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ON CONNECTION BETWEEN CHARACTERESTIC FUNCTIONS AND THE CARATHEODORI CLASS FUNCTIONS

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Abstract. Connection of characteristic functions $S(z)$ of nonunitary operator T with the functions of Caratheodori class is established. It was demonstrated that the representing measures from integral representation of the function of Caratheodori's class are defined by restrictions of spectral measures of unitary dilation, of a restricted operator T on the corresponding defect subspaces.

Introduction. The main tool to investigate a nonself-adjoint operator A is the characteristic function $S(\lambda)$ [2, 3]. For the operators of the class in question it is important that a homographic transformation over $S(\lambda)$ reduces to Nevanlinna function class corresponding to the restriction of the real part of resolvent A_R of operator the A onto the non-hermicity space of the main operator A. Problems of a similar nature concerned with a nonunitary operator T are

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studied in the present work. We have shown that homographic transformation of a characteristic function $S(Z)$ [1, 4], corresponding to the operator T, results in two operator-functions belonging to the Caratheodory class. It was ascertained that for these functions there exist upper and lower majorization inside and outside the unit disk.

1. Let T be a bounded linear operator defined in Hilbert space a H. An ensemble

$$\triangle_T = \left(\sigma_E, H \oplus E, V = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix}, H \oplus F, \sigma_E\right) \tag{1}$$

is called a unitary metric colligation [1, 3] if the operator

$$V = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix} : H \oplus E \to H \oplus F$$

satisfies the formulas

$$V^* \begin{bmatrix} I & 0 \\ 0 & \sigma_F \end{bmatrix} V = \begin{bmatrix} I & 0 \\ 0 & \sigma_E \end{bmatrix}; V \begin{bmatrix} I & 0 \\ 0 & \sigma_E^{-1} \end{bmatrix} V^* = \begin{bmatrix} I & 0 \\ 0 & \sigma_F^{-1} \end{bmatrix} \tag{2}$$

where σ_E and σ_F are self-adjoint invertible operators defined in the Hilbert spaces E and F respectively. It is known [1] that for any bounded operator T there always exists Hilbert spaces E and F and the corresponding operators

$$\Phi : E \to H, \Psi : H \to F, K : E \to F, \sigma_E : E \to F, \sigma_F : F \to F,$$

so that (2) holds.
The function

$$S_{\triangle_T} = K + \Psi(ZI - T)^{-1}\Phi,$$

is called the characteristic function of the colligation $\triangle_T$ [1, 4].
Assume that the point $z = 1$ does not belong to the spectrum of the operator T.
Let us define the operator A,

$$A = i(I + T)(I - T)^{-1}, \tag{3}$$

which is the Cayley transformation of the operator T [3]. The inverse transformation is

$$T = (A - iI)(A + iI)^{-1}. \tag{4}$$

Remind [3] that the ensemble

$$\triangle \ = (A, H, \varphi, E, \sigma), \tag{5}$$

is called a local colligation if

$$A - A^* = i\varphi^* \sigma \varphi, \tag{6}$$

where A is a linear bounded operator in the Hilbert space H, σ is a self-adjoint operator in the Hilbert space E and $\varphi : H \to E$.
Let us include the operator A (3) into a local colligation

$$\triangle_A = (A, H, \varphi, \tilde{E}, \sigma_A), \tag{7}$$

where,

$$A - A^* = i\varphi^* \sigma_A \varphi, \tag{8}$$

and the characteristic function [3] of the colligation $\triangle_A$ is equal to

$$S_{\triangle_A(\lambda)} = I - i\varphi(A - \lambda I)^{-1} \varphi^* \sigma_A. \tag{9}$$

It is readily seen, that

$$\begin{cases} I - T^*T = 2(A^* - iI)^{-1} \dfrac{A - A^*}{i} (A + iI)^{-1}, \\ \\ I - TT^* = 2(A - iI)^{-1} \dfrac{A - A^*}{i} (A^* + iI)^{-1}. \end{cases} \tag{10}$$

Taking into account the colligation relation (2) for $\triangle_T(1)$ and for $\triangle_A$ (8) we have

$$(11) \qquad \begin{cases} 2L^*\varphi^*\sigma\varphi L = \Psi^*\sigma_F\Psi, \\ 2L\varphi^*\sigma_A\varphi L^* = \Psi^*\sigma_E^{-1}\Psi^*, \end{cases}$$

where,

$$L = (A + iI)^{-1} = \frac{1}{2i}(I - T).$$

We call the colligation $\triangle_A(7)$ *minimal* if the non-hermicity of the space $\overline{(A - A^*)H}$ coincides with the canal subspace $\overline{\varphi^*\tilde{E}}$ [3],

$$\overline{(A - A^*)H} = \overline{\varphi^*\tilde{E}}\ .$$

In a similar manner we will call the unitary metric colligation $\triangle_T(1)$ minimal if

$$\begin{cases} D_T = \overline{(T^*T - I)H} = \overline{\Psi^*F}; \\ D_{T^*} = \overline{(TT^* - I)H} = \overline{\Phi E}. \end{cases}$$

From (10) it follows that in the case of minimality of the colligations $\triangle_T$ and $\triangle_A$, the spaces D_T and $\overline{(A - A^*)H}$ as well as D_{T^*} and $\overline{(A - A^*)H}$ are isomorphic. This property allows us to define operators R_E and R_F by formulas

$$(12) \qquad \begin{array}{ll} R_E\Phi^*L^{*^{-1}}h \stackrel{def}{=} \sqrt{2}\varphi h; & (E \ \to \ \tilde{E}), \\ R_F\Psi L^{-1}h \stackrel{def}{=} -\sqrt{2}\varphi h; & (F \ \to \ \tilde{E}), \end{array}$$

where $h \in H$.

Theorem 1. *Suppose that the point $z = 1$ is a regular one for the main operator T of the colligation $\triangle_T$ (1). If the operator A is the Cayley transformation (3) of the operator T, then the characteristic functions of the minimal colligations $\triangle_T$ (1) and $\triangle_A$ (4) are related by the formula*

$$S_{\triangle_A(\lambda)}R_E = R_F S_{\triangle_T}\left(\frac{\lambda - i}{\lambda + i}\right)\sigma_E^{-1}.$$

Proof. By virtue of minimality of colligations $\triangle_T$ and $\triangle_A$ it is sufficient to verify (13) on the respective dense sets. Really,

$$
\begin{aligned}
& R_F S_{\triangle_T}(\frac{\lambda-i}{\lambda+i})\sigma_E^{-1}\Phi^* L^{*^{-1}} h \\
= \ & R_F \left\{ K\sigma_E^{-1}\Phi^* + \Psi\left(\frac{\lambda-i}{\lambda+i} - (A-iI)(A+iI)^{-1}\right)^{-1} \Phi\sigma_E^{-1}\Phi^* \right\} L^{*^{-1}} h \\
= \ & R_F \left\{ K\sigma_E^{-1}\Phi^* L^{*^{-1}} + \frac{1}{2i}\Psi L^{-1}(\lambda I - A)^{-1} L^{-1}\Phi\sigma_E^{-1}\Phi^* L^{*^{-1}} + \right. \\
& \qquad \left. + \frac{1}{2i}\Psi L^{-1}\Phi\sigma_E^{-1}\Phi^* L^{*^{-1}} \right\} h.
\end{aligned}
$$

Using the colligation relation (2) $\Psi T^* + K\ \sigma_E^{-1}\Phi^* = 0$ and (4) we obtain

$$
K\sigma_E^{-1}\Phi^* L^{*^{-1}} + \Psi(A^* + iI) = 0.
$$

Since

$$
\frac{1}{2i}\Psi L^{-1}\Phi\sigma_E^{-1}\Phi^* L^{*^{-1}} = \frac{1}{i}\Psi\varphi^*\sigma_A\varphi = -\Psi(A - A^*),
$$

by virtue of (11) and (8). Thus

$$
\begin{aligned}
& R_F S_{\triangle_T}(\frac{\lambda-i}{\lambda+i})\sigma_E^{-1}\Phi^* L^{*^{-1}} h \\
= \ & R_F \left\{ -\Psi(A+iI) + i\Psi L^{-1}(A-\lambda I)^{-1}\varphi^*\sigma_A\varphi \right\} h \\
= \ & -R_F\Psi L^{-1} h + iR_F\Psi L^{-1}\left[(A-\lambda I)^{-1}\varphi^*\sigma_A\varphi h\right] \\
= \ & \sqrt{2}\varphi h - i\sqrt{2}\varphi(A-\lambda I)^{-1}\varphi^*\sigma_A\varphi h.
\end{aligned}
$$

Thus

$$
R_F S_{\triangle_T}(\frac{\lambda-i}{\lambda+i})\sigma_E^{-1}\Phi^* L^{*^{-1}} = S_{\triangle_A}(\lambda) R_E \Phi^* L^{*^{-1}} h. \qquad \square
$$

It is necessary to note that the relation between characteristic functions (13) of Cayley transformation (3) allows to study nonbounded operators. Really, if we consider a symmetric densely defined operator with a regular point $\lambda = -1$, then its Cayley transformation (4) reduces to an isometric operator with corresponding domain and range. Realizing (nonunitary!) continuation of this isometry onto the whole space H we obtain the operator T.

2. As is well known [2, 3] transition to the diagonal of an open system, associated with a local colligation, reduces to the Nevanlinna function class. This function class is related to characteristic functions by the homographic transformation and is the restriction of the real part of the main operator resolvents onto the canal subspace.

Let us give an appropriate analogue of this transformation for unitary metric colligations.

Let us write down an equation for an open system $F_\triangle = \{R_\triangle, S_\triangle\}$ [1]

$$F_\triangle : \begin{cases} R_\triangle : x_{n+1} = Tx_n + \Phi u_n, \\ S_\triangle : v_n = \Psi x_n + K u_n, \end{cases} \tag{14}$$

and for a dual system $F_\triangle^+ = \{R_\triangle^+, S_\triangle^+\}$ [1]

$$F_\triangle^+ : \begin{cases} R_\triangle^+ : x_n = T^* x_{n+1} + \Psi^* \sigma_F v_n, \\ S_\triangle^+ : u_n = \sigma_E^{-1} \Phi^* x_{n+1} + \sigma_E^{-1} K^* \sigma_F v_n, \end{cases} \tag{15}$$

which are associated with a unitary metric colligation $\triangle$ (1). Let us apply the operator T to the mapping equation $R_\triangle^+$ (15). Then by making use of the colligation relation $T\Psi^* + \Phi\sigma_E^{-1}K = 0$ one may derive that

$$TT^* x_{n+1} = Tx_n + \Phi\sigma_E^{-1} K^* \sigma_F v.$$

By summing this equality and the equation $R_\triangle(14)$ we find out that

$$B_{T^*} x_{n+1} = 2Tx_n + \Phi(u_n + K^+ v_n),$$

where, $B_{T^*} = TT^* + I$ is a self-adjoint invertible operator and $K^+ = \sigma_E^{-1} K^* \sigma_F$ is an adjoint to K operator relative to σ_E and σ_F forms (i.e. $\langle \sigma_F K u, v\rangle = \langle \sigma_E u, K^+ v\rangle$).

Let a pair of linear mappings $F_E(d) = \{R_E(d), V_E(d)\}$ be the E- *diagonal of the open systems* $F_\triangle$ and $F_\triangle^+$, –

$$F_E(d) : \begin{cases} R_E(d) : B_{T^*} x_{n+1} = 2Tx_n + \Phi u_n^-, \\ V_E(d) : u_n^+ = \sigma_E^{-1} \Phi^* x_{n+1}, \end{cases} \tag{16}$$

where the input u_n^- and the output u_n^+ of the E-diagonal $F_E(d)$ are equal to

$$u_n^- = u_n + K^+ v_n, \qquad u_n^+ = u_n - K^- v_n.$$

"Fourier transform" of the diagonal $F_E(d)$ (16) ($h_n \to z^n h_0$) leads to the function

$$V_E(z) = z\Phi^*(zB_{T^*} - 2T)^{-1}\Phi, \tag{17}$$

which (in the virtue of $u_0 = S_{\triangle^+}(z)v_0$) is a homographic transformation of the function $S_{\triangle^+}(z)$, –

$$V_E(z)[S_{\triangle^+}(z) + K^+] = \sigma_E[S_{\triangle^+}(z) - K^+]. \tag{18}$$

Let us evaluate $Re\,\langle\sigma_E u_n^+, u_n^-\rangle$. It is easy to see that

$$\begin{aligned} Re\left\langle\sigma_E u_n^+, u_n^-\right\rangle &= \langle\sigma_E u_n, u_n\rangle - \left\langle\sigma_E K^+ v_n, K^+ v_n\right\rangle \\ &= \langle\sigma_E u_n, u_n\rangle - \langle\sigma_F v_n, v_n\rangle + \left\langle(\sigma_F - (K^+)^*\sigma_E K^+)v_n, v_n\right\rangle. \end{aligned}$$

Using the isometry (2) one may easily reduce the right-hand side to the form

$$= \|x_{n+1}\|^2 - \|x_n\|^2 + \left\langle(\sigma_F - (K^+)^*\sigma_E K^+)v_n, v_n\right\rangle.$$

Since $V_E(z)u_0^- = \sigma_E u_0^+$ and $u_0 = S_{\triangle^+}(z)v_0$, where $v_0 = v_0(z)$ and $u_0^\pm = u_0^\pm(z)$ then

$$\left\langle ReV_E(z)u_0^-, u_0^-\right\rangle = \left(|z|^2 - 1\right)\|x_0\|^2 + \|\Psi^*\sigma_F v_0\|^2 \tag{19}$$

It means, that if $|z| > 1$ the real part $ReV_E(z)$ is majorized below by a Hermitian-positive majorant, such that

$$\|\Psi^*\sigma_F v_0\|^2 = \left\langle(\sigma_F - (K^+)^*\sigma_E K^+)v_0, v_0\right\rangle \geq 0.$$

Formula (19) leads to

$$\begin{aligned} &\left(S_{\triangle^+}(z) + K^+\right)^* ReV_E(z)\left(S_{\triangle^+}(z) + K^+\right) \\ =\ &S_{\triangle^+}^*(z)\sigma_E S_{\triangle^+}(z) - \sigma_F - \left((K^+)^*\sigma_E K^+ - \sigma_F\right). \end{aligned} \tag{20}$$

Thus the above majorization is such that for $|z| > 1$ implies

(21) $ReV_E(z) \geq \left(\left[S_{\triangle^+}(z)+K^+\right]^*\right)^{-1}\left(\sigma_F-(K^+)^*\sigma_E K^+\right)\left(S_{\triangle^+}(z)+K^+\right)^{-1}.$

3. Let us apply the operator T^* to the equation R of the system (14), then

$$TT^*x_n = T^*x_{n+1} + \Psi^*\sigma_E K u_n,$$

(since $T^*\Phi + \Psi^*\sigma_F K = 0$).
By summing this relation with the equation $R_\triangle^+$ (15) we derive that

$$B_T x_{n+1} = 2T^*x_{n+1} + \Psi^*\sigma_F(Ku_n + v_n),$$

where $B_T = T^*T + 1$.
Let a pair of linear mappings

$$F_F(d) : \begin{cases} R_E(d): \ B_T x_n = 2T^*x_{n+1} + \Psi^*\sigma_F v_n^-, \\ V_E(d): \ v_n^+ = \Psi x_n, \end{cases} \tag{22}$$

where,

$$v_n^- = Ku_n + v_n, \qquad v_n^+ = v_n - Ku_n.$$

be the F-diagonal of the open systems $F_\triangle$ and $F_\triangle^*$.
The operator-function

$$V_F(z) = \Psi(B_T - 2zT^*)^{-1}\Psi^*, \tag{23}$$

which is a homographic transformation of the function $S_\triangle(z)$,

$$V_F(z)\sigma_F[K + S_\triangle(z)] = [S_\triangle(z) - K]. \tag{24}$$

corresponds to the mapping $V_F(d)$.
Evidently,

$$\begin{aligned} Re\left\langle \sigma_F v_n^-, v_n^+\right\rangle &= \langle \sigma_F v_n, v_n\rangle - \langle K^*\sigma_F K u_n, u_n\rangle \\ &= \langle \sigma_F v_n, v_n\rangle - \langle \sigma_E u_n, u_n\rangle + \|\Phi x_n\|^2 = \end{aligned}$$

using (2) allows to transform the right-hand side to the form

$$= \|x_n\|^2 - \|x_{n+1}\|^2 + \|\Phi u_n\|^2 .$$

Since

$$V_F(z)\sigma_F v_0^- = v_0^+, \text{ and } v_0 = S_\triangle(z)u_0,$$

then,

$$\left\langle ReV_F(z)\sigma_E v_0^-, \sigma_F v_0^- \right\rangle = \left(1 - |z|^2\right) \|x_0\|^2 + \|\Phi u_0\|^2 . \tag{25}$$

In that case, if $|z| < 1$, $ReV_F(z)$ is majorized below by the Hermitian-positive majorant, which generates the expression

$$\|\Phi u_0\|^2 = \langle (\sigma_E - K^* \sigma_F K)u_0, u_0\rangle \geq 0.$$

Thus there exists

$$\begin{aligned} & [K + S_\triangle(z)]^* \sigma_F ReV_F(z) \sigma_F [K + S_\triangle(z)] \\ = \; & S_\triangle^*(z)\sigma_F S_\triangle(z) - \sigma_E - (K^*\sigma_F K - \sigma_E), \end{aligned} \tag{26}$$

and the above majorization means that

$$ReV_F(z) \geq \left([K + S_\triangle(z)]^*\right)^{-1} (\sigma_E - K^*\sigma_F K) \left(K + S_{\triangle^+}(z)\right)^{-1}, \tag{27}$$

if $|z| < 1$.
To sum up the reasoning of the sections **2, 3** we may formulate the following theorem.

Theorem 2. *The homographic transformation* (18) ((24)) *leads to the transfer mapping* $V_E(z)(V_F(z))$ *of the* $E(F)$*-diagonal of the open systems* (16) ((22)). *In that case the real part of* $ReV_E(z)$ $(ReV_F(z))$ *outside the disk* $|z| > 1$ *or inside the disk* $|z| < 1$ *has from below the Hermitian-positive majorant* (21) ((27)) *from below. Furthermore* $V_E(z)$ *and* $V_F(z)$ *are related by the formula*

$$V_F(z)\sigma_F K = K\sigma_E^{-1} V_E^* \left(\frac{1}{\bar{z}}\right). \tag{28}$$

To derive (28) it is necessary to determine $S_{\triangle}(z)$ from (24)

$$S_{\triangle}(z) = -K + 2\left(I - V_F(z)\sigma_F\right)^{-1} K,$$

and the function $S_{\triangle^+}(z)$ from (18)

$$S_{\triangle^+}(z) = -K^+ + 2\left(\sigma_E - V_E(z)\right)^{-1} \sigma_E K^+.$$

Using this equality by virtue of the equation $S_{\triangle^+}(z) = \sigma_E^{-1} S^*(\frac{1}{\bar{z}})\sigma_F$, we obtain

$$S_{\triangle}(z) = -K + 2K\left(I - \sigma_E^{-1} V_E^*\left(\frac{1}{\bar{z}}\right)\right)^{-1},$$

that leads to the formula (28) after equating of the right-hand parts.

Remark 1. It follows from the formula (20) that the majorization (21)of the real part of the function $V_E(z)$ guarantees realization of the J-properties of the characteristic function $S_{\triangle}(z)$ outside the disk $|z| > 1$, and from formula (26) it follows that $V_F(z)$ guarantees realization of the J-properties of the $S_{\triangle}(z)$ inside the disk $|z| < 1$ in the virtue of the lower majorant (27).

Remark 2. Recall [2, 5] that a function $f(z)$ belongs to the Caratheodori class (C) if $f(z)$ maps the disk $|z| < 1$ into the half plane $Re\omega \geq 0$. Such functions have the corresponding Riesz-Herglotz integral representation [2, 5]. It follows from (21) and (27) that $E(F)$-diagonals $V_E^*\left(\frac{1}{\bar{z}}\right)$ $(V_F(z))$ are the operator-functions of the Caratheodori type, but with slightly more rigid condition concerning the respective lower majorant.

4. It is not difficult to see that

$$\begin{aligned} \tilde{S}(z) &= S_{\triangle}^*(0)\sigma_F S_{\triangle}(z) = \sigma_E - z\Phi^*(zI - T)^{-1}\Phi, \\ \tilde{S}'(z) &= S_{\triangle}(z)\sigma_E^{-1} S_{\triangle}^*(0) = \sigma_F^{-1} - z\Psi(zI - T)^{-1}\Psi^*. \end{aligned} \tag{29}$$

Elementary calculations show that

$$\begin{aligned} W(z) &= -2\tilde{S}(z) + \tilde{S}(0) + \sigma_E = \Phi^*(T + zI)(-T + zI)^{-1}\Phi, \\ W'(z) &= -2\tilde{S}'(z) + \tilde{S}'(0) + \sigma_F^{-1} = \Psi(T + zI)(-T + zI)^{-1}\Psi^*. \end{aligned} \tag{30}$$

Let T is a contraction , $\|T\| \leq 1$. Include the operator T into the minimal unitary metric colligation

$$\triangle = \left(I_E, H \oplus E, V = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix}, H \oplus F, I_F \right), \tag{31}$$

where,

$$\begin{array}{rcl} E & = & D_{T^*} = \overline{(I - TT^*)H}., \quad F = D_T = \overline{(I - T^*T)H} \quad , \quad \Phi = |I - TT^*|^{\frac{1}{2}}, \\ \Psi & = & \Phi = |I - TT^*|^{\frac{1}{2}}, \quad K = -T^* \mid_E, (\text{see } [1, 4]). \end{array}$$

Recall some information concerning function theory and dilation theory [4]. For the function $\omega = f(z)$ of the class C (Caratheodori), analytic with $|z| < 1$ and performing mapping the disk $|z| < 1$ into the half plane $Re\omega \geq 0$, the Riesz-Herglotz integral representation [2, 5] is valid

$$f(z) = i\, Im\, f(0) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta), \tag{32}$$

where $\sigma(\theta)$ is a non-decreasing function of constrained variation, which in essence is defined unambiguously by $f(z)$.

A subspace $L \subset G$ is called *wandering* for isometric V in G if $V^k L \perp V^s L$ where $k, s \in Z, (k \neq s)$.

Theorem 3 [4]. *Let U in G is a minimal unitary dilation for completely nonunitary contraction of T in H. Then the subspaces*

$$L = \overline{(U - T)H} \quad , \quad L^* = \overline{(U^* - T^*)H}$$

are the wandering subspaces for U and moreover

$$G = span \left\{ l^2_Z(L), l^2_Z(L^*) \right\} = \cdots \oplus U^* L^* \oplus L^* \oplus H \oplus L \oplus UL \oplus \cdots,$$

where

$$l^2_Z(L) = \sum_{-\infty}^{\infty} \oplus U^k L, \quad l^2_Z(L^*) = \sum_{-\infty}^{\infty} \oplus U^k L^*.$$

It is evident that the function

$$W(\xi) = W(\xi^{-1}) = \Phi^*(I + \xi T)(I - \xi T)^{-1}\Phi$$

is analytical function if $|\xi| < 1$.
If

$$U^* = \int_{-\pi}^{\pi} e^{i\varphi} dE(\varphi)$$

is the spectral representation of U^*, then

$$\begin{aligned} (I + \xi T)(I - \xi T)^{-1} &= P_H (I + \xi U)(I - \xi U)^{-1} \mid_H \\ &= P_H (U^* + \xi)(U^* - \xi)^{-1} \mid_H \\ &= P_H \int_{-\pi}^{\pi} \frac{e^{i\varphi} + \xi}{e^{i\varphi} - \xi} dE(\varphi) \mid_H \end{aligned}$$

Thus $\hat{W}(\xi)$ belongs to the Caratheodori class since

$$W(\xi) = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + \xi}{e^{i\varphi} - \xi} dF(\varphi), \tag{33}$$

where $F(\varphi) = \Phi^* E(\varphi)\Phi$ is nonnegative, non-decreasing operator- function in the space E if $\varphi \in [-\pi, \pi]$.
It is evident that

$$L_* = UL^* = \overline{(I - UT^*)H},$$

and in the virtue of inclusion in the colligation (31) it is clear that $\overline{\Phi E} = P_H L_*$. From the representation of the subspace G in Theorem 3 and wandering property of the subspace L^* we have $L_* \subset H \oplus L$. Thus, every vector $f_{L_*} \in L_*$ may be expanded into the sum

$$f_{L_*} = f_H \oplus f_L.$$

It is evident that

$$(U^* + I\xi)(U^* - I\xi)^{-1} = \sum_0^\infty \xi^k U^k + \sum_1^\infty \xi^{k+1} U^{k+1}, \quad (|\xi| < 1)\,.$$

Therefore

$$(34)\ \ (U^*{+}I\xi)(U^*{-}I\xi)^{-1} f_{L_*} = (U^*{+}I\xi)(U^*{-}I\xi)^{-1} f_H + (U^*{+}I\xi)(U^*{-}I\xi)^{-1} f_L,$$

thus in the virtue of the above remarks

$$\begin{aligned}(U^* + \xi)(U^* - \xi)^{-1} f_{L_*} &\in l^2_Z(L_*),\\ (U^* + I\xi)(U^* - I\xi)^{-1} f_L &\in l^2_{Z_+}(L) = \sum_0^\infty U^k L.\end{aligned}$$

Therefore

$$P_H(U^* + I\xi)(U^* - I\xi)^{-1} f_{L_*} = P_H(U^* + I\xi)(U^* - I\xi)^{-1} f_H.$$

Thus we conclude that "bordering" of $\Phi^* E(\varphi)\Phi$ means restriction of the spectral measure $E(\varphi)$ on $l^2_Z(L_*)$.

Theorem 4. *Functions $W(\xi) = W(\xi^{-1})$ and $\hat{W}'(\xi) = W'(\xi^{-1})$, halomorphic inside the unit disk $|\xi| < 1$ belong to the Caratheodori class. For $\hat{W}(\xi)$ and $\hat{W}'(\xi)$ there exists Riesz-Herglotz integral representation* (33) *on the measures $dF(\varphi) = \Phi^* dE(\varphi)\Phi$ and $dF'(\varphi) = \Psi dE(\varphi)\Psi^*$ respectively. The spectral measure $dE(\varphi)$ of the unitary minimal dilation U of the operator T in the case when $dF(\varphi)$ describes the restriction of the operator U onto a subspace of the double-sided shift operator $l^2_Z(L_*)$ and in the case of $dF'(\varphi)$ onto $l^2_Z(L)$.*

Remark 3. It is reasonable that the representing measures $dF(\varphi)$ and $dF'(\varphi)$ must be connected with each other by some concordance condition. The assertion may be deduced by the space G structure (Theorem 3.). The problem of dilation constructing by the representing measures $dF(\varphi)$ and $dF'(\varphi)$ taking into account the above mentioned concordance conditions of these measures is quite natural and interesting.

Remark 4. For non-selfadjoint operators, the resolvent narrowing on the defect subspace leads to Nevanlinna function class, and exactly the same type of problems has been studied in the work of Naboko S. N. [6] in the very complete fashion. In this case, there appear two functions $W(z)$ and $W'(z)$ (30) from Caratheodori class, which gives us two representative measures $dF(\varphi)$ and $dF'(\varphi)$, and this leads us to new non-standard type of problems in this field.

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