)$ is norm-SLD in the pointwise topology. We establish that if $C(K_1)$ has a τ_p-Kadec renorming and K_2 belongs to the class of spaces obtained by closing the class of compact metrizable spaces under inverse limits of transfinite continuous sequences of retractions, then $C(K_1 \times K_2)$ has a τ_p-Kadec renorming.

In [18], the authors establish, under certain conditions, the three-space property for a sequential version of the Kadec property. (A property of Banach spaces is a three-space property if X has the property whenever Y and X/Y do, where Y is a subspace of X.) A Banach space is said to have the *Kadec-Klee property* if every weakly convergent sequence on the unit sphere is strongly convergent. (The terminology is not used consistently in the literature. In particular, in [5] a norm which has the Kadec-Klee property is what we have called a Kadec norm.) A norm is *locally uniformly rotund* (LUR) if whenever x_n, $n \in \mathbb{N}$, and x are on the unit sphere and $\lim \|x_n + x\| = 2$ we have $\lim x_n = x$. As pointed out in [1], if the norm in a Banach space X is LUR and τ is a tvs topology on X such that the unit ball is τ-closed (for example the weak topology), then the norm is necessarily τ-Kadec. In [18], it is shown that if X is a Banach space, Y is a subspace of X, Y has the Kadec-Klee property and X/Y has an LUR renorming, then X has the Kadec-Klee property. We show, solving a problem raised in [18], that the Kadec-Klee property can be replaced by the Kadec property in their result. It is not known whether the existence of a Kadec renoming is a three-space property. Ribarska has shown in [21] that being norm-SLD in the weak topology is a three-space property. Her proof also shows that for spaces $L \subseteq K$, if $C(L)$ and $C_0(K \setminus L)$ are norm-SLD in the pointwise topology, then so is $C(K)$.

We write lsc, usc for lower semi-continuous, upper semi-continuous, respectively. Given a map $f\colon X \to Y$, a *level set* of f is any set of the form $\{x \in X\colon f(x) = y_0\}$, where $y_0 \in Y$ is fixed. Given a normed space $(X, \|\cdot\|)$ we denote by $\overline{\mathrm{B}}_X$ and S_X the closed unit ball and the unit sphere of X respectively. A closed (resp. open) ball centered at x and with radius $r > 0$ is denoted by $\overline{\mathrm{B}}(x, r)$ (resp. $\mathrm{B}(x, r)$). Similarly, for a set $A \subseteq X$, $\mathrm{B}(A, r)$ denotes $\{x \in X\colon \operatorname{dist}(x, A) < r\} = A + \mathrm{B}(0, r)$.

2. Preliminaries.

We begin with a standard fact.

Proposition 2.1. *Let K and L be compact spaces, and let $\varphi\colon K \to L$ be a continuous surjection. Then the map $T\colon C(L) \to C(K)$ defined by $T(f) = f\varphi$ is a linear isometry and a τ_p-homeomorphism onto its range. In particular, if $C(K)$ has an equivalent τ_p-Kadec norm, then so does $C(L)$.*

Proof. T is clearly linear. We have $\|T(f)\|_\infty = \|f\varphi\|_\infty = \|f\|_\infty$ because φ is onto, so T is an isometry. The fact that T is a τ_p-homeomorphism onto its range follows from the fact that φ is onto and from the equality $T(f)(x) = f(\varphi x)$ for $x \in K$. □

The following Proposition is given as [1, Proposition 1] for the case where τ is generated by a total subspace of X^*. As pointed out in [20, Proposition 4], the proof works for any linear topology.

Proposition 2.2. *Let X be a Banach space whose norm is τ-Kadec. Then the norm is τ-lsc, i.e., the unit ball is τ-closed.*

Proposition 2.3. (Cf. [20, Lemma 1].) *Let X be a Banach space, $x_0 \in S_X$, τ a weaker linear topology on X with respect to which the norm is τ-Kadec at x_0 (i.e., the norm and τ neighborhoods of x_0 are the same). Then for any $r > 0$, there exists $\delta > 0$ and a neighborhood $U \in \tau$ of x_0 such that*

$$U \cap \mathrm{B}(0, 1 + \delta) \subseteq \mathrm{B}(x_0, r).$$

P r o o f. Find a neighborhood $W \in \tau$ of x_0 such that $W \cap \mathrm{S}_X \subseteq \mathrm{B}(x_0, r/2)$. By the τ-continuity of the addition, there are $V, V' \in \tau$ such that $x_0 \in V$, $0 \in V'$ and $V + V' \subseteq W$. Fix $\delta > 0$ such that $\delta \leqslant r/2$ and $\mathrm{B}(0, \delta) \subseteq V'$. Then $V \cap (\mathrm{S}_X + \mathrm{B}(0, \delta)) \subseteq \mathrm{B}(x_0, r)$. Indeed, if $y \in V$ and $\|y - z\| < \delta$ for some $z \in \mathrm{S}_X$ then $z \in (V + V') \cap \mathrm{S}_X \subseteq \mathrm{B}(x_0, r/2)$ so $\|y - x_0\| \leqslant \|y - z\| + \|z - x_0\| < r/2 + \delta \leqslant r$. As closed balls are τ-closed (Proposition 2.2), we may assume that $V \cap \overline{\mathrm{B}}(0, 1 - \delta) = \emptyset$. Then $V \cap \mathrm{B}(0, 1 + \delta) \subseteq \mathrm{B}(x_0, r)$. □

We shall need the simple facts about lower semi-continuous maps given by the next three propositions and their corollaries.

Proposition 2.4. *Let X be a topological space and let $f, g \colon X \to \mathbb{R}$ be functions whose sum is identically equal to a constant value $k \in \mathbb{R}$. For any $x \in X$, if f is lsc at x, then g is usc at x.*

P r o o f. Fix $\varepsilon > 0$ and find a neighborhood V of x such that $f(x') > f(x) - \varepsilon$ for $x' \in V$. Thus $g(x') = k - f(x') < k - f(x) + \varepsilon = g(x) + \varepsilon$ whenever $x' \in V$. □

Corollary 2.5. *Let X be a topological space.*

(a) *If $f, g \colon X \to \mathbb{R}$ are lsc, then the restrictions of f and g to any level set for $f + g$ are continuous.*

(b) *If* $f_n\colon X \to \mathbb{R}$, $n \in \mathbb{N}$, *are nonnegative lsc functions such that* $\sum_{n\in\mathbb{N}} f_n$ *converges pointwise, then the restriction of each* f_n *to a level set for* $\sum_{n\in\mathbb{N}} f_n$ *is continuous.*

Proof. (a) Applying Proposition 2.4 to the restrictions of f and g to a level set $S = \{x \in X : f(x) + g(x) = k\}$ shows that because these functions are lsc at every point, they are also usc at every point.

(b) Apply part (a) to $f = f_n$ and $g = \sum_{m\neq n} f_m$. □

It will be useful to have a slightly stronger version of Corollary 2.5(b).

Proposition 2.6. *Let* X *be a topological space,* $\{f_n\}_{n\in\omega}$ *a sequence of nonnegative lsc real-valued functions on* X *such that* $\theta(x) = \sum_{n\in\omega} f_n(x)$ *is finite for every* $x \in X$. *Assume* $\{x_\sigma\}_{\sigma\in\Sigma}$ *is a net in* X *converging to* $x \in X$ *and* $\lim_{\sigma\in\Sigma} \theta(x_\sigma) = \theta(x)$ *for every* $\sigma \in \Sigma$. *Then* $\lim_{\sigma\in\Sigma} f_k(x_\sigma) = f_k(x)$ *for every* $k \in \omega$.

Proof. Fix $k \in \omega$ and let $g = \sum_{n\neq k} f_n$. Observe that g is lsc as the supremum of a set of lsc functions. Fix $\varepsilon > 0$. There exists σ_0 such that $\theta(x_\sigma) - \theta(x) \leqslant \varepsilon/2$, $f_k(x_\sigma) > f_k(x) - \varepsilon$ and $g(x_\sigma) > g(x) - \varepsilon/2$ for $\sigma \geqslant \sigma_0$. Fix $\sigma \geqslant \sigma_0$ and suppose $f_k(x_\sigma) \not< f_k(x) + \varepsilon$. Then

$$\theta(x_\sigma) = f_k(x_\sigma) + g(x_\sigma) > f_k(x) + \varepsilon + g(x) - \varepsilon/2 = \theta(x) + \varepsilon/2,$$

so $\theta(x_\sigma) - \theta(x) > \varepsilon/2$, a contradiction. □

Proposition 2.7. *Let* X *be a topological space,* $n \in \mathbb{N}$, $f_i\colon X \to \mathbb{R}$ *for* $1 \le i \le n$. *Let* $x \in X$. *Suppose* $\sum f_i \le 0$, $\sum f_i(x) = 0$, *and each* f_i *is lsc at* x. *Then each* f_i *is continuous at* x.

Proof. Fix i and $\varepsilon > 0$. For y in some neighborhood of x we have

$$f_i(x) - \varepsilon < f_i(y) \le -\sum_{j\neq i} f_j(y) < -\Big(\sum_{j\neq i}(f_j(x) - \varepsilon/(n-1))\Big) = f_i(x) + \varepsilon.$$

□

Corollary 2.8. *Let* X *be a topological space,* $n \in \mathbb{N}$, $f_i\colon X \to \mathbb{R}$ *for* $1 \le i \le n$, $h\colon X \to \mathbb{R}$. *Let* $x \in X$. *Suppose* $\sum f_i \le h$, $\sum f_i(x) = h(x)$, *each* f_i *is lsc at* x *and* h *is usc at* x. *Then* h *and each* f_i *is continuous at* x.

Proof. $f_0 + \cdots + f_{n-1} - h \le 0$, $f_0(x) + \cdots + f_{n-1}(x) - h(x) = 0$ and $-h$ is lsc at x. □

An *inverse sequence* is a family of mappings $p^\beta_\alpha\colon X_\beta \to X_\alpha$, $\alpha < \beta < \kappa$, where κ is a limit ordinal, such that $\alpha < \beta < \gamma \implies p^\beta_\alpha p^\gamma_\beta = p^\gamma_\beta$. Usually, the maps p^β_α are surjections. We refer the reader to [6, Section 2.5] for the basic properties of inverse systems. We recall here some of the relevant terminology.

We write $\mathbb{S} = \{X_\alpha; p^\beta_\alpha\colon \alpha < \beta < \kappa\}$ and we call p^β_α's the *bonding mappings* of $\mathbb{S}$. The *inverse limit* of $\mathbb{S}$, denoted by $\varprojlim \mathbb{S}$ is defined to be the subspace of the product $\prod_{\alpha<\kappa} X_\alpha$ consisting of all x such that $p^\beta_\alpha(x(\beta)) = x(\alpha)$ for every $\alpha < \beta < \kappa$. If each X_α is compact then $\varprojlim \mathbb{S} \neq \emptyset$. If moreover each p^β_α is a surjection then the projection $p_\alpha\colon \varprojlim \mathbb{S} \to X_\alpha$ is also a surjection. From a category-theoretic perspective, the inverse limit of $\mathbb{S}$ is a space X together with a family of continuous maps (called *projections*) $\{p_\alpha\colon \alpha < \kappa\}$ which has the property that for every space Y and a family of continuous maps $\{f_\alpha\colon \alpha < \kappa\}$ such that $p^\beta_\alpha f_\beta = f_\alpha$ holds for every $\alpha < \beta < \kappa$, there exists a unique continuous map $h\colon Y \to X$ such that $p_\alpha h = f_\alpha$ for every $\alpha < \kappa$. The limit is uniquely determined in the sense that if X' with projections p'_α, $\alpha < \kappa$, is another, then the unique continuous map $h\colon X' \to X$ such that $p_\alpha h = p'_\alpha$ for all $\alpha < \kappa$ is a homeomorphism. The definition of $\varprojlim \mathbb{S}$ given above is one of the possibilities. We will use the property that $\varprojlim\{X_\alpha; p^\beta_\alpha\colon \alpha < \beta < \kappa\}$ is isomorphic to $\varprojlim\{X_\alpha; p^\beta_\alpha\colon \alpha < \beta,\ \alpha, \beta \in C\}$ for every cofinal set $C \subseteq \kappa$.

An inverse sequence $\mathbb{S} = \{X_\alpha; p^\beta_\alpha\colon \alpha < \beta < \kappa\}$ is *continuous* if for every limit ordinal $\delta < \kappa$ the space X_δ together with $\{p^\delta_\alpha\colon \alpha < \delta\}$ is homeomorphic to $\varprojlim\{X_\alpha; p^\beta_\alpha\colon \alpha < \beta < \delta\}$.

A *retraction* is a continuous map $f\colon X \to Y$ which has a right inverse, i.e. a continuous map $j\colon Y \to X$ with $fj = \mathrm{id}_Y$. Note that j is an embedding and f restricted to $j[Y]$ is a homeomorphism.

Finally, we point out that many of our results about Banach spaces equipped with a weaker linear topology τ with respect to which the norm is lsc have conclusions which assert the existence of an equivalent norm with a certain property. In all such results, the assumption that the norm is τ-lsc can be weakened to the assumption that the τ-closure of the unit ball is bounded, since the Minkowski functional of this closure provides an equivalent τ-lsc norm.

3. Finite products of linearly ordered spaces. In this section we show that $C(L_0 \times \cdots \times L_{n-1})$ has a τ_p-Kadec renorming, whenever $L_0, \ldots, L_{n-1}$ are compact linearly ordered spaces. In Theorem 4.9, this result will be extended to arbitrary products.

Lemma 3.1. *If X is a compact linearly ordered space, (Y, d) is a metric* *space, $f\colon X \to Y$ is continuous, and for each $m \in \omega$ we set*

$$v_m(f) = \sup\Big\{\sum_{i<m} d(f(a_i), f(a_{i+1}))\colon a_0 \leqslant a_1 \leqslant \ldots \leqslant a_m\Big\},$$

where 0 *and* 1 *denote the first and last elements of X, then*

$$\lim_{m\to\infty} v_{m+1}(f) - v_m(f) = 0.$$

Proof. Fix $\varepsilon > 0$. Let $\mathcal{I}$ be a finite cover of X by open intervals I such that $f[I]$ has diameter $< \varepsilon$. Fix any $m > |\mathcal{I}|$. By compactness, we can choose $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_m \leqslant a_{m+1}$ so that $v_{m+1}(f) = \sum_{i<m+1} d(f(a_i), f(a_{i+1}))$. For some $I \in \mathcal{I}$ and $i_0 < m+1$, we have $a_{i_0}, a_{i_0+1} \in I$. Suppose $i_0 < m$. Then

$$\begin{aligned}d(f(a_{i_0}), f(a_{i_0+1})) + d(f(a_{i_0+1}), f(a_{i_0+2})) &\leqslant d(f(a_{i_0}), f(a_{i_0+2})) + 2d(f(a_{i_0}), f(a_{i_0+1}))\\ &< d(f(a_{i_0}), f(a_{i_0+2})) + 2\varepsilon\end{aligned}$$

and we get

$$\begin{aligned}v_m(f) &\geq d(f(a_0), f(a_1)) + \cdots + d(f(a_{i_0-1}), f(a_{i_0})) + d(f(a_{i_0}), f(a_{i_0+2}))\\ &\quad + d(f(a_{i_0+2}), f(a_{i_0+3})) + \cdots + d(f(a_m), f(a_{m+1}))\\ &> \sum_{i<m+1} d(f(a_i), f(a_{i+1})) - 2\varepsilon\\ &= v_{m+1}(f) - 2\varepsilon,\end{aligned}$$

which gives $0 \leqslant v_{m+1}(f) - v_m(f) < 2\varepsilon$. If $i_0 = m$, replace the triple $(f(a_{i_0}), f(a_{i_0+1}), f(a_{i_0+2}))$ by the triple $(f(a_{i_0-1}), f(a_{i_0}), f(a_{i_0+1}))$ in the argument above. □

Let L be a linearly ordered space. We say that points $x, y \in L$ are *adjacent* if $x \neq y$ and no point is strictly between x, y.

Theorem 3.2. *Assume L_i, $i < n$ are compact linearly ordered spaces and $D_i \subseteq L_i$ is dense in L_i and contains all pairs of adjacent points for each $i < n$. Then $C(\prod_{i<n} L_i)$ has an equivalent $\tau_p(D)$-Kadec norm, where $D = \prod_{i<n} D_i$.*

(See Theorem 4.9 for the case of arbitrary products.)

Proof. For $f \in C(\prod_{i<n} L_i)$, we will need to consider expressions of the form

$$f(x_0, x_1, \ldots, x_{k-1}, a, x_{k+1}, \ldots, x_{n-1}). \tag{3.1}$$

For notational convenience, we sometimes permute the arguments so that a comes first. Letting $h_k\colon L_k \times \prod_{\ell<n,\, \ell\neq k} L_\ell \to \prod_{\ell<n} L_\ell$ be given by

$$h_k(a,\, x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}) = (x_0, x_1, \ldots, x_{k-1},\, a,\, x_{k+1}, \ldots, x_{n-1}),$$

we can then write

$$f(h_k(a,\, x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n-1}))$$

instead of (3.1).

For each $k < n$ and $m \in \omega$, define $v^k_m(f)$ on $C(\prod_{i<n} L_i)$ by letting

$$v^k_m(f) = \sup\Big\{\sum_{i<m} \|f(h_k(a^k_i,\, \cdot)) - f(h_k(a^k_{i+1},\, \cdot))\|_\infty \colon a^k_1 \leqslant \ldots \leqslant a^k_m\Big\}.$$

The function v^k_m is a $\tau_p(D)$-lsc seminorm and

$$\lim_{m\to\infty}\ v^k_{m+1}(f) - v^k_m(f) = 0,$$

by Lemma 3.1.

Define $|\cdot|$ on $C(\prod_{i<n} L_i)$ as follows.

$$|f| = \|f\|_\infty + \sum_{k<n}\sum_{m\in\omega} \frac{1}{m\cdot 2^m} v^k_m(f).$$

It is readily seen that $|\cdot|$ is a norm on $C(\prod_{i<n} L_i)$ and is equivalent to the sup norm. We now verify that it is a $\tau_p(D)$-Kadec norm. Since the terms in the definition of $|f|$ are all $\tau_p(D)$-lsc functions of f, Corollary 2.5(b) implies that they are all $\tau_p(D)$-continuous functions of f when restricted to $S := \{f\colon |f| = 1\}$. Fix $f \in S$ and $\varepsilon > 0$.

For each $k < n$, the map $x \mapsto f(h_k(x,\, \cdot))$ is continuous (with the norm topology on the range), so there is a finite collection $\mathcal{I}_k$ of open intervals covering L_k such that the diameter in $C(\prod_{\ell<n,\, \ell\neq k} L_\ell)$ of $\{f(h_k(x,\, \cdot))\colon x \in I\}$ is less than ε for each $I \in \mathcal{I}_k$. We may assume that $\inf I \in D_k \cup \{0\}$ and $\sup I \in D_k \cup \{1\}$ for each $I \in \mathcal{I}_k$. Let $A_k = \{\inf I\colon I \in \mathcal{I}_k\} \cup \{\sup I\colon I \in \mathcal{I}_k\}$. Then $A_k \subseteq D_k \cup \{0, 1\}$.

Let $m \in \omega$ be such that for each $k < n$, $v^k_{m+3}(f) - v^k_m(f) < \varepsilon$.

For each $k < n$, fix $a^k_0 \leqslant a^k_1 \leqslant \ldots \leqslant a^k_m$ in D_k such that

$$v^k_m(f) < \sum_{i<m} \|f(h(a^k_i,\, \cdot)) - f(h(a^k_{i+1},\, \cdot))\|_\infty + \delta,$$

where $\delta = \varepsilon/(m+4)$. Let $H_k = \{a_i^k \colon i \leqslant m\} \cup (A_k \cap D_k)$.

Fix a $\tau_p(D)$-open neighborhood U of f such that for $g \in S \cap U$ we have, for all $k < n$, that $v_{m+i}^k(g)$ is strictly within ε of $v_{m+i}^k(f)$ for $i \leqslant 3$. This gives

$$|v_{m+i}^k(g) - v_{m+i}^k(f)| < \varepsilon \quad \text{and} \quad |v_{m+i}^k(f) - v_m^k(f)| < \varepsilon$$

and hence

$$|v_{m+i}^k(g) - v_m^k(f)| < 2\varepsilon.$$

For each $k < n$ and for each pair of elements $a < b$ of D_k, choose $x = x_{a,b}^k$ and $y = y_{a,b}^k$ in D such that $x(k) = a$, $y(k) = b$, $x(\ell) = y(\ell)$ for all $\ell \neq k$ and

$$\|f(h_k(a,\cdot)) - f(h_k(b,\cdot))\|_\infty < |f(x) - f(y)| + \delta.$$

Write

$$\overline{H}_k = H_k \cup \{z(k) \colon z = x_{a,b}^\ell \text{ or } z = y_{a,b}^\ell \text{ for some } \ell < n \text{ and some } a < b \text{ in } H_\ell\}.$$

Then $\overline{H}_k \subseteq D_k$. Let $g \in U$ agree sufficiently closely with f on $H = \prod_{k<n} \overline{H}_k$ so that $|g(h) - f(h)| < \varepsilon$ for each $h \in H$ and the following condition is satisfied.

(∗) For each $k < n$, for each $i_0 < m$, and any choice of elements of H_k of the form
$$a_{i_0}^k = b_0 \leqslant b_1 \leqslant b_2 \leqslant b_3 = a_{i_0+1}^k$$
we have, for each $j_0 < 3$,
$$\begin{aligned}&\sum_{i \neq i_0} |g(x^k_{a_i^k, a_{i+1}^k}) - g(y^k_{a_i^k, a_{i+1}^k})| + \sum_{j \neq j_0} |g(x^k_{b_j, b_{j+1}}) - g(y^k_{b_j, b_{j+1}})| \\ &> \sum_{i \neq i_0} |f(x^k_{a_i^k, a_{i+1}^k}) - f(y^k_{a_i^k, a_{i+1}^k})| + \sum_{j \neq j_0} |f(x^k_{b_j, b_{j+1}}) - f(y^k_{b_j, b_{j+1}})| - \varepsilon.\end{aligned}$$

Assume also that for each $k < n$ we have

$$(*_1) \qquad \sum_{i<m} |g(x^k_{a_i^k, a_{i+1}^k}) - g(y^k_{a_i^k, a_{i+1}^k})| > \sum_{i<m} |f(x^k_{a_i^k, a_{i+1}^k}) - f(y^k_{a_i^k, a_{i+1}^k})| - \varepsilon.$$

From (∗) it follows that for any $x \in [b_{j_0}, b_{j_0+1}]$, writing

$$s = \|f(h_k(b_{j_0},\cdot)) - f(h_k(x,\cdot))\|_\infty + \|f(h_k(x,\cdot)) - f(h_k(b_{j_0+1},\cdot))\|_\infty$$

and

$$t = \|g(h_k(b_{j_0},\cdot)) - g(h_k(x,\cdot))\|_\infty + \|g(h_k(x,\cdot)) - g(h_k(b_{j_0+1},\cdot))\|_\infty$$

we have

$$\begin{aligned}
v^k_m(f) + 2\varepsilon - t &> v^k_{m+3}(g) - t\\
&\geqslant \sum_{i\neq i_0} \|g(h_k(a^k_i,\cdot)) - g(h_k(a^k_{i+1},\cdot))\|_\infty\\
&\quad + \sum_{j\neq j_0} \|g(h_k(b_j,\cdot)) - g(h_k(b_{j+1},\cdot))\|_\infty\\
&\geqslant \sum_{i\neq i_0} |g(x^k_{a^k_i,a^k_{i+1}}) - g(y^k_{a^k_i,a^k_{i+1}})| + \sum_{j\neq j_0} |g(x^k_{b_j,b_{j+1}}) - g(y^k_{b_j,b_{j+1}})|\\
&> \sum_{i\neq i_0} |f(x^k_{a^k_i,a^k_{i+1}}) - f(y^k_{a^k_i,a^k_{i+1}})| + \sum_{j\neq j_0} |f(x^k_{b_j,b_{j+1}}) - f(y^k_{b_j,b_{j+1}})| - \varepsilon\\
&> \sum_{i\neq i_0} (\|f(h_k(a^k_i,\cdot)) - f(h_k(a^k_{i+1},\cdot))\|_\infty - \delta)\\
&\quad + \left(\sum_{j\neq j_0} (\|f(h_k(b_j,\cdot)) - f(h_k(b_{j+1},\cdot))\|_\infty - \delta) + s\right) - s - \varepsilon\\
&\geqslant \sum_{i<m} \|f(h_k(a^k_i,\cdot)) - f(h_k(a^k_{i+1},\cdot))\|_\infty - s - \varepsilon - (m+3)\delta\\
&> v^k_m(f) - s - 2\varepsilon
\end{aligned}$$

and hence $t < s + 4\varepsilon$, i.e., for any $x \in [b_{j_0}, b_{j_0+1}]$,

$$\begin{aligned}
(**)\quad &\|g(h_k(b_{j_0},\cdot)) - g(h_k(x,\cdot))\|_\infty + \|g(h_k(x,\cdot)) - g(h_k(b_{j_0+1},\cdot))\|_\infty\\
&< \|f(h_k(b_{j_0},\cdot)) - f(h_k(x,\cdot))\|_\infty + \|f(h_k(x,\cdot)) - f(h_k(b_{j_0+1},\cdot))\|_\infty + 4\varepsilon.
\end{aligned}$$

Consider a point $p \in \prod_{k<n} L_k$. Define

$$T = \{k < n \colon p_k \notin \overline{H}_k\}.$$

We will show by induction on $r = |T|$ that $|g(p) - f(p)| < (7r+1)\varepsilon$. This is true if $r = 0$ since then $p \in H$. For the inductive step, suppose $|T| = r+1$. Choose an open neighborhood of p of the form $\prod_{k<n} I_k$, where $I_k \in \mathcal{I}_k$ for each $k < n$. For each $k < n$, let $-1 \leqslant i_0(k) \leqslant m$ be such that $a^k_{i_0(k)} \leqslant p_k \leqslant a^k_{i_0(k)+1}$, where $a^k_{-1} = 0$, $a^k_{m+1} = 1$. Define $r_k = \max\{a^k_{i_0(k)}, \inf I_k\}$ and $s_k = \min\{a^k_{i_0(k)+1}, \sup I_k\}$. Pick any $k \in T$. Assume first that $-1 < i_0(k) < m$, so in particular $r_k, s_k \in D_k$ and hence $r_k, s_k \in \overline{H}_k$. If q_1, q_2 denote the modifications of p obtained by replacing the k-th coordinate of p by r_k and s_k respectively, then $|g(q_i) - f(q_i)| < (7r+1)\varepsilon$, $i = 1, 2$, by the induction hypothesis. Using $(**)$ with $j_0 = 1$ and "$a^k_{i_0(k)} \leqslant r_k \leqslant p_k \leqslant s_k \leqslant a^k_{i_0(k)+1}$" in the place of "$a^k_{i_0} = b_0 \leqslant b_1 \leqslant x \leqslant b_2 \leqslant b_3 = a^k_{i_0+1}$" we get

$$\begin{aligned}
|g(q_1) - g(p)| + |g(p) - g(q_2)| &\leqslant \|g(h_k(r_k, \cdot)) - g(h_k(p_k, \cdot))\|_\infty \\
&\quad + \|g(h_k(p_k, \cdot)) - g(h_k(s_k, \cdot))\|_\infty \\
&< \|f(h_k(r_k, \cdot)) - f(h_k(p_k, \cdot))\|_\infty \\
&\quad + \|f(h_k(p_k, \cdot)) - f(h_k(s_k, \cdot))\|_\infty + 4\varepsilon \\
&< 6\varepsilon
\end{aligned}$$

and hence

$$\begin{aligned}
|g(p) - f(p)| &\leqslant |g(p) - g(q_1)| + |g(q_1) - f(q_1)| + |f(q_1) - f(p)| \\
&< 6\varepsilon + (7r+1)\varepsilon + \varepsilon = (7(r+1)+1)\varepsilon.
\end{aligned}$$

Assume now that $i_0(k) = m$ (the case $i_0(k) = -1$ is similar). We have $a^k_0 \leq \cdots \leq a^k_m \leq p_k$. Let q denote the modification of p obtained by replacing the k-th coordinate with a^k_m. Then

$$\begin{aligned}
|f(q) - f(p)| &\leqslant \|f(h_k(a^k_m, \cdot)) - f(h_k(p_k, \cdot))\|_\infty \\
&\leqslant v^k_{m+1}(f) - \sum_{i<m} \|f(h_k(a^k_i, \cdot)) - f(h_k(a^k_{i+1}, \cdot))\|_\infty \\
&< v^k_{m+1}(f) - v^k_m(f) + \delta < 2\varepsilon.
\end{aligned}$$

Similarly, using $(*_1)$, we get

$$\begin{aligned}
|g(q) - g(p)| &\leqslant \|g(h_k(a^k_m, \cdot)) - g(h_k(p_k, \cdot))\|_\infty \\
&\leqslant v^k_{m+1}(g) - \sum_{i<m} \|g(h_k(a^k_i, \cdot)) - g(h_k(a^k_{i+1}, \cdot))\|_\infty \\
&< v^k_{m+1}(f) + \varepsilon - \sum_{i<m} |g(x^k_{a^k_i, a^k_{i+1}}) - g(y^k_{a^k_i, a^k_{i+1}})| \\
&< v^k_{m+1}(f) + \varepsilon - \sum_{i<m} |f(x^k_{a^k_i, a^k_{i+1}}) - f(y^k_{a^k_i, a^k_{i+1}})| + \varepsilon \\
&< v^k_{m+1}(f) - \sum_{i<m} \|f(h_k(a^k_i, \cdot)) - f(h_k(a^k_{i+1}, \cdot))\|_\infty + m\delta + \varepsilon \\
&< v^k_{m+1}(f) - v^k_m(f) + (m+1)\delta + 2\varepsilon < 4\varepsilon.
\end{aligned}$$

Thus $|f(p) - g(p)| < 6\varepsilon + |f(q) - g(q)|$ and by the induction hypothesis, $|f(q) - g(q)| < (7r+1)\varepsilon$. Hence also in this case we get $|f(p) - g(p)| < (7(r+1)+1)\varepsilon$.

Finally, $\|f-g\|_\infty < (7n+1)\varepsilon$ which completes the proof. □

Remark 3.3. The above result is no longer valid if we drop the requirement that the sets D_i contain all pairs of adjacent points. For example, if L is the double arrow line and D is a countable dense set then $\tau_p(D)$ is second countable, while $C(L)$ is not second countable, and the same is true when restricted to any sphere of $C(L)$.

We also cannot replace the assumption on the sets D_i by "dense countably compact". It is shown in [3, Example 5.17] that the space of continuous functions on $D = (\omega_1 + \omega_1^*)^{\omega_1}$ endowed with the topology induced by the lexicographic order (ω_1^* means ω_1 with the reversed order) is not norm-SLD for the pointwise topology. In particular, it has no τ_p-Kadec renorming. On the other hand, D is a countably compact linearly ordered space. If we take L to be the Čech-Stone compactification of D, then L is linearly ordered—it is obtained from the Dedekind completion of D by doubling the points which are not endpoints and are not in D—and $C(L)$ is isomorphic to $C(D)$ via the restriction map. Since this map is also a $(\tau_p(D), \tau_p)$-homeomorphism, $C(L)$ has no $\tau_p(D)$-Kadec renorming.

4. Inverse limits and projectional resolutions of the identity.

In this section we show the existence of a τ_p-Kadec renorming on a space $C(K)$ when K is a suitable inverse limit of spaces K' for which $C(K')$ has a τ_p-Kadec renorming. As an application, we obtain in particular that $C(K \times L)$ has a τ_p-Kadec renorming, whenever $C(K)$ has a τ_p-Kadec norm and L is a Valdivia compact space.

We begin with a technical lemma inspired by a very useful result of Troyanski. (See [5, VII Lemma 1.1].)

Lemma 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and let τ be a linear topology on X such that the unit ball of X is τ-closed. Fix a function $h\colon \mathbb{N} \to \mathbb{N}$. Suppose there are*

(a) *families $\mathcal{F}_0, \mathcal{F}_1, \dots$ of bounded (τ,τ)-continuous linear operators on X such that for each n, $\mathcal{F}_n$ is uniformly bounded,*

(b) *for each $T \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, an equivalent τ-Kadec norm $|\cdot|_T$ on the range of T, and*

(c) *for each* $n \in \mathbb{N}$ *and* $T \in \mathcal{F}_n$, *a set* $S_n(T) \subseteq \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n$ *of cardinality at most* $h(n)$,

so that

(d) *for each* $x \in X$ *and each* $\varepsilon > 0$, *we can find* $n \in \mathbb{N}$ *and* $T \in \mathcal{F}_n$ *such that* $\|x - T_0 x\| < \varepsilon$ *for some* $T_0 \in S_n(T)$ *and* $|Tx|_T > \sup\{|T'x|_{T'} : T' \in \mathcal{F}_n,\, T' \neq T\}$.

Then there exists an equivalent τ*-Kadec norm on* X.

Proof. We may assume that $|\cdot|_T \leqslant \|\cdot\|$ for each $T \in \bigcup_{n\in\omega} \mathcal{F}_n$. Define

$$|x|_{k,n} = \sup\{|Tx|_T + \frac{1}{k} \sum_{T' \in S_n(T)} |T'x|_{T'} + \|x - T'x\| : T \in \mathcal{F}_n\}$$

and

$$|x| = \|x\| + \sum_{k,n<\omega} \beta_{k,n} |x|_{k,n},$$

where $\beta_{k,n} > 0$ are such that $\beta_{k,n}|x|_{k,n} \leqslant 2^{-(k+n)}\|x\|$. (These constants exist because for each fixed n, the operators in $\mathcal{F}_n$ are uniformly bounded and the sets $S_n(T)$, $T \in \mathcal{F}_n$, are bounded in cardinality.)

It is clear that $|\cdot|$ is equivalent to $\|\cdot\|$. We will show that $|\cdot|$ is τ-Kadec. It is τ-lsc since $\|\cdot\|$ and all the $|\cdot|_{k,n}$ are (use (b) and Proposition 2.2). Thus, by Corollary 2.5(b), on $S := \{x \in X : |x| = 1\}$, each of these functions is τ-continuous. Fix $x \in S$ and $\varepsilon > 0$. By (d), there are $n \in \mathbb{N}$ and $T \in \mathcal{F}_n$ such that $\|x - T_0 x\| < \varepsilon$ for some $T_0 \in S_n(T)$ and

$$\delta = |Tx|_T - \sup\{|T'x|_{T'} : T' \in \mathcal{F}_n,\, T' \neq T\} > 0.$$

Choose k so that

$$\frac{h(n)}{k} \sup\{2\|T'\| + 1 : T' \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n\} \cdot \|x\| < \delta.$$

Then

$$|x|_{k,n} = |Tx|_T + \frac{1}{k} \sum_{T' \in S_n(T)} |T'x|_{T'} + \|x - T'x\|.$$

(To see this, consider the effect on the expression on the right-hand side of the equation of replacing T by some other $\widetilde{T} \in \mathcal{F}_n$. The first term drops by at least δ (by definition of δ). By the choice of k, the second term cannot make up for the decrease.) By Proposition 2.3. and the (τ,τ)-continuity of T_0, there is an $\eta > 0$ and there is a $U \in \tau$ containing x such that if $|T_0 y|_{T_0}$ is within η of $|T_0 x|_{T_0}$ and $y \in U$ then $\|T_0 y - T_0 x\| < \varepsilon$.

From the τ-lsc of each of the terms in the expression for $|x|_{k,n}$ as functions of x and the τ-continuity of $|\cdot|_{k,n}$ on S, it follows from Corollary 2.8 that $y \mapsto |T_0 y|_{T_0}$ and $y \mapsto \|y - T_0 y\|$ are continuous at x on S. Thus, by shrinking U to a smaller τ-neighborhood of x, we may arrange that $y \mapsto |T_0 y|_{T_0}$ and $y \mapsto \|y - T_0 y\|$ vary by less than $\min\{\eta, \varepsilon\}$ on $U \cap S$. Since $\|x - T_0 x\| < \varepsilon$, this means in particular that $\|y - T_0 y\| < 2\varepsilon$ for $y \in U \cap S$.

For $y \in U \cap S$, we have

$$\|y - x\| \leqslant \|y - T_0 y\| + \|T_0 y - T_0 x\| + \|T_0 x - x\| < 2\varepsilon + \varepsilon + \varepsilon = 4\varepsilon.$$

This completes the proof. □

Remark 4.2. The above lemma, as well as its corollaries, could be stated in a more general form saying that on each TX there is a weaker linear topology τ_T for which T is (τ, τ_T)-continuous and TX has a τ_T-Kadec renorming. The proofs require only minor changes.

Theorem 4.3. *Let X be a Banach space and let $\{P_n \colon X \to X\}_{n\in\mathbb{N}}$ be a uniformly bounded sequence of projections such that $\bigcup_{n\in\mathbb{N}} P_n X$ is dense in X. Let τ be a weaker linear topology on X such that the unit ball is τ-closed. If for each $n \in \mathbb{N}$, P_n is (τ,τ)-continuous and there exists a τ-Kadec renorming of $P_n X$, then there exists a τ-Kadec renorming of X.*

Proof. We apply Lemma 4.1 with $\mathcal{F}_n = \{P_n\}$ and $S_n(P_n) = \{P_n\}$. Condition (d) of Lemma 4.1 reduces in this case to the fact that for every $x \in X$ and $\varepsilon > 0$ there exists $n \in \omega$ such that $\|x - P_n x\| < \varepsilon$. To see that this is true, fix $x \in X$ and $\varepsilon > 0$ and set $\delta = \varepsilon/(1+M)$, where M is a constant which bounds the norms of all P_n's. Then, by assumption, there are $n \in \mathbb{N}$ and $y \in P_n X$ such that $\|x - y\| < \delta$. We have $y = P_n y$ and hence $\|P_n x - y\| \leqslant \|P_n\| \cdot \|x - y\| < M\delta$. Thus

$$\|x - P_n x\| \leqslant \|x - y\| + \|y - P_n x\| < \delta + M\delta = \varepsilon. \qquad \square$$

Theorem 4.4. *Let $(X, \|\cdot\|)$ be a Banach space and assume that $\{T_\alpha \colon X \to X\}_{\alpha<\kappa}$ is a sequence of uniformly bounded linear operators on X such that for each $x \in X$,*

(i) *the sequence $\{\|T_\alpha x\|\}_{\alpha<\kappa}$ belongs to $c_0(\kappa)$,*

(ii) *for every* $\varepsilon > 0$ *there exists a finite set* $A \subseteq \kappa$ *such that*

$$\Big\| x - \sum_{\alpha \in A} T_\alpha x \Big\| < \varepsilon,$$

(iii) $T_\alpha X \cap T_\beta X = \{0\}$ *whenever* $\alpha \neq \beta$.

Assume further that τ *is a linear topology on* X *such that the the unit ball of* X *is* τ*-closed and for each* $\alpha < \kappa$, $T_\alpha X$ *has a* τ*-Kadec renorming and* T_α *is* (τ, τ)*-continuous. Then* X *has an equivalent* τ*-Kadec norm.*

Proof. Let $Q_A = \sum_{\alpha \in A} T_\alpha$ and define $\mathcal{F}_n = \{Q_A \colon A \in [\kappa]^n\}$, $S(Q_A) = \{Q_{A'} \colon A' \subseteq A\}$ (so $S(Q_A)$ has cardinality at most $2^{|A|}$). If $\|\cdot\|_\alpha$ is a τ-Kadec norm on $T_\alpha X$ then $\|\cdot\|_{Q_A} = \sum_{\alpha \in A} \|\cdot\|_\alpha$ is a τ-Kadec norm on $Q_A X$. We may assume that $\|\cdot\|_\alpha \leqslant \|\cdot\|$ for each $\alpha < \kappa$. We need to check condition (d) of Lemma 4.1. Fix $x \in X$, $\varepsilon > 0$. By (ii) there exists $A_0 \in [\kappa]^{<\omega}$ such that $\|x - Q_{A_0} x\| < \varepsilon$. By (i), there exists a finite set $A \supseteq A_0$ such that

$$\max_{\alpha \notin A} \|T_\alpha x\|_\alpha < \min_{\alpha \in A} \|T_\alpha x\|_\alpha.$$

It follows that $\|Q_A x\|_{Q_A} > \sup\{\|Q_B x\|_{Q_B} \colon |B| = |A| \ \&\ B \neq A\}$. Thus, by Lemma 4.1, we get a τ-Kadec renorming of X. □

Theorem 4.5. *Assume* X *is a Banach space and* $\{P_\alpha \colon X \to X\}_{\alpha \leqslant \kappa}$ *is a sequence of projections such that*

(a) $P_0 = 0$, $P_\kappa = \mathrm{id}_E$ *and* $P_\beta P_\alpha = P_\alpha = P_\alpha P_\beta$ *whenever* $\alpha \leqslant \beta \leqslant \kappa$.

(b) *There is* $M < +\infty$ *such that* $\|P_\alpha\| \leqslant M$ *for every* $\alpha < \kappa$.

(c) *If* $\lambda \leqslant \kappa$ *is a limit ordinal then* $\bigcup_{\xi < \lambda} P_\xi E$ *is dense in* $P_\lambda E$.

Assume that τ *is a linear topology on* X *such that the unit ball of* X *is* τ*-closed and for each* $\alpha < \kappa$, $(P_{\alpha+1} - P_\alpha)X$ *has a* τ*-Kadec renorming and* $P_{\alpha+1} - P_\alpha$ *is* (τ, τ)*-continuous. Then* X *has a* τ*-Kadec renorming.*

Proof. Let $T_\alpha = P_{\alpha+1} - P_\alpha$. A standard and well known argument (see e.g. [5, pp. 236, 284]) shows that $\{T_\alpha\}_{\alpha < \kappa}$ satisfies the assumptions of Theorem 4.4. We write out the proof of condition (ii) for the sake of completeness because it is not given explicitly in [5].

Proceed by induction on limit ordinals $\lambda < \kappa$. If $\lambda = \omega$ then $P_\omega x = \lim_{n \to \infty} P_n x = \sum_{n \in \omega} (P_{n+1} x - P_n x) = \sum_{n \in \omega} T_n x$ (recall that $P_0 = 0$), so

$\sum_{n<k} T_n x$ can be taken arbitrarily close to $P_\omega x$. Now let $\lambda > \omega$ and assume the statement is true for limit ordinals below λ (and for every $\varepsilon > 0$). There exists $\xi_0 < \lambda$ such that $\|P_\lambda x - P_\beta x\| < \varepsilon/2$ for $\xi \geqslant \xi_0$. If there is a limit ordinal β such that $\xi_0 \leqslant \beta < \lambda$ then, by induction hypothesis

$$(***) \qquad \Big\|P_\beta x - \sum_{\alpha\in A} T_\alpha x\Big\| < \varepsilon/2.$$

for some finite set $A \subseteq \beta$ and we have $\|P_\lambda x - \sum_{\alpha\in A} T_\alpha x\| < \varepsilon$. Otherwise, $\xi_0 = \beta+n$, where $\beta \geqslant \omega$ is a limit ordinal and again $(***)$ holds for some finite set $A \subseteq \beta$. Now we have $P_{\beta+n}x - P_\beta x = \sum_{\alpha=\beta}^{\beta+n-1} T_\alpha x$ and hence $\|P_\lambda x - \sum_{\alpha\in B} T_\alpha x\| \leqslant \|P_\lambda x - P_{\beta+n}x\| + \|P_\beta x - \sum_{\alpha\in A} T_\alpha x\| < \varepsilon$, where $B = A \cup \{\beta, \beta+1, \ldots, \beta+n-1\}$. □

A sequence $\{P_\alpha : \alpha \leqslant \kappa\}$ satisfying conditions (a), (b) and (c) of the above theorem with $M = 1$ and such that the density of $P_\alpha X$ is $\leqslant |\alpha| + \aleph_0$, is called a *projectional resolution of the identity* (PRI) on X, see [5] or [7].

The following proposition is a purely category-theoretic property of inverse limits. It is standard but we do not know a reference for it, so we write out the proof.

Proposition 4.6. *Let $\{X_\alpha; p_\alpha^\beta : \alpha < \beta < \kappa\}$ be a continuous inverse sequence of topological spaces such that each $p_\alpha^{\alpha+1}$ is a retraction and let X, with projections $\{p_\alpha : \alpha < \kappa\}$, be the inverse limit of the sequence. Then there exists a collection of continuous embeddings $\{i_\alpha^\beta : X_\alpha \to X_\beta\}_{\alpha<\beta<\kappa}$, such that*

(1) $p_\alpha^\beta i_\alpha^\beta = \mathrm{id}_{X_\alpha}$ *for all* $\alpha < \beta < \kappa$ *and* $i_\gamma^\beta i_\alpha^\gamma = i_\alpha^\beta$ *for all* $\alpha < \gamma < \beta < \kappa$.

Moreover, there exist continuous embeddings $i_\alpha : X_\alpha \to X$ such that

(2) $p_\alpha i_\alpha = \mathrm{id}_{X_\alpha}$ *and* $i_\beta i_\alpha^\beta = i_\alpha$, *whenever* $\alpha < \beta < \kappa$.

Proof. We can treat (2) as a special case of (1) by allowing $\beta = \kappa$ in (1) and setting $X_\kappa = X$ and $p_\alpha^\kappa = p_\alpha$ for $\alpha < \kappa$. We construct the maps i_α^β by induction on $\beta \leq \kappa$. Assume i_ξ^η have been constructed for every $\xi < \eta < \beta$; for convenience we set $i_\xi^\xi = \mathrm{id}_{X_\xi}$. Suppose first that β is a successor, i.e. $\beta = \delta + 1$.

Fix any continuous map $i_\delta^{\delta+1} : X_\delta \to X_{\delta+1}$ which is a right inverse of $p_\delta^{\delta+1}$. For $\alpha < \delta$, define $i_\alpha^{\delta+1} = i_\delta^{\delta+1} i_\alpha^\delta$. To see that (1) holds, observe that

$$p_\alpha^{\delta+1} i_\alpha^{\delta+1} = p_\alpha^{\delta+1} i_\delta^{\delta+1} i_\alpha^\delta = p_\alpha^\delta p_\delta^{\delta+1} i_\delta^{\delta+1} i_\alpha^\delta = p_\alpha^\delta i_\alpha^\delta = \mathrm{id}_{X_\alpha},$$

and

$$i_\gamma^{\delta+1} i_\alpha^\gamma = i_\delta^{\delta+1} i_\gamma^\delta i_\alpha^\gamma = i_\delta^{\delta+1} i_\alpha^\delta = i_\alpha^{\delta+1}.$$

Suppose now that β is a limit ordinal. Fix $\alpha < \beta$. Observe that for $\alpha \leqslant \xi < \eta < \beta$ we have

$$p^\eta_\xi i^\eta_\alpha = p^\eta_\xi i^\eta_\xi i^\xi_\alpha = i^\xi_\alpha.$$

Since X_β together with $\{p^\beta_\xi \colon \xi \in [\alpha, \beta)\}$ is the limit of $\{X_\xi; p^\eta_\xi \colon \alpha \leqslant \xi < \eta < \beta\}$, there exists a unique continuous map $i^\beta_\alpha \colon X_\alpha \to X_\beta$ such that

$$p^\beta_\xi i^\beta_\alpha = i^\xi_\alpha$$

holds for every $\xi \in [\alpha, \beta)$. In particular $p^\beta_\alpha i^\beta_\alpha = \mathrm{id}_{X_\alpha}$. Thus we have defined mappings i^β_α, for $\alpha < \beta$. It remains to check that $i^\beta_\gamma i^\gamma_\alpha = i^\beta_\alpha$ for $\alpha < \gamma < \beta$. To see this, observe that for $\xi \in [\gamma, \beta)$ we have

$$p^\beta_\xi (i^\beta_\gamma i^\gamma_\alpha) = i^\xi_\gamma i^\gamma_\alpha = i^\xi_\alpha,$$

and for $\xi \in [\alpha, \gamma)$ we have

$$p^\beta_\xi (i^\beta_\gamma i^\gamma_\alpha) = p^\gamma_\xi p^\beta_\gamma (i^\beta_\gamma i^\gamma_\alpha) = p^\gamma_\xi i^\gamma_\alpha = i^\xi_\alpha.$$

Since i^β_α is the unique map satisfying $p^\beta_\xi i^\beta_\alpha = i^\xi_\alpha$ for $\xi \in [\alpha, \beta)$, we get $i^\beta_\gamma i^\gamma_\alpha = i^\beta_\alpha$. □

Lemma 4.7. *Let $\{K; p_\alpha : \alpha < \kappa\}$ be the inverse limit of the continuous inverse sequence of compact spaces*

$$\{K_\alpha; p^\beta_\alpha \colon \alpha < \beta < \kappa\}$$

in which the bonding maps $p^{\alpha+1}_\alpha$ are retractions.

(a) *If for each $\alpha < \kappa$, $C(K_\alpha)$ has a τ_p-Kadec renorming, then $C(K)$ has a τ_p-Kadec renorming.*

(b) *Let $\{i^\beta_\alpha \colon \alpha < \beta < \kappa\}$ and $\{i_\alpha \colon \alpha < \kappa\}$ be collections of right inverses satisfying* (1) *and* (2) *of Proposition* 4.6. *Assume that $D \subseteq K$ is dense, and for each $\alpha < \kappa$, $i_\alpha p_\alpha[D] \subseteq D$ and $C(K_\alpha)$ has a $\tau_p(p_\alpha[D])$-Kadec renorming. Then $C(K)$ has a $\tau_p(D)$-Kadec renorming.*

Proof. (a) (Cf. the proof of [5, VI Theorem 7.6].) Let $\{i^\beta_\alpha \colon \alpha < \beta < \kappa\}$ and $\{i_\alpha \colon \alpha < \kappa\}$ be collections of right inverses given by Lemma 4.6. Let $R_\alpha = i_\alpha p_\alpha$. R_α is a retraction of K onto $i_\alpha[K_\alpha]$. If $\alpha < \beta$ then

$$R_\alpha R_\beta = i_\alpha\, p_\alpha\, i_\beta\, p_\beta = i_\alpha\, p^\beta_\alpha\, p_\beta\, i_\beta\, p_\beta = i_\alpha\, p^\beta_\alpha\, p_\beta = i_\alpha\, p_\alpha = R_\alpha$$

and

$$R_\beta R_\alpha = i_\beta\, p_\beta\, i_\alpha\, p_\alpha = i_\beta\, p_\beta\, i_\beta\, i_\alpha^\beta\, p_\alpha = i_\beta\, i_\alpha^\beta\, p_\alpha = i_\alpha\, p_\alpha = R_\alpha.$$

We also have $R_\alpha R_\alpha = R_\alpha$.

Let $P_\alpha\colon C(K) \to C(K)$ be given by $P_\alpha(f) = fR_\alpha$.

$C(K_\alpha)$ can be identified with the range of P_α via the linear map T defined by $Tg = gp_\alpha$. T is norm-preserving, and in particular one-to-one, because p_α maps onto K_α. From $Tg = gp_\alpha = gp_\alpha i_\alpha p_\alpha = gp_\alpha R_\alpha = P_\alpha(gp_\alpha)$ and $P_\alpha(f) = fR_\alpha = fi_\alpha p_\alpha = T(fi_\alpha)$, we see that the range of T is indeed the same as the range of P_α. Note that $T^{-1}(h) = hi_\alpha$. T is a τ_p-homeomorhism because for $x \in K$ and $y \in K_\alpha$, the maps $g \mapsto (Tg)(x) = g(p_\alpha x)$ and $h \mapsto (T^{-1}h)(y) = h(i_\alpha y)$ are τ_p-continuous. It follows from our assumption that the range of P_α has an equivalent τ_p-Kadec norm.

Then $\{P_\alpha\colon \alpha < \kappa\}$ is a sequence of projections of norm one satisfying the condition

$$\alpha < \beta \implies P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha.$$

For any $x \in K$, the map $f \mapsto P_\alpha(f)(x)$ is τ_p-continuous since it coincides with $f \mapsto f(R_\alpha x)$. Hence, P_α is (τ_p, τ_p)-continuous.

We now check that $\bigcup_{\alpha<\beta} P_\alpha C(K)$ is dense in $P_\beta C(K)$ for every limit ordinal $\beta \leqslant \kappa$. It will then follow that $\{P_\alpha\}_{\alpha<\kappa}$ satisfies the assumptions of Theorem 4.5 and the proof of (a) will be complete. We show that for each $f \in C(K)$,

$$\lim_{\alpha\to\beta} P_\alpha(f) = P_\beta(f).$$

Fix $\varepsilon > 0$. K_β has a base consisting of open sets of the form $(p_\alpha^\beta)^{-1}[U]$ where $\alpha < \beta$ and U is open in K_α. Hence, K_β is covered by finitely many such sets on which the oscillation of $f\, i_\beta$ is at most ε. By replacing the finitely many α's involved here by the largest of them, we may assume that they are all equal to some $\alpha_0 < \beta$. (If $\alpha < \alpha' < \beta$ and U is open in K_α, then since $p_\alpha^\beta = p_\alpha^{\alpha'} p_{\alpha'}^\beta$, we have $(p_\alpha^\beta)^{-1}[U] = (p_{\alpha'}^\beta)^{-1}[V]$ where $V = (p_\alpha^{\alpha'})^{-1}[U]$.) Thus we have open sets $U_1, \ldots, U_n$ in K_{α_0} such that the sets

$$(p_{\alpha_0}^\beta)^{-1}[U_1], \ldots, (p_{\alpha_0}^\beta)^{-1}[U_n]$$

cover K_β and on each of them the oscillation of $f\, i_\beta$ is at most ε. For any α such that $\alpha_0 \le \alpha < \beta$ and for any $x \in K$, letting $j \in \{1, \ldots, n\}$ be such that $p_{\alpha_0}(x) \in U_j$, we have

$$p_{\alpha_0}^\beta(i_\alpha^\beta p_\alpha(x)) = p_{\alpha_0}^\alpha\, p_\alpha(x) = p_{\alpha_0}(x) \in U_j,$$

so that

$$i_\alpha^\beta\, p_\alpha(x) \in (p_{\alpha_0}^\beta)^{-1}[U_j].$$

Clearly we also have

$$p_\beta(x) \in (p_{\alpha_0}^\beta)^{-1}[U_j],$$

and hence

$$\begin{aligned}|P_\alpha(f)(x) - P_\beta(f)(x)| &= |fR_\alpha(x) - fR_\beta(x)| = |f\, i_\alpha\, p_\alpha(x) - f\, i_\beta\, p_\beta(x)| \\ &= |f\, i_\beta(i_\alpha^\beta p_\alpha(x)) - f\, i_\beta(p_\beta(x))| \le \varepsilon.\end{aligned}$$

This completes the proof of (a).

The proof of (b) is obtained by making suitable adjustments to the proof of (a). We check that T is a $(\tau_p(p_\alpha[D]), \tau_p(i_\alpha\, p_\alpha[D]))$-homeomorphism. When $d \in D$, $g \mapsto (Tg)(i_\alpha\, p_\alpha(d)) = g(p_\alpha\, i_\alpha\, p_\alpha(d)) = g(p_\alpha(d))$ is $\tau_p(p_\alpha[D])$-continuous and $h \mapsto (T^{-1}h)(p_\alpha(d)) = h(i_\alpha\, p_\alpha(d))$ is $\tau_p(i_\alpha\, p_\alpha[D])$-continuous. Hence, our assumption gives that the range of P_α has an equivalent $\tau_p(i_\alpha\, p_\alpha[D])$-Kadec norm. It follows that the range of P_α has an equivalent $\tau_p(D)$-Kadec norm. For any $d \in D$, the map $f \mapsto P_\alpha(f)(d) = f(R_\alpha(d)) = f(i_\alpha\, p_\alpha(d))$ is $\tau_p(D)$-continuous. Hence, P_α is $(\tau_p(D), \tau_p(D))$-continuous. Finally, the fact that D is dense ensures that the unit ball of $C(K)$ is $\tau_p(D)$-closed. The rest of the proof is as for (a). □

Given a family of spaces $\{X_\alpha\}_{\alpha<\kappa}$, their product $\prod_{\alpha<\kappa} X_\alpha$ is the limit of a continuous inverse sequence of smaller products $\prod_{\xi<\alpha} X_\xi$, with the usual projections as bonding maps. This leads to the following.

Corollary 4.8. *Let $\{K_\alpha : \alpha < \kappa\}$ be a family of compacta and assume that for every finite $S \subseteq \kappa$, $C(\prod_{\alpha\in S} K_\alpha)$ has a τ_p-Kadec renorming. Then $C(\prod_{\alpha<\kappa} K_\alpha)$ has a τ_p-Kadec renorming.*

Proof. Proceed by induction on the cardinality of the index set, which we can assume is infinite. The induction hypothesis ensures that for each $\beta < \kappa$, $C(\prod_{\alpha<\beta} K_\alpha)$ has a τ_p-Kadec renorming. Now apply Lemma 4.7(a). □

In [13] an analogous result on the σ-fragmentability (with a version on LUR renormability) of products is proved. In [2] it is shown that the property of having a τ_p-lsc LUR renorming is productive in the sense that $C(\prod_{\alpha<\kappa} K_\alpha)$ has a τ_p-lsc LUR renorming if (and trivially only if) each $C(K_\alpha)$ has a τ_p-lsc LUR renorming. It is unknown whether the property of having a τ_p-Kadec renorming is productive in this sense.

Lemma 4.7 allows us to generalize Theorem 3.2 to infinite products.

Theorem 4.9. *Let $\{L_\alpha\colon \alpha < \kappa\}$ be a collection of compact linearly ordered spaces and for each $\alpha < \kappa$ let D_α be a dense subset of L_α which contains all pairs of adjacent points. Then $C(\prod_{\alpha<\kappa} L_\alpha)$ has an equivalent $\tau_p(\prod_{\alpha<\kappa} D_\alpha)$-Kadec norm.*

Proof. Proceed by induction on the cardinality of the index set. Theorem 3.2 takes care of the case $\kappa < \omega$. Assume that κ is an infinite cardinal and write $K = \prod_{\alpha<\kappa} L_\alpha$ and $K_\alpha = \prod_{\xi<\alpha} L_\xi$ for $\alpha < \kappa$. Note that K, equipped with the usual projections $p_\alpha\colon K \to K_\alpha$, is the inverse limit of the continuous inverse sequence $\{K_\alpha; p_\alpha^\beta\colon \alpha < \beta < \kappa\}$, where the p_α^β's are the usual projections. Fix a base point $d_\alpha \in D_\alpha$ for each $\alpha < \kappa$. For $\alpha < \beta < \kappa$, define embeddings

$$i_\alpha^\beta\colon \prod_{\xi<\alpha} L_\xi \to \prod_{\xi<\beta} L_\xi$$

by $i_\alpha^\beta(x)(\xi) = x(\xi)$ for $\xi < \alpha$ and $i_\alpha^\beta(x)(\xi) = d_\xi$ for $\alpha \le \xi < \beta$. By the induction hypothesis, $C(K_\alpha)$ has an equivalent $\tau_p(\prod_{\xi<\alpha} D_\xi)$-Kadec norm for each α, κ. The assumptions of part (b) of Lemma 4.7 are satisfied with $D = \prod_{\alpha<\kappa} D_\alpha$. □

Denote by $\mathcal{R}$ the minimal class of compact spaces which contains all metric compacta and is closed under limits of continuous inverse sequences of retractions. More formally, $\mathcal{R}$ is the smallest class of spaces which satisfies the following conditions:

1. Every metrizable compact space is in $\mathcal{R}$.
2. If $\mathbb{S} = \{X_\alpha; p_\alpha^\beta\colon \alpha < \beta < \kappa\}$ is a continuous inverse sequence such that each X_α is in $\mathcal{R}$ and each $p_\alpha^{\alpha+1}$ is a retraction, then every space homeomorphic to $\varprojlim \mathbb{S}$ belongs to $\mathcal{R}$.

Note that every Valdivia compact space belongs to $\mathcal{R}$ (see e.g. [15]). Also, for every ordinal ξ, the compact linearly ordered space $\xi + 1$ belongs to $\mathcal{R}$. If $\xi \geqslant \aleph_2$ then $\xi + 1$ is not Valdivia compact (see [15]). It is easy to see that class $\mathcal{R}$ is closed under products and direct sums.

Theorem 4.10. (a) *Assume K is a compact space such that $C(K)$ has a τ_p-Kadec renorming and assume $L \in \mathcal{R}$. Then $C(K \times L)$ has a τ_p-Kadec renorming.*

(b) *For every $L \in \mathcal{R}$, $C(L)$ has a τ_p-lsc LUR renorming.*

Proof. (a) Denote by $\mathcal{R}_0$ the class of all spaces $L \in \mathcal{R}$ such that $C(K \times L)$ has a τ_p-Kadec renorming. It suffices to show that $\mathcal{R}_0$ contains all metric compacta and is closed under limits of continuous inverse sequences of retractions.

The latter fact follows from Lemma 4.7, because if $L = \varprojlim\{L_\alpha; p_\alpha^\beta\colon \alpha < \beta < \kappa\}$ then $K \times L = \varprojlim\{K \times L_\alpha; q_\alpha^\beta\colon \alpha < \beta < \kappa\}$, where $q_\alpha^\beta = \mathrm{id}_K \times p_\alpha^\beta$. It remains to show that $\mathcal{R}_0$ contains all metric compacta. As every compact metric space is a continuous image of the Cantor set, it is enough to show that $C(K \times 2^\omega)$ has a τ_p-Kadec renorming.

We have $2^\omega = \varprojlim\{2^n; p_n^m\colon n < m < \omega\}$ so

$$K \times 2^\omega = \varprojlim\{K \times 2^n; q_n^m\colon n < m < \omega\},$$

where $q_n^m = \mathrm{id}_K \times p_n^m$. Clearly, $C(K \times 2^n)$ has a τ_p-Kadec renorming being a finite power of $C(K)$, so again Lemma 4.7 gives a τ_p-Kadec renorming of $C(K \times 2^\omega)$.

(b) It is enough to check that the class of all compact spaces K for which $C(K)$ has a τ_p-lsc LUR renorming is closed under inverse limits of retractions. Assume $K = \varprojlim \mathbb{S}$, where $\mathbb{S} = \{K_\alpha; r_\alpha^\beta\colon \alpha < \beta < \kappa\}$ is a continuous inverse sequence of retractions and for each $\alpha < \kappa$, $C(K_\alpha)$ has a τ_p-lsc LUR renorming. As in the proof of Lemma 4.7(a), there is a sequence of projections $\{P_\alpha\colon \alpha < \kappa\}$ on $C(K)$ such that P_α is adjoint to the retraction $r_\alpha\colon K \to K_\alpha$. Now apply Proposition VII.1.6 and Remark VII.1.7 from [5] to obtain a τ_p-lsc LUR renorming of $C(K)$. In fact, [5, Proposition VII.1.6] deals with projectional resolutions of the identity, but no assumption about the density of $\operatorname{im} P_\alpha$ is used in the proof. □

Remark 4.11. Note that by Proposition 2.1, Theorem 4.10(a) applies also when L is a continuous image of a space from $\mathcal{R}$. (If L' is a continuous image of L, then $K \times L'$ is a continuous image of $K \times L$.)

Example 4.12. In [23] an example of a compact, non-separable ccc space of countable π-character which has a continuous map onto the Cantor set in such a way that the fibers are relatively small linearly ordered spaces (their order type is an ordinal less than the additivity of Lebesgue measure). This space belongs to $\mathcal{R}$.

As in [23], we use Boolean algebraic language and work with the Boolean algebra whose Stone space is the required example.

Let $\mathbb{N}$ denote the set of positive natural numbers and denote by $\mathbb{N}[i]$ the set of all numbers of the form $2^i(2j-1)$, where $j \in \mathbb{N}$. Define $K = \{x \subseteq \mathbb{N}\colon (\forall\, i)\ |x[i]| \leqslant i\}$, where $x[i] = x \cap \mathbb{N}[i]$, and

$$Z = \{x \in K\colon \lim_{i\to\infty} |x[i]|/i = 0\}.$$

Denote by $\subseteq^*$ the *almost inclusion relation*, i.e. $a \subseteq^* b$ if $a \setminus b$ is finite. Define

$$T = \{(t, n)\colon n \in \mathbb{N},\ t \in K \text{ and } t \subseteq n\}.$$

We are going to define a subalgebra of $\mathcal{P}(T)/\operatorname{fin}$, where fin is the ideal of finite subsets of T. Let

$$T_{(t,n)} = \{(s,m) \in T \colon m \geqslant n \text{ and } s \cap n = t\}$$

and

$$T_a = \{(s,m) \in T \colon a \cap m \subseteq s\}.$$

Define $\mathbb{B}_0$ to be the subalgebra of $\mathcal{P}(T)/\operatorname{fin}$ generated by the classes of the sets $T_{(t,n)}$, $(t,n) \in T$. Then $\mathbb{B}_0$ is a countable free Boolean algebra. In what follows we shall identify subsets of T with their equivalence classes in $\mathcal{P}(T)/\operatorname{fin}$. The context should make it clear when classes are intended.

By [8, p. 151], there exists a sequence $A = \{a_\alpha \colon \alpha < \kappa\}$ of elements of Z such that $\alpha < \beta \implies a_\alpha \subseteq^* a_\beta$ and for every $a \in K$ there is $\alpha < \kappa$ such that $a_\alpha \not\subseteq^* a$. Moreover κ equals the additivity of the Lebesgue measure, so $\kappa > \aleph_0$. Let $\mathbb{B}_\alpha$ be the subalgebra of $\mathcal{P}(T)/\operatorname{fin}$ generated by

$$\mathbb{B}_0 \cup \{T_a \colon a \in K \ \& \ (\exists\, \xi < \alpha)\ a =^* a_\xi\}.$$

Finally, let $\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$ and let X be the Stone space of $\mathbb{B}$. It has been shown in [23] that X is a non-separable ccc space with countable π-character. Moreover, the inclusion $\mathbb{B}_0 \subseteq \mathbb{B}$ induces, by duality, a map from X onto the Cantor set such that all fibers are well-ordered of size $< \kappa$.

Theorem 4.13. *$X \in \mathcal{R}$ and consequently $C(X)$ has a τ_p-lsc LUR renorming.*

Proof. We will show by induction on $\alpha < \kappa$ that $\operatorname{Ult}(\mathbb{B}_\alpha) \in \mathcal{R}$ for every $\alpha < \kappa$ and that each quotient mapping $r_\alpha \colon \operatorname{Ult}(\mathbb{B}_{\alpha+1}) \to \operatorname{Ult}(\mathbb{B}_\alpha)$ induced by $\mathbb{B}_\alpha \subseteq \mathbb{B}_{\alpha+1}$ is a retraction. The latter property is equivalent to the existence of a retraction $h \colon \mathbb{B}_{\alpha+1} \to \mathbb{B}_\alpha$, i.e. a homomorphism such that $h \restriction \mathbb{B}_\alpha = \mathrm{id}_{\mathbb{B}_\alpha}$.

Fix $\alpha < \kappa$ and assume $\operatorname{Ult}(\mathbb{B}_\alpha) \in \mathcal{R}$. Given Boolean algebras $\mathbb{A} \subseteq \mathbb{B}$ and $x \in \mathbb{B} \setminus \mathbb{A}$ we will denote by $\mathbb{A}[x]$ the algebra generated by $\mathbb{A} \cup \{x\}$ ($\mathbb{A}[x]$ is called a *simple extension* of $\mathbb{A}$). Note the following

Claim 4.14. *Assume $a \subseteq a'$ are in K and $a' \setminus a$ is finite. Then $T_{a'} \in \mathbb{B}_0[T_a]$.*

Proof. Let $n \in \omega$ be such that $a' \setminus a \subseteq n$. Let $\mathcal{S} = \{s \subseteq n \colon s \in K$ and $a' \cap n \subseteq s\}$. Then $T_{a'} = T_a \cap \bigcup_{s \in \mathcal{S}} T_{(s,n)}$. $\square$

Define $\mathbb{B}_{\alpha+1}^{-1} = \mathbb{B}_\alpha$ and $\mathbb{B}_{\alpha+1}^{n+1} = \mathbb{B}_{\alpha+1}^{n}[T_{a_\alpha \setminus n}]$. By the above claim, $\mathbb{B}_{\alpha+1} = \bigcup_{n\in\omega} \mathbb{B}_{\alpha+1}^n$. We need to check that $\mathbb{B}_{\alpha+1}^n$ is a retract of $\mathbb{B}_{\alpha+1}^{n+1}$ and that $\operatorname{Ult}(\mathbb{B}_\alpha^{n+1}) \in \mathcal{R}$ for every $n \geqslant -1$.

Note that, by Sikorski's extension criterion (see e.g. [17, p. 67]), if $\mathbb{A}$ is a Boolean algebra and $\mathbb{A}[x]$ is a simple extension of $\mathbb{A}$ then $\mathbb{A}$ is a retract of $\mathbb{A}[x]$ iff there exists $c \in \mathbb{A}$ such that for every $a_0, a_1 \in \mathbb{A}$ with $a_0 \leqslant x \leqslant a_1$ we have $a_0 \leqslant c \leqslant a_1$. This holds for example, if $\{a \in \mathbb{A} \colon a \leqslant x\}$ has a least upper bound in $\mathbb{A}$.

We will need the following easy fact about our Boolean algebra. We leave the verification to the reader. Part (a) is like Claim 1 from the proof of [23, Theorem 8.4].

Claim 4.15. (a) *The sets $T_a \cap T_{(t,n)}$, where $a =^* a_\xi$ for some $\xi < \alpha$, are dense in $\mathbb{B}_\alpha$.*

(b) *For every nonnegative integer n, every element of $\mathbb{B}^n_{\alpha+1}$ is a finite sum of elements of the form $T_a \cap T_{(t,n)} \cap \neg T_{b_0} \cap \cdots \cap \neg T_{b_{k-1}}$, where $b_i =^* a_{\eta_i}$ for some $\eta_i \leqslant \alpha$ and $a =^* a_\xi$ for some $\xi < \alpha$ or $a = a_\alpha \setminus i$ where $i < n$.*

(c) *If $x = T_a \cap T_{(t,n)}$ and $0_{\mathbb{B}} < x \leqslant T_b$ then $b \subseteq a \cup t$.*

We consider separately the cases $n = -1$ and $n > -1$.

Case 1. $n = -1$. By Claim 4.15 (a) and (c), no non-zero element of $\mathbb{B}_\alpha$ is below T_{a_α}. Thus $\mathbb{B}_\alpha = \mathbb{B}^{-1}_{\alpha+1}$ is a retract of $\mathbb{B}^0_{\alpha+1}$. To see that $\mathrm{Ult}(\mathbb{B}^0_{\alpha+1}) \in \mathcal{R}$ it is enough to show that $\mathbb{B}_\alpha/\mathcal{I}$ is countable (and hence its Stone space is second countable), where $\mathcal{I} = \{x \in \mathbb{B}_\alpha \colon x \cap T_{a_\alpha} = 0_{\mathbb{B}}\}$, because $\mathrm{Ult}(\mathbb{B}^0_{\alpha+1})$ is the direct sum of $\mathrm{Ult}(\mathbb{B}_\alpha)$ and $\mathrm{Ult}(\mathbb{B}_\alpha/\mathcal{I})$. Let $q \colon \mathbb{B}_\alpha \to \mathbb{B}_\alpha/\mathcal{I}$ be the quotient map. Observe that for $\xi < \alpha$, $q(T_{a_\xi \cap a_\alpha \setminus n}) = 1_{\mathbb{B}_\alpha/\mathcal{I}}$, because $T_{a_\alpha} \leqslant T_{a_\xi \cap a_\alpha \setminus n}$. Now, by Claim 4.14, $\mathbb{B}_\alpha$ is generated by $\mathbb{B}_0 \cup \{T_a \colon a = a_\xi \cap a_\alpha \setminus n \ \&\ n \in \omega \ \&\ \xi < \alpha\}$. It follows that $\mathbb{B}_\alpha/\mathcal{I}$ is countable.

Case 2. $n > -1$. By Claim 4.15, we have $\sup\{x \in \mathbb{B}^n_{\alpha+1} \colon x \leqslant T_{a_\alpha \setminus n}\} = T_{a_\alpha \setminus (n-1)} \in \mathbb{B}^n_{\alpha+1}$. Hence $\mathbb{B}^n_{\alpha+1}$ is a retract of $\mathbb{B}^{n+1}_{\alpha+1}$. In order to see that $\mathrm{Ult}(\mathbb{B}^{n+1}_{\alpha+1}) \in \mathcal{R}$ it is enough to show that, as in Case 1, the quotient algebra $\mathbb{B}^n_{\alpha+1}/\mathcal{I}$ is countable, where $\mathcal{I} = \{x \in \mathbb{B}^n_{\alpha+1} \colon x \cap T_{a_\alpha \setminus n} = 0_{\mathbb{B}}\}$. This can be done by an argument similar to the one used as in Case 1. We now have new generators of the form $T_{a_\alpha \setminus i}$, $i < n$, but only finitely many of them, so the quotient $\mathbb{B}^n_{\alpha+1}/\mathcal{I}$ is still countable. □

Remark 4.16. If the additivity of the Lebesgue measure is $> \aleph_2$ then the space X from the above example is not a continuous image of a Valdivia compact space. Indeed, let κ denote the additivity of the Lebesgue measure and suppose that X is a continuous image of a Valdivia compact space. Let $h \colon X \to 2^\omega$ be a continuous map such that all fibers of h are well ordered of order type $< \kappa$ (see [23]). One can show that in fact there are fibers of arbitrary large order type below κ (see the proof of Claim 4 in [23, p. 74]). Hence, assuming $\kappa > \aleph_2$, there is $p \in 2^\omega$ such that $F = h^{-1}(p)$ has order type $> \aleph_2$. Observe that F is a G_δ subset of X and therefore it is also a continuous image of a Valdivia compact space (see [15]). On the other hand, a well ordered continuous image of a Valdivia compact space has order type $< \aleph_2$ (see [16]).

It can be shown that X is Valdivia compact if $\kappa = \aleph_1$. We do not know whether X is Valdivia compact if $\kappa = \aleph_2$.

5. A three-space property. We show that the three-space property for Kadec renormings holds under the assumption that the quotient space has an LUR renorming. This solves a problem raised in [18] where it is shown that a Banach space E has a Kadec-Klee renorming provided some subspace F has a Kadec-Klee renorming and E/F has an LUR renorming.

We begin with an auxiliary lemma on extending Kadec norms.

Lemma 5.1. *Let E be a Banach space and let F be a closed subspace of E. Assume τ is a weaker linear topology on E such that F and the unit ball of E are τ-closed and F has an equivalent τ-Kadec norm. Then there exists an equivalent τ-lsc norm $\|\cdot\|$ on E which is τ-Kadec on F, i.e. for every $y \in F$ with $\|y\| = 1$ and for every $\varepsilon > 0$ there exists $V \in \tau$ such that $y \in V$ and $\mathrm{S}_E \cap V \subseteq \mathrm{B}(y, \varepsilon)$, where S_E denotes the unit sphere of E with respect to $\|\cdot\|$.*

Proof. We use ideas from [20]. Let $\|\cdot\|_0$ be the original norm of E which, as we may assume, is τ-lsc and let $B \subseteq F$ denote the unit closed ball with respect to a given τ-Kadec norm. Let $G_n = \mathrm{cl}_\tau \mathrm{B}_{\|\cdot\|_0}(B, 1/n)$. Then each G_n is a convex, bounded, symmetric neighborhood of the origin in E. Denote by p_n the Minkowski functional of G_n and define

$$\|x\| = \sum_{n>0} \alpha_n p_n(x),$$

where $\{\alpha_n\}_{n\in\omega}$ is a sequence of positive reals making the above series convergent. Then $\|\cdot\|$ is an equivalent norm on E which is τ-lsc, because each p_n is τ-lsc. We show that $\|\cdot\|$ is τ-Kadec on F.

Fix $y \in F$ with $\|y\| = 1$ and fix $\varepsilon > 0$. By Proposition 2.3, find a τ-neighborhood W of y and $r > 1$ such that

$$y \in W \cap rB \subseteq B(y, \varepsilon/4).$$

We claim that there exist a smaller τ-neighborhood U of y, $n \in \mathbb{N}$ and $\gamma > 0$ such that

$$U \cap (r+\gamma)G_n \subseteq \mathrm{B}(y,\varepsilon). \tag{5.1}$$

First, find $W_0 \in \tau$ such that $y \in W_0$ and $W_0 + \mathrm{B}(0,\delta) \subseteq W$ for some $\delta > 0$. Then $W_0 \cap \mathrm{B}(rB,\delta) \subseteq \mathrm{B}(y,\varepsilon/4+\delta)$. Indeed, if $w \in W_0$ and $\|w-z\| < \delta$ for some $z \in rB$ then $z \in rB \cap (W_0 + \mathrm{B}(0,\delta)) \subseteq rB \cap W \subseteq \mathrm{B}(y,\varepsilon/4)$. Find $n \in \omega$ so small that $r/n \leqslant \delta$ and assume that $\delta < \varepsilon/4$. Then $W_0 \cap \mathrm{B}(rB,r/n) \subseteq \mathrm{B}(y,\varepsilon/2)$. Next, find $W_1, V \in \tau$ such that $y \in W_1$, $0 \in V = -V$ and $W_1 + V \subseteq W_0$. Then

$$W_1 \cap \mathrm{cl}_\tau(\mathrm{B}(rB,r/n)) \subseteq \mathrm{B}(y,\varepsilon/2) + V.$$

Indeed, if $w \in W_1 \cap \mathrm{cl}_\tau(\mathrm{B}(rB,r/n))$ then there is $z \in \mathrm{B}(rB,r/n)$ such that $z - w \in V$, so $z \in W_1 + V \subseteq W_0$ and hence $z \in \mathrm{B}(y,\varepsilon/2)$. As V can be an arbitrarily small τ-neighborhood of 0, it follows that

$$W_1 \cap \mathrm{cl}_\tau(\mathrm{B}(rB,r/n)) \subseteq \mathrm{cl}_\tau \mathrm{B}(y,\varepsilon/2) = \overline{\mathrm{B}}(y,\varepsilon/2).$$

The last equality follows from the fact that closed balls are τ-closed. Note that

$$\mathrm{cl}_\tau(\mathrm{B}(rB,r/n)) = r\,\mathrm{cl}_\tau(\mathrm{B}(B,1/n)) = rG_n.$$

Thus we have $W_1 \cap rG_n \subseteq \overline{\mathrm{B}}(y,\varepsilon/2)$. Finally, find a τ-neighborhood U of y and $\eta > 0$ such that $U + \mathrm{B}(0,\eta) \subseteq W_1$ and $\eta < \varepsilon/2$. Let $\gamma > 0$ be such that $\gamma G_n \subseteq \mathrm{B}(0,\eta)$. Fix $u \in U \cap (r+\gamma)G_n$. Then there is $z \in rG_n$ such that $u - z \in \gamma G_n \subseteq \mathrm{B}(0,\eta)$, so $z \in U + \mathrm{B}(0,\eta) \subseteq W_1$ and hence $z \in \overline{\mathrm{B}}(y,\varepsilon/2)$. Thus $u \in \overline{\mathrm{B}}(y,\varepsilon/2+\eta) \subseteq \mathrm{B}(y,\varepsilon)$. This finishes the proof of (5.1).

Now, using the fact that each p_n is τ-continuous on the $\|\cdot\|$-unit sphere, we may assume, shrinking U if necessary, that $p_n(x) < p_n(y)+\gamma$ whenever $x \in U$ and $\|x\| = 1$. Note that $p_n(y) \leqslant r$, since $r^{-1}y \in G_n$. Thus, if $x \in U$ and $\|x\| = 1$ then $p_n(x) < r+\gamma$ which means that $x \in (r+\gamma)G_n$ and hence $\|x-y\| < \varepsilon$. This the completes proof. □

Remark 5.2. If, in the above lemma, τ is the weak topology then the norm defined by

$$\|x\| = \|x\|_0 + \mathrm{dist}(x,F)$$

is Kadec on F, where $\|\cdot\|_0$ is any equivalent norm such that $(F, \|\cdot\|_0 \restriction F)$ has the Kadec property. This idea was used in [18]. In general, we do not know whether $\mathrm{dist}(\,\cdot\,,F)$ is τ-lsc.

The following lemma, stated for sequences instead of nets, is due to Haydon [10, Proposition 1.2] and it is a variation of a lemma of Troyanski (see [5, p. 271]) which is an important tool for obtaining LUR renormings.

Lemma 5.3. *Let X be topological space, let S be a set and let $\varphi_s, \psi_s \colon X \to [0,+\infty)$ be lower semi-continuous functions such that $\sup_{s\in S}(\varphi_s(x)+\psi_s(x)) < +\infty$ for every $x \in X$. Define*

$$\varphi(x) = \sup_{s\in S} \varphi_s(x), \quad \theta_m(x) = \sup_{s\in S}(\varphi_s(x) + 2^{-m}\psi_s(x)), \quad \theta(x) = \sum_{m\in\omega} 2^{-m}\theta_m(x).$$

Assume further that $\{x_\sigma\}_{\sigma\in\Sigma}$ is a net converging to $x \in X$ and $\theta(x_\sigma) \to \theta(x)$. Then there exists a finer net $\{x_\gamma\}_{\gamma\in\Gamma}$ and a net $\{i_\gamma\}_{\gamma\in\Gamma} \subseteq S$ such that

$$\lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x_\gamma) = \lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x) = \lim_{\gamma\in\Gamma} \varphi(x_\gamma) = \varphi(x)$$

and

$$\lim_{\gamma\in\Gamma}(\psi_{i_\gamma}(x_\gamma) - \psi_{i_\gamma}(x)) = 0.$$

Proof. By Proposition 2.6, we have $\lim_{\sigma\in\Sigma}\theta_m(x_\sigma) = \theta_m(x)$ for every $m \in \omega$. Thus, given $m \in \omega$, we can choose $j(m) \in S$ and $\sigma(m) \in \Sigma$ such that

$$\varphi_{j(m)}(x) + 2^{-m}\psi_{j(m)}(x) > \sup_{\sigma\geqslant\sigma(m)} \theta_m(x_\sigma) - 2^{-2m}$$

and

$$\varphi_{j(m)}(x_\sigma) > \varphi_{j(m)}(x) - 2^{-2m} \quad \text{and} \quad \psi_{j(m)}(x_\sigma) > \psi_{j(m)}(x) - 2^{-2m}$$

hold for $\sigma \geqslant \sigma(m)$. We may also assume that $\sigma(m_1) \leqslant \sigma(m_2)$ whenever $m_1 < m_2$. Define

$$\Gamma = \{(\sigma,m) \in \Sigma\times\omega \colon \sigma \geqslant \sigma(m)\}.$$

Consider Γ with the coordinate-wise order and define $h\colon \Gamma \to \Sigma$ by setting $h(\sigma,m) = \sigma$. Finally, define $i(\gamma) = j(m)$, where $\gamma = (\sigma,m) \in \Gamma$. Fix $\gamma = (\sigma,m) \in \Gamma$. We have, knowing that $i(\gamma) = j(m)$ and $\sigma \geqslant \sigma(m)$,

$$\varphi_{i(\gamma)}(x)+2^{-m}\psi_{i(\gamma)}(x) > \sup_{\xi\geqslant\sigma(m)} \theta_m(x_\xi)-2^{-2m} \geqslant \varphi_{i(\gamma)}(x_{h(\gamma)})+2^{-m}\psi_{i(\gamma)}(x_{h(\gamma)})-2^{-2m}.$$

The last inequality holds because $h(\gamma) = \sigma \geqslant \sigma(m)$. It follows that

$$|\varphi_{i(\gamma)}(x) - \varphi_{i(\gamma)}(x_{h(\gamma)})| < 2^{-2m+1} \quad \text{and} \quad |\psi_{i(\gamma)}(x) - \psi_{i(\gamma)}(x_{h(\gamma)})| < 2^{-m+1},$$

because $\varphi_{i(\gamma)}(x_{h(\gamma)}) > \varphi_{i(\gamma)}(x) - 2^{-2m}$ and $\psi_{i(\gamma)}(x_{h(\gamma)}) > \psi_{i(\gamma)}(x) - 2^{-2m}$. This shows that

$$\lim_{\gamma\in\Gamma} |\varphi_{i(\gamma)}(x) - \varphi_{i(\gamma)}(x_{h(\gamma)})| = 0 \quad \text{and} \quad \lim_{\gamma\in\Gamma} |\psi_{i(\gamma)}(x) - \psi_{i(\gamma)}(x_{h(\gamma)})| = 0. \tag{5.2}$$

We also have

$$\begin{aligned}\varphi_{i(\gamma)}(x) + 2^{-m}\sup_{s\in S}\psi_s(x) \geqslant \varphi_{i(\gamma)}(x) + 2^{-m}\psi_{i(\gamma)}(x) &\geqslant \sup_{\xi\geqslant\sigma(m)} \theta_m(x_\xi) - 2^{-2m}\\ &\geqslant \limsup_{\eta\in\Gamma}\varphi(x_{h(\eta)}) - 2^{-2m} \geqslant \liminf_{\eta\in\Gamma}\varphi(x_{h(\eta)}) - 2^{-2m}\\ &\geqslant \varphi(x) - 2^{-2m} \geqslant \varphi_{i(\gamma)}(x) - 2^{-2m}.\end{aligned}$$

Thus, passing to the limit, we get

$$\begin{aligned}\liminf\nolimits_{\gamma\in\Gamma}\varphi_{i(\gamma)}(x) \geqslant \limsup\nolimits_{\gamma\in\Gamma}\varphi(x_{h(\gamma)}) \geqslant \liminf\nolimits_{\gamma\in\Gamma}\varphi(x_{h(\gamma)})\\ \geqslant \varphi(x) \geqslant \limsup\nolimits_{\gamma\in\Gamma}\varphi_{i(\gamma)}(x).\end{aligned} \tag{5.3}$$

By (5.2) and (5.3), the proof is complete. □

Theorem 5.4. *Assume E is a Banach space and τ is a weaker linear topology on X such that the unit ball of E is τ-closed. Assume further that F is a closed subspace of E which has a τ-Kadec renorming and the quotient E/F has a τ'-lsc LUR renorming for some Hausdorff locally convex linear topology τ' on E/F such that the quotient map is (τ,τ') continuous. Then E has a τ-Kadec renorming.*

Note that since the unit ball of E/F under the LUR renorming is closed with respect to the weak topology on E/F generated by the τ'-continuous linear functionals, we could have equivalently assumed that τ' is the weak topology on E/F generated by a total subspace of E/F.

Proof. The assumptions imply that F is τ-closed, being the pre-image of a singleton under the quotient map. Let $\|\cdot\|$ be an equivalent τ-lsc norm on E which is τ-Kadec on F (Lemma 5.1). Denote by $|\cdot|_q$ the quotient norm on E/F. Let $|\cdot|$ be an LUR norm on E/F which is τ'-lsc. Write $\widehat{x}$ for $x+F$, i.e. the image of x under the quotient map.

Let $b\colon E/F \to E$ be a continuous selection for the quotient map obtained by Bartle-Graves Theorem so that for each $y \in E/F$, $b(y) \in y$, the range of b on the unit sphere of E/F is bounded in norm by a positive constant M, and $b(ty) = tb(y)$ whenever $t \geqslant 0$ (see [5, VII Lemma 3.2 and its proof]). Let $S = \{a \in E/F\colon |a| = 1\}$.

Since the unit ball for $|\cdot|$ is τ'-closed, the τ'-continuous functionals of unit norm for the dual norm $|\cdot|^*$ to $|\cdot|$ form a norming set for $(E/F, |\cdot|)$. For each $a \in S$ choose a τ'-continuous functional $f_a \in (E/F)^*$ such that $f_a(a) = 1$ and $|f_a|^* \leq 2$. Note that if $\|f_a\|$ denotes the norm of f_a with respect to $|\cdot|_q$, then the values $\|f_a\|$ are bounded. (We have $|f_a(y)| \leq |f_a|^* \, |y| \leq 2|y| \leq 2K|y|_q$ for some constant K and hence $\|f_a\| \leq 2K$.) By enlarging the constant M introduced above, we may assume that $\|f_a\| \leq M$ for each $a \in S$. Define $P_a x = f_a(\widehat{x}) b(a)$ and let ψ_a be the seminorm given by

$$\psi_a(x) = \|x - P_a x\|.$$

Note that ψ_a is τ-lsc, because P_a is a τ-continuous functional. Next, define

$$\varphi_a(x) = \inf\{r > 0 \colon |r^{-1}\widehat{x} + a| \leqslant 2\}.$$

Observe that φ_a is the Minkowski functional of the set $H_a = \{x \in E \colon |\widehat{x} + a| \leqslant 2\}$. H_a is a convex set containing 0 as an internal point, so φ_a satisfies the triangle inequality and is positively homogeneous. Because $x \mapsto \widehat{x}$ is (τ, τ')-continuous and $|\cdot|$ is τ'-lsc, H_a is a τ-closed set and thus φ_a is τ-lsc. Both families $\{\varphi_a \colon a \in S\}$ and $\{\psi_a \colon a \in S\}$ are pointwise bounded, specifically $\varphi_a(x) \leqslant |\widehat{x}|$ and $\psi_a(x) \leqslant (M^2 + 1)\|x\|$. Applying Lemma 5.3 we get a τ-lsc function θ satisfying the assertion of that lemma and such that $\|x\|_\theta = \theta(x) + \theta(-x)$ defines a τ-lsc semi-norm on E. Define $\|\cdot\|_K$ on E by

$$\|x\|_K = \|x\| + |\widehat{x}| + \|x\|_\theta.$$

This is a norm equivalent to $\|\cdot\|$. It is τ-lsc since each of the three terms defines a τ-lsc function of x. By Corollary 2.5, the restriction to the unit sphere for $\|\cdot\|_K$ of each of these three functions is τ-continuous.

We will show that $\|\cdot\|_K$ is a τ-Kadec norm on E.

Fix $x \in E$ with $\|x\|_K = 1$ and fix a net $\{x_\sigma\}_{\sigma\in\Sigma}$ which τ-converges to x and $\|x_\sigma\|_K = 1$ for every $\sigma \in \Sigma$. We will be done if we find a finer net converging in norm. We may assume that $x \notin F$, so that $\widehat{x} \neq 0$. Since $\|\cdot\|_\theta$ is τ-continuous on the sphere, $\lim_{\sigma\in\Sigma} \|x_\sigma\|_\theta = \|x\|_\theta$. From the definition of $\|\cdot\|_\theta$ and Proposition 2.6, we have $\lim_{\sigma\in\Sigma} \theta(x_\sigma) = \theta(x)$, so by Lemma 5.3 we get a finer net, which we still denote by $\{x_\sigma\}_{\sigma\in\Sigma}$ and a net $\{a_\sigma\}_{\sigma\in\Sigma}$ such that

$$\lim_{\sigma\in\Sigma} (\|x_\sigma - P_{a_\sigma} x_\sigma\| - \|x - P_{a_\sigma} x\|) = 0 \tag{5.4}$$

and

$$\lim_{\sigma\in\Sigma}\varphi_{a_\sigma}(x_\sigma)=\lim_{\sigma\in\Sigma}\varphi_{a_\sigma}(x)=\lim_{\sigma\in\Sigma}\sup_{a\in S}\varphi_a(x_\sigma)=\sup_{a\in S}\varphi_a(x). \tag{5.5}$$

Now observe that $\sup_{a\in S}\varphi_a(x)=|\widehat{x}|$. Indeed, we have $||\widehat{x}|^{-1}\widehat{x}+a|\leqslant 2$, so $\varphi_a(x)\leqslant|\widehat{x}|$ for every $a\in S$. On the other hand, if $a=|\widehat{x}|^{-1}\widehat{x}$, then

$$|r^{-1}\widehat{x}+a|=(r^{-1}+|\widehat{x}|^{-1})\cdot|\widehat{x}|=r^{-1}|\widehat{x}|+1,$$

so $|r^{-1}\widehat{x}+a|\leqslant 2$ iff $r\geqslant|\widehat{x}|$ which shows that $\varphi_a(x)=|\widehat{x}|$.

Let $t=|\widehat{x}|^{-1}$.

Claim 5.5. *$\lim_{\sigma\in\Sigma}a_\sigma=t\widehat{x}$.*

Proof. By (5.5) we have $\lim_{\sigma\in\Sigma}\varphi_{a_\sigma}(x)=|\widehat{x}|=t^{-1}$. This means that for every ε such that $0<\varepsilon<t^{-1}$ there exists $\sigma(\varepsilon)\in\Sigma$ such that $|r^{-1}\widehat{x}+a_\sigma|\geqslant 2$ whenever $r\leqslant t^{-1}-\varepsilon$ and $\sigma\geqslant\sigma(\varepsilon)$. Observe that $r^{-1}\widehat{x}$ has norm close to 1, when r is close to $|\widehat{x}|^{-1}$. By LUR, this implies that a_σ must be close to $t\widehat{x}$. More formally, fix $\sigma\geqslant\sigma(\varepsilon)$ and let $r=t^{-1}-\varepsilon$ and observe that

$$\begin{aligned}2&\leqslant|t\widehat{x}+a_\sigma+(r^{-1}-t)\widehat{x}|\\&\leqslant|t\widehat{x}+a_\sigma|+(r^{-1}-t)t^{-1}=|t\widehat{x}+a_\sigma|+\Big(\frac{1}{1-\varepsilon t}-1\Big).\end{aligned}$$

It follows that $\liminf_{\sigma\in\Sigma}|t\widehat{x}+a_\sigma|\geqslant 2$. As $|t\widehat{x}|=1$, the LUR property of $|\cdot|$ implies $\lim_{\sigma\in\Sigma}a_\sigma=t\widehat{x}$. □

By the (τ,τ')-continuity of the quotient map and the τ-continuity of $x\mapsto|\widehat{x}|$ on the unit sphere, we have τ'-$\lim_{\sigma\in\Sigma}\widehat{x}_\sigma=\widehat{x}$ and $\lim_{\sigma\in\Sigma}|\widehat{x}_\sigma|=|\widehat{x}|$. As $|\cdot|$ is a τ'-lsc LUR norm, it is τ'-Kadec and hence by Proposition 2.3 $\lim_{\sigma\in\Sigma}|\widehat{x}_\sigma-\widehat{x}|=0$.

Claim 5.6. *$\lim_{\sigma\in\Sigma}P_{a_\sigma}x=b(\widehat{x})$.*

Proof.

$$\begin{aligned}t\|P_{a_\sigma}x-b(\widehat{x})\|&=\|f_{a_\sigma}(t\widehat{x})b(a_\sigma)-f_{a_\sigma}(a_\sigma)b(t\widehat{x})\|\\&\leqslant\|f_{a_\sigma}(t\widehat{x}-a_\sigma)\cdot b(t\widehat{x})\|+\|f_{a_\sigma}(t\widehat{x})(b(a_\sigma)-b(t\widehat{x}))\|\\&\leqslant M\Big(|t\widehat{x}-a_\sigma|_q\cdot\|b(t\widehat{x})\|+\|tx\|\cdot\|b(a_\sigma)-b(t\widehat{x})\|\Big).\end{aligned}$$

By Claim 5.5, we have $\lim_{\sigma\in\Sigma}|t\widehat{x}-a_\sigma|_q=0$ and $\lim_{\sigma\in\Sigma}\|b(a_\sigma)-b(t\widehat{x})\|=0$ and hence the claim holds. □

Claim 5.7. *$\lim_{\sigma\in\Sigma}\|P_{a_\sigma}x_\sigma-P_{a_\sigma}x\|=0$.*

Proof. We have

$$\|P_{a_\sigma}x_\sigma - P_{a_\sigma}x\| = \|f_{a_\sigma}(\widehat{x}_\sigma - \widehat{x})b(a_\sigma)\| \leqslant \|f_{a_\sigma}\| \cdot |\widehat{x}_\sigma - \widehat{x}|_q \cdot \|b(a_\sigma)\| \leqslant M^2 |\widehat{x}_\sigma - \widehat{x}|_q,$$

from which the claim follows since $\lim_{\sigma\in\Sigma} |\widehat{x}_\sigma - \widehat{x}| = 0$ as explained above. □

In order to finish the proof of the theorem, note that (5.4) and Claim 5.6 give

$$\lim_{\sigma\in\Sigma}(\|x_\sigma - P_{a_\sigma}x_\sigma\| - \|x - b(\widehat{x})\|) = 0.$$

Because τ is weaker than the norm topology, Claim 5.6 and Claim 5.7 give $\tau\text{-}\lim_{\sigma\in\Sigma} P_{a_\sigma}x_\sigma = b(\widehat{x})$ and hence $\tau\text{-}\lim_{\sigma\in\Sigma}(x_\sigma - P_{a_\sigma}x_\sigma) = x - b(\widehat{x})$. Thus $\lim_{\sigma\in\Sigma} \|(x_\sigma - P_{a_\sigma}x_\sigma) - (x - b(\widehat{x}))\| = 0$, because $\|\cdot\|$ is τ-Kadec on F and $x - b(\widehat{x}) \in F$. (If $x - b(\widehat{x}) = 0$, use the last displayed equation above instead of this argument.) Therefore we have

$$\|x_\sigma - x\| \leqslant \|(x_\sigma - P_{a_\sigma}x_\sigma) - (x - b(\widehat{x}))\| + \|P_{a_\sigma}x_\sigma - P_{a_\sigma}x\| + \|P_{a_\sigma}x - b(\widehat{x})\|.$$

Since all three of terms on the right tend to 0, we are done. □

Corollary 5.8. *Assume X is a locally compact space such that $C_0(X)$ has a τ_p-Kadec renorming and K is a compactification of X such that $C(K \setminus X)$ has a τ_p-lsc LUR renorming. Then $C(K)$ has a τ_p-Kadec renorming.*

Proof. Define $T\colon C(K) \to C(K \setminus X)$ by setting $Tf = f \restriction (K \setminus X)$. Then T is a bounded, pointwise continuous linear operator onto $C(K \setminus X)$ and $\ker T = C_0(X)$. Thus $C(K \setminus X)$ is isomorphic to $C(K)/C_0(X)$. Apply Theorem 5.4 for $E = C(K)$, $F = C_0(X)$ and τ, τ' the respective pointwise convergence topologies. □

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