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A CHARACTERIZATION THEOREM FOR THE K -FUNCTIONAL ASSOCIATED WITH THE ALGEBRAIC VERSION OF TRIGONOMETRIC JACKSON INTEGRALS

T. Zapryanova

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ABSTRACT. The purpose of this paper is to present a characterization of a certain Peetre K -functional in $L_p[-1, 1]$ norm, for $1 \leq p \leq 2$ by means of a modulus of smoothness. This modulus is based on the classical one taken on a certain linear transform of the function.

1. Introduction.

1.1. Notations. Let X be a normed space. For a given “differential” operator D we set $X \cap D^{-1}(X) = \{g \in X : Dg \in X\}$. Let X be one of the spaces $L_p[-1, 1]$, $1 \leq p < \infty$ or $C[-1, 1]$. In this case we denote the norm in X by $\|\cdot\|_p$, $1 \leq p \leq \infty$, where $\|\cdot\|_\infty$ means the uniform norm. Two examples of the operator D are

$$D_1g := \varphi(\varphi g')', \quad D_2g := \varphi^2 g'', \quad \text{where } \varphi(x) = \sqrt{1-x^2}.$$

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We define for every $f \in X$ and $t > 0$ the K -functionals

$$(1.1) \quad K(f, t; X, Y, D_1) := \inf \left\{ \|f - g\|_p + t \|\varphi(\varphi g')'\|_p : g \in Y \right\},$$

$$(1.2) \quad K(f, t; X, Y, D_2) := \inf \left\{ \|f - g\|_p + t \|\varphi^2 g''\|_p : g \in Y \right\},$$

where Y is a given subspace of $X \cap D_1^{-1}(X)$ or $X \cap D_2^{-1}(X)$, respectively.

The K -functional (1.1) with $X = C[-1, 1]$, $Y = C^2[-1, 1]$ is equivalent to the approximation error of Jackson type operator $G_{s,n}$ in uniform norm (such equivalence was established in [5]), while the K -functional (1.2) with $X=L_p[-1, 1]$, $Y = C^2[-1, 1]$ is well-known and is equivalent to the approximation error of Bernstein polynomials in the interval $[0, 1]$ ($p = \infty$) and characterizes the best polynomial approximations ($1 \leq p \leq \infty$).

We recall that the operator $G_{s,n} : C[-1, 1] \rightarrow \Pi_{sn-s}$ is defined by (see [4])

$$G_{s,n}(f, x) = \pi^{-1} \int_{-\pi}^{\pi} f(\cos(\arccos x + v)) K_{s,n}(v) dv,$$

where

$$K_{s,n}(v) = c_{n,s} \left(\frac{\sin(nv/2)}{\sin(v/2)} \right)^{2s}, \quad \pi^{-1} \int_{-\pi}^{\pi} K_{s,n}(v) dv = 1.$$

Π_r denotes the set of all algebraic polynomials of degree not exceeding r (r is natural number).

Notation $\Phi(f, t) \sim \Psi(f, t)$ means that there is a positive constant γ , independent of f and t , such that $\gamma^{-1}\Psi(f, t) \leq \Phi(f, t) \leq \gamma\Psi(f, t)$.

By c we denote positive constants, independent of f and t , that may differ at each occurrence.

For r – natural number we denote

$$C^r[a, b] = \left\{ f : f, f', \dots, f^{(r)} \in C[a, b] \text{ (continuous function in } [a, b]) \right\}$$

1.2. Known results. The idea for the equivalence of the approximation errors of a given sequence of operators and the values of proper K -functionals was studied systematically in [1]. Such equivalence was established for the algebraic version of trigonometric Jackson integrals $G_{s,n}$ and K -functionals (1.1) in uniform norm in [5] (see Theorem A).

Theorem A. For $s \geq 2$ and every $f \in C[-1, 1]$ we have

$$\|f - G_{s,n}f\|_{\infty} \sim K \left(f, \frac{1}{n^2}; C[-1, 1], C^2, D_1 \right).$$

Using a linear transform of functions in [3] Ivanov compares the K -functional

$$(1.3) \quad K(f, t; X, Y, D_3) := \inf \left\{ \|f - g\|_p + t \|(\psi g')'\|_p : g \in Y \right\},$$

with the already characterized K -functional

$$(1.4) \quad K(f, t; X, Y, D_4) := \inf \left\{ \|f - g\|_p + t \|\psi g''\|_p : g \in Y \right\},$$

where $\psi(x) = x(1 - x)$; X is one of the spaces $L_p[0, 1]$, $1 \leq p < \infty$ or $C[0, 1]$; Y is a given subspace of $X \cap D_3^{-1}(X)$ or $X \cap D_4^{-1}(X)$, respectively; $D_3g := (\psi g')'$, $D_4g := \psi g''$.

Ivanov proved the following

Theorem B. *For every $t \in (0, 1]$ and $f \in L_1[0, 1]$ we have*

$$K(f, t; L_1[0, 1], C^2, D_3) \sim K(Bf, t; L_1[0, 1], C^2, D_4) + tE_0(f)_1,$$

where

$$(Bf)(x) = f(x) + \int_{1/2}^x \left(\frac{x}{y^2} - \frac{1-x}{(1-y)^2} \right) f(y) dy$$

and $E_0(f)_1$ denotes the best approximation of f in $L_1[0, 1]$ by constant.

1.3. New results. The aim of this paper is to define a modulus that is equivalent to the K -functional (1.1) for $1 \leq p \leq 2$. We apply the method presented in [3].

First, let us note that the K -functionals $K(f, t; L_p[-1, 1], C^2, D_1)$ and $K(f, t; L_p[-1, 1], C^2, D_2)$ are not equivalent. The inequality $K(f, t; L_p[-1, 1], C^2, D_2) \leq cK(f, t; L_p[-1, 1], C^2, D_1)$ is not true for a fixed c , every f , every $t \in (0, 1]$ and $1 \leq p \leq 2$ because of functions like (with small positive ε)

$$f_\varepsilon(x) = \begin{cases} \arcsin x, & x \in [-1 + \varepsilon, 1 - \varepsilon]; \\ ax^3 + bx + d, & x \in [1 - \varepsilon, 1]; \\ ax^3 + bx - d, & x \in [-1, -1 + \varepsilon]; \end{cases}$$

where a, b, d are chosen such that $f \in C^2$.

But these K -functionals can become equivalent if in the one of them instead f stays Af for appropriate operator A .

Let $f \in L_1[-1, 1]$. For every $-1 < x < 1$ we define the value of the operator A by

$$(1.5) \quad (Af)(x) = f(x) + \frac{1}{2} \int_0^x \left(\frac{1+x}{(1+y)^2} - \frac{1-x}{(1-y)^2} \right) f(y) dy.$$

Using operator (1.5) we prove

Theorem 1. *For every $t \in (0, 1]$ and $f \in L_p[-1, 1]$, $1 \leq p \leq 2$, we have*

$$K(f, t; L_p[-1, 1], C^2, D_1) \sim K(Af, t; L_p[-1, 1], C^2, D_2).$$

We mention that in Theorem 1 there is no additional term $tE_0(f)_p$ in the equivalence relation, while in Theorem B there is. Moreover, the equivalence in Theorem B is valid only for $p = 1$, while Theorem 1 holds for $1 \leq p \leq 2$. Although the operators D_1 and D_3 are similar we cannot reduce one to another. We can write the operator $D_1g(x)$ of the form:

$$(D_1g)(x) = (1 - x^2)g''(x) - xg'(x).$$

On the other hand, the analogue of D_3 for the interval $[-1, 1]$ is

$$\tilde{D}_3G(y) = (1 - y^2)G''(y) - 2yG'(y),$$

i.e. \tilde{D}_3G differs from D_1G by constant multiplier 2 in the term containing G' .

From Theorem 1 and characterizations of some weighted Peetre K -functionals in terms of weighted moduli established in [2, Ch. 2, Theorem 2.1.1] we get

Corollary 1. *For $f \in L_p[-1, 1]$, $t \in (0, 1]$ and $1 \leq p \leq 2$ with $\phi = \sqrt{1 - x^2}$ we have*

$$K(f, t; L_p[-1, 1], C^2, D_1) \sim \omega_\phi^2(Af, \sqrt{t})_p,$$

where ω_ϕ^2 is Ditzian-Totik modulus of smoothness, introduced in [2].

The equivalence in Theorem 1 is no longer true for $2 < p < \infty$ as the following example shows. Let $F(x) = \arcsin x$. We have $E_0(F)_p \sim 1$ and thus $ct \leq K(F, t; L_p[-1, 1], C^2, D_1)$ for $2 < p < \infty$ (see Lemma 4). On the other hand $K(AF, t; L_p[-1, 1], C^2, D_2) = 0$ for every p because $AF(x) = x$, i. e. $AF \in C^2[-1, 1]$ and $D_2(AF) = \varphi^2(x)(AF)'' = 0$.

The connection between the K -functionals of f and Af with D_1 and D_2 as differential operators respectively, is not so satisfactory when $2 < p < \infty$. We have

Theorem 2. *For every $t \in (0, 1]$ and $f \in L_p[-1, 1]$, $2 < p < \infty$, we have*

$$K(f, t; L_p[-1, 1], C^2, D_1) \leq c \left[K(Af, t^{\frac{1}{p} + \frac{1}{2}}; L_p[-1, 1], C^2, D_2) + t^{\frac{1}{p} + \frac{1}{2}} E_0(f)_p \right],$$

$$K(Af, t; L_p[-1, 1], C^2, D_2) + tE_0(f)_p \leq cK(f, t; L_p[-1, 1], C^2, D_1).$$

The proof of Theorem 1 follows the scheme. First we establish in Lemma 2 the equivalence

$$K(f, t; L_p[-1, 1], Z_1, D_1) \sim K(Af, t; L_p[-1, 1], Z_2, D_2) \text{ for } 1 \leq p < \infty,$$

where Z_1 and Z_2 are suitable subspaces of C^2 (see Definition 2). On the other hand these variations of Y produce K -functionals equal to the K -functionals we compare in Theorem 1. In Lemma 1 and Lemma 3 b) respectively we prove that

$$K(f, t; L_p[-1, 1], Z_1, D_1) = K(f, t; L_p[-1, 1], C^2, D_1) \text{ for } 1 \leq p < \infty \text{ and}$$

$$K(F, t; L_p[-1, 1], Z_2, D_2) = K(F, t; L_p[-1, 1], C^2, D_2) \text{ for } 1 \leq p \leq 2.$$

The last two relation we obtain using Lemma 2 from [3, p.116]. We state this lemma, as we use it several times.

Definition 1. For given $Y \subset X \cap D^{-1}(X)$ and a positive number γ we define $S_\gamma(Y)$ as the set of all $g \in X \cap D^{-1}(X)$ such that for every $\varepsilon > 0$ there is $h \in Y$ such that $\|g - h\| < \varepsilon$ and $\|Dh\| < \gamma \|Dg\| + \varepsilon$.

Lemma B. Let $Y_1, Y_2 \subset X \cap D^{-1}(X)$ and $\rho > 0$. Then for a given positive γ the following statements are equivalent:

- i) $K(f, t; X, Y_1, D) \leq K(f, \gamma t; X, Y_2, D)$ for every $f \in X, 0 < t$.
- ii) $K(f, t; X, Y_1, D) \leq K(f, \gamma t; X, Y_2, D)$ for every $f \in X, 0 < t \leq \rho$.
- iii) $Y_2 \subset S_\gamma(Y_1)$.

In particular, i) with $\gamma = 1$ holds when $Y_2 \subset Y_1$.

Theorems 1 and 2 are proved in Section 3.

2. Properties of the operators. In the next statement we collect some properties of operator A .

Theorem 3. a) A is a linear operator, satisfying $\|Af\|_p \leq 2\|f\|_p$ for every $1 \leq p \leq \infty$.

b) $Af = f$ for every $f \in \Pi_0$.

c) If $f, f' \in AC_{loc}(-1, 1)$, then $(Af)(0) = f(0)$, $(Af)'(0) = f'(0)$ and

$$\varphi^2(x)(Af)''(x) = \varphi(x)(\varphi(x)f'(x))', \quad -1 < x < 1.$$

Proof. We write (1.5) as

$$(Af)(x) = f(x) + \frac{1}{2} \int_{-1}^1 R(x, y) f(y) dy,$$

where the kernel $R : (-1, 1) \times (-1, 1) \rightarrow \mathbb{R}$ is given as follows: $R(0, y) = 0$; for $x \in (-1, 0)$ we have $R(x, y) = (1 - x)(1 - y)^{-2} - (1 + x)(1 + y)^{-2}$ if $y \in (x, 0)$ and $R(x, y) = 0$ if $y \notin (x, 0)$; for $x \in (0, 1)$ we have $R(x, y) = (1 + x)(1 + y)^{-2} - (1 - x)(1 - y)^{-2}$ if $y \in (0, x)$ and $R(x, y) = 0$ if $y \notin (0, x)$.

Set $x_0 = \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$. Then for a fixed $x \in (-1, 0)$ the kernel $R(x, y)$ is negative for $y \in (x, x_0)$ and positive for $y \in (x_0, 0)$; for a fixed $x \in (0, 1)$ the kernel $R(x, y)$ is positive for $y \in (0, x_0)$ and negative for $y \in (x_0, x)$. Thus

$$\int_{-1}^1 |R(x, y)| dy = 2 - 2\sqrt{1-x^2} \leq 2.$$

Hence $\|Af\|_\infty \leq 2\|f\|_\infty$.

Set $y_0 = \frac{(1+y)^2 - (1-y)^2}{(1+y)^2 + (1-y)^2}$. Then for a fixed $y \in (-1, 0)$ the kernel $R(x, y)$ is positive for $x \in (-1, y_0)$ and negative for $x \in (y_0, y)$; for a fixed $y \in (0, 1)$ the kernel $R(x, y)$ is negative for $x \in (y, y_0)$ and positive for $x \in (y_0, 1)$. Thus for $y \neq 0$ we have

$$\int_{-1}^1 |R(x, y)| dx = 1 - \frac{4}{(1+y)^2 + (1-y)^2} + \frac{2}{\max\{(1+y)^2, (1-y)^2\}} \leq 1.$$

Hence $\|Af\|_1 \leq \frac{3}{2}\|f\|_1$. Now the Riesz-Thorin theorem proves a).

Part b) follows from $\int_{-1}^1 R(x, y) dy = 0$.

Part c) follows from (1.5) by direct computation. \square

The operator A is invertible and we give an explicit formula for its inverse operator A^{-1} . Let every $f \in L_1[-1, 1]$ and $-1 < x < 1$ we set

$$(A^{-1}f)(x) = f(x) + \int_0^x \left(\frac{y}{1-y^2} + \frac{\arcsin y - \arcsin x}{(1-y^2)^{\frac{3}{2}}} \right) f(y) dy.$$

In the next statement we collect some properties of A^{-1} .

Theorem 4. a) A^{-1} is a linear operator, $\|A^{-1}f\|_p \leq c\|f\|_p$ for every $1 \leq p < \infty$.

b) $A^{-1}f = f$ for every $f \in \Pi_0$

c) $A^{-1}Af = AA^{-1}f = f$ for every $f \in L_1[-1, 1]$.

d) If $f, f' \in AC_{loc}(-1, 1)$, then $(A^{-1}f)(0) = f(0)$, $(A^{-1}f)'(0) = f'(0)$

and

$$\varphi(x)(\varphi(x)(A^{-1}f)'(x))' = \varphi^2(x)f''(x), \quad -1 < x < 1.$$

Proof. a) We have

$$|\arcsin 1 - \arcsin z| \leq c\sqrt{1 - z^2} \quad \text{for } z \in [0, 1]$$

and

$$|\arcsin(-1) - \arcsin z| \leq c\sqrt{1 - z^2} \quad \text{for } z \in [-1, 0].$$

As $0 < y < x$ and $\arcsin z$ is increasing $|\arcsin y - \arcsin 1| > |\arcsin y - \arcsin x|$.
Now we estimate

$$\begin{aligned} & \int_0^1 \left| \int_0^x \frac{1}{(1 - y^2)^{\frac{3}{2}}} (\arcsin y - \arcsin x) f(y) dy \right|^p dx \\ & \leq \int_0^1 \left(\int_0^x \frac{1}{(1 - y^2)^{\frac{3}{2}}} |(\arcsin y - \arcsin 1) f(y)| dy \right)^p dx \\ & \leq c \int_0^1 \left(\int_0^x \frac{\sqrt{1 - y^2}}{(1 - y^2)^{\frac{3}{2}}} |f(y)| dy \right)^p dx = c \int_0^1 \left(\int_0^x \frac{|f(y)|}{1 - y^2} dy \right)^p dx \\ & \leq c \int_0^1 \left(\int_0^1 \frac{|f(y)|}{1 - y} dy \right)^p dx \quad (y \rightarrow 1 - y) \\ & = c \int_0^1 \left(\int_x^1 \frac{|f(1 - y)|}{y} dy \right)^p dx \quad (\text{Hardy inequality}) \\ & \leq c \int_0^1 \left(y \frac{|f(1 - y)|}{y} \right)^p dy = c \int_0^1 |f(1 - y)|^p dy = c \|f\|_{p[0,1]}^p. \end{aligned}$$

Similarly, using Hardy inequality we get

$$\int_0^1 \left| \int_0^x \frac{y}{1-y^2} f(y) dy \right|^p dx \leq c \int_0^1 \left(\int_0^x \frac{|f(y)|}{1-y} dy \right)^p dx \leq c \|f\|_{p[0,1]}^p,$$

$$\int_{-1}^0 \left| \int_0^x \frac{1}{(1-y^2)^{\frac{3}{2}}} (\arcsin y - \arcsin x) f(y) dy \right|^p dx \leq c \|f\|_{p[-1,0]}^p \quad \text{and}$$

$$\int_{-1}^0 \left| \int_0^x \frac{y}{1-y^2} f(y) dy \right|^p dx \leq c \|f\|_{p[-1,0]}^p.$$

From these inequalities we get

$$\left\{ \int_{-1}^1 \left| \int_0^x \left(\frac{y}{1-y^2} + \frac{\arcsin y - \arcsin x}{(1-y^2)^{\frac{3}{2}}} \right) f(y) dy \right|^p dx \right\}^{\frac{1}{p}} \leq c \|f\|_{p[-1,1]}.$$

This proves a).

Part b) follows from

$$\int_0^x \left(\frac{y}{1-y^2} + \frac{1}{(1-y^2)^{\frac{3}{2}}} (\arcsin y - \arcsin x) \right) dy = 0.$$

Finally, c) and d) can be obtained by direct computation. \square

The action of the operators A and A^{-1} on the function $f(x) = x$ is given below:

$$(A(\cdot))(x) = \frac{1}{2}(1+x) \ln(1+x) - \frac{1}{2}(1-x) \ln(1-x),$$

$$(A^{-1}(\cdot))(x) = \arcsin x.$$

Definition 2. Set $Z_1 = \{f \in C^2[-1, 1] : f'(-1) = 0, f'(1) = 0\}$,

$$Z_2 = \left\{ f \in C^2[-1, 1] : \int_{-1}^0 \frac{x}{\sqrt{1-x^2}} f'(x) dx = 0, \int_0^1 \frac{x}{\sqrt{1-x^2}} f'(x) dx = 0 \right\}.$$

Theorem 5. a) $(Af)''(x)$ is continuous at $x = -1$ and at $x = 1$ for every $f \in Z_1$.

b)
$$\int_{-1}^0 \frac{x}{\sqrt{1-x^2}} (Af)'(x) dx = \int_0^1 \frac{x}{\sqrt{1-x^2}} (Af)'(x) dx = 0 \text{ for every } f \in Z_1.$$

c) $(A^{-1}f)''(x)$ is continuous at $x = -1$ and at $x = 1$ for every $f \in Z_2$.

d) $(A^{-1}f)'(-1) = (A^{-1}f)'(1) = 0$ for every function $f \in Z_2$.

e) $A(Z_1) = Z_2$ and $A^{-1}(Z_2) = Z_1$.

Proof. For every function $f \in Z_1$ we have that $f'(x) = (x-1)f''(1) + o(1-x)$ and $f'(x) = (x+1)f''(-1) + o(1+x)$. From Theorem 3 c) we have

$$(Af)''(x) = f''(x) - \frac{x}{1-x^2} f'(x),$$

which together with the above representations gives $(Af)''(x) = \frac{3}{2}f''(1) + o(1)$ for x close to 1 and $(Af)''(x) = \frac{3}{2}f''(-1) + o(1)$ for x close to -1 . This proves a).

We write the derivative of Af as

$$\begin{aligned} (Af)'(x) &= f'(x) - \frac{x}{1-x^2} f(x) + \int_0^x \frac{1+y^2}{(1-y^2)^2} f(y) dy \\ &= f'(x) - \frac{x}{1-x^2} f(x) + \int_0^x f(y) d\frac{y}{1-y^2} = f'(x) - \int_0^x \frac{y}{1-y^2} f'(y) dy. \end{aligned}$$

Then we have

$$\begin{aligned} &\int_0^1 \frac{x}{\sqrt{1-x^2}} (Af)'(x) dx \\ &= \lim_{\varepsilon \rightarrow 0+} \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} (Af)'(x) dx \\ &= \lim_{\varepsilon \rightarrow 0+} \left(\int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} f'(x) dx - \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} \int_0^x \frac{y}{1-y^2} f'(y) dy dx \right). \end{aligned}$$

We consider the last term.

$$\begin{aligned} \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} \int_0^x \frac{y}{1-y^2} f'(y) dy dx &= \int_0^{1-\varepsilon} \frac{y}{1-y^2} f'(y) \left(\int_y^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} dx \right) dy \\ &= \int_0^{1-\varepsilon} \frac{y}{1-y^2} (\sqrt{1-y^2} - \sqrt{2\varepsilon - \varepsilon^2}) f'(y) dy. \end{aligned}$$

Then

$$\int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} (Af)'(x) dx = \sqrt{2\varepsilon - \varepsilon^2} \int_0^{1-\varepsilon} \frac{y}{1-y^2} f'(y) dy.$$

For every $f \in Z_1$, $\left| \int_0^1 \frac{y}{1-y^2} f'(y) dy \right| \leq c$ and hence eventually

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^{1-\varepsilon} \frac{x}{\sqrt{1-x^2}} (Af)'(x) dx = \int_0^1 \frac{x}{\sqrt{1-x^2}} (Af)'(x) dx = 0.$$

Similar arguments prove the other claim of b).

We have

$$\begin{aligned} (A^{-1}f)'(x) &= f'(x) - \frac{1}{\sqrt{1-x^2}} \int_0^x \frac{f(y)}{(1-y^2)^{\frac{3}{2}}} dy + \frac{x}{1-x^2} f(x) \\ (2.1) \quad &= f'(x) + \frac{1}{\sqrt{1-x^2}} \int_0^x \frac{y}{\sqrt{1-y^2}} f'(y) dy. \end{aligned}$$

$$\begin{aligned} (A^{-1}f)''(x) &= f''(x) + \frac{x}{1-x^2} f'(x) + \frac{x}{(1-x^2)^{\frac{3}{2}}} \int_0^x \frac{y}{\sqrt{1-y^2}} f'(y) dy \\ (2.2) \quad &= f''(x) + \frac{x}{1-x^2} (A^{-1}f)'(x). \end{aligned}$$

For every $f \in Z_2$ and $x \in [0, 1]$ we rewrite (2.1) as

$$(A^{-1}f)'(x) = f'(x) - \frac{1}{\sqrt{1-x^2}} \int_x^1 \frac{y}{\sqrt{1-y^2}} f'(y) dy.$$

Using Taylor's expansion of f' around 1 we get from the above

$$(A^{-1}f)'(x) = f'(x) - \frac{1}{\sqrt{1-x^2}} \int_x^1 \frac{y}{\sqrt{1-y^2}} (f'(1) + (y-1)f''(1) + o(1-y)) dy.$$

Now we compute the last integral.

$$\int_x^1 \frac{y(1-y)}{\sqrt{1-y^2}} dy = \frac{1}{2}(-\arcsin 1 + \arcsin x - x\sqrt{1-x^2} + 2\sqrt{1-x^2}).$$

As

$$\lim_{x \rightarrow 1-0} \frac{-\arcsin 1 + \arcsin x - x\sqrt{1-x^2} + 2\sqrt{1-x^2}}{(1-x^2)^{\frac{3}{2}}} = \frac{1}{3}$$

we can write

$$-\arcsin 1 + \arcsin x - x\sqrt{1-x^2} + 2\sqrt{1-x^2} = \frac{1}{3}(1-x^2)^{\frac{3}{2}} + o((1-x^2)^{\frac{3}{2}}).$$

Above computations and Taylor's expansion of f' around 1 imply

$$(2.3) \quad (A^{-1}f)'(x) = -\frac{2}{3}(1-x)f''(1) + o(1-x).$$

Equations (2.2) and (2.3) give $(A^{-1}f)''(x) = \frac{2}{3}f''(1) + o(1)$ for x close to 1. In a similar way we get

$$(2.4) \quad (A^{-1}f)'(x) = \frac{2}{3}(1+x)f''(-1) + o(1+x).$$

Hence $(A^{-1}f)''(x) = \frac{2}{3}f''(-1) + o(1)$ for x close to -1 , which proves c).

Part d) follows from (2.3) and (2.4).

For every $f \in Z_1$ from a) we get $(Af)'$, $Af \in AC[-1, 1]$ and hence $Af \in C^2$. Now using b) we get $Af \in Z_2$, i.e., $A(Z_1) \subset Z_2$. Similarly, from c) and d) we get $A^{-1}(Z_2) \subset Z_1$. Using Theorem 4 c) we get $Z_1 = A^{-1}(A(Z_1)) \subset A^{-1}(Z_2)$ and $Z_2 = A(A^{-1}(Z_2)) \subset A(Z_1)$. Hence $A^{-1}(Z_2) = Z_1$ and $A(Z_1) = Z_2$. \square

3. Proofs of the Theorems.

Lemma 1. a) For every $t > 0$ and $f \in L_p[-1, 1]$, $1 \leq p < \infty$, we have

$$K(f, t; L_p[-1, 1], C^2, D_1) = K(f, t; L_p[-1, 1], Z_1, D_1).$$

b) For every $t > 0$ and $f \in C[-1, 1]$ we have

$$K(f, t; C[-1, 1], C^2, D_1) \sim K(f, t; C[-1, 1], Z_1, D_1)$$

Proof. Let $\mu \in C^\infty(\mathbb{R})$ be such that $\mu(x) = 1$ for $x \leq 0$, $\mu(x) = 0$ for $x \geq 1$ and $0 < \mu(x) < 1$ for $0 < x < 1$. For given $\delta \in \left(0, \frac{1}{2}\right)$ we set $\mu_{-1}(x) = \mu\left(\frac{1+x}{\delta}\right)$ and $\mu_1(x) = \mu\left(\frac{1-x}{\delta}\right)$ for every $x \in [-1, 1]$. Thus,

$\text{supp } \mu_{-1}(x) = [-1, -1 + \delta]$, $\text{supp } \mu_1(x) = [1 - \delta, 1]$ and $\left\| \mu_j^{(k)} \right\|_\infty = O(\delta^{-k})$ for $j = -1, 1$ and $k = 1, 2$.

Let $g \in C^2[-1, 1]$. For $x \in [-1, 1]$ set

$$(3.1) \quad G(x) = [1 - \mu_{-1}(x) - \mu_1(x)]g(x) + \mu_{-1}(x)g(-1) + \mu_1(x)g(1)$$

Then $G \in Z_1$. From $G(x) - g(x) = \mu_{-1}(x)[g(-1) - g(x)] + \mu_1(x)[g(1) - g(x)]$ we get $\|G - g\|_p \leq 2^{1/p} \|G - g\|_\infty \leq 2^{1/p} \omega_1(g, \delta)_\infty = O(\delta)$.

From (3.1) we obtain

$$(3.2) \quad \begin{aligned} (D_1G)(x) &= [1 - \mu_{-1}(x) - \mu_1(x)](D_1g)(x) \\ &\quad - 2\varphi^2(x)[\mu'_{-1}(x) - \mu'_1(x)]g'(x) \\ &\quad + (D_1\mu_1)(x)[g(1) - g(x)] + (D_1\mu_{-1})(x)[g(-1) - g(x)]. \end{aligned}$$

From (3.2) for $1 \leq p < \infty$ we get $\|D_1G\|_p \leq \|D_1g\|_p + O(\delta^{1/p})$, which proves part a) in view of Lemma 2 in [3, p. 116].

For $p = \infty$ (3.2) implies

$$\|D_1G\|_\infty \leq \|D_1g\|_\infty + c[|g'(-1)| + |g'(1)|] + O(\delta) \leq c\|D_1g\|_\infty + O(\delta),$$

because of $|g'(-1)| = |(D_1g)(-1)|$ and $|g'(1)| = |(D_1g)(1)|$. Applying Lemma 2 again in [3, p. 116] we prove part b). \square

Lemma 2. For every $t > 0$ and $f \in L_p[-1, 1]$, $1 \leq p < \infty$, we have

$$K(f, t; L_p[-1, 1], Z_1, D_1) \sim K(Af, t; L_p[-1, 1], Z_2, D_2).$$

Proof. For a given $g \in Z_1$ we set $G = Ag \in Z_2$ (see Theorem 5 e)). Then Theorem 4 c), a) and d) implies $\|f - g\|_p = \|A^{-1}(Af - Ag)\|_p \leq c\|Af - G\|_p$ and $\|D_1g\|_p = \|D_1A^{-1}G\|_p = \|D_2G\|_p$. Hence,

$$\|f - g\|_p + t\|D_1g\|_p \leq c(\|Af - G\|_p + t\|D_2G\|_p),$$

which gives $K(f, t; L_p, Z_1, D_1) \leq cK(Af, t; L_p, Z_2, D_2)$.

For a given $G \in Z_2$ we set $g = A^{-1}G \in Z_1$ (see Theorem 5 e)). Using Theorem 3 a), c) and Theorem 4 c), we get

$$\|Af - G\|_p = \|A(f - A^{-1}G)\|_p \leq 2\|f - g\|_p, \quad \|D_2G\|_p = \|D_2Ag\|_p = \|D_1g\|_p.$$

Hence,

$$\|Af - G\|_p + t\|D_2G\|_p \leq 2(\|f - g\|_p + t\|D_1g\|_p),$$

which gives $K(Af, t; L_p[-1, 1], Z_2, D_2) \leq 2K(f, t; L_p[-1, 1], Z_1, D_1)$. \square

From Lemmas 1, 2 we obtain

Corollary 2. For every $t > 0$ and $f \in L_p[-1, 1]$, $1 \leq p < \infty$, we have

$$K(f, t; L_p[-1, 1], C^2, D_1) \sim K(Af, t; L_p[-1, 1], Z_2, D_2).$$

Lemma 3. a) For every $t \in (0, 1]$ and $F \in L_p[-1, 1]$, $2 < p < \infty$, we have

$$K(F, t; L_p[-1, 1], Z_2, D_2) \leq c \left[K(F, t^{\frac{1}{p} + \frac{1}{2}}; L_p[-1, 1], C^2, D_2) + t^{\frac{1}{p} + \frac{1}{2}} E_0(F)_p \right].$$

b) For every $t \in (0, 1]$ and $F \in L_p[-1, 1]$, $1 \leq p \leq 2$, we have

$$K(F, t; L_p[-1, 1], Z_2, D_2) = K(F, t; L_p[-1, 1], C^2, D_2).$$

Proof. For $\delta \in (0, 1/2)$ we set $\mu(x) = (1 - x\delta^{-1})_+^3$, where $(y)_+ = y$ if $y \geq 0$ and $(y)_+ = 0$ if $y \leq 0$. For $g \in C^2[-1, 1]$ we set

$$(3.3) \quad G(x) = g(x) + \alpha\mu(x + 1) + \beta\mu(1 - x)$$

where

$$\alpha = \frac{\delta}{3} \frac{\int_{-1}^0 \frac{y}{\sqrt{1-y^2}} g'(y) dy}{\int_{-1}^0 \frac{y}{\sqrt{1-y^2}} \left(1 - \frac{y+1}{\delta}\right)_+^2 dy}, \quad \beta = -\frac{\delta}{3} \frac{\int_0^1 \frac{y}{\sqrt{1-y^2}} g'(y) dy}{\int_0^1 \frac{y}{\sqrt{1-y^2}} \left(1 - \frac{1-y}{\delta}\right)_+^2 dy}.$$

Parameters α , β and δ are chosen in such way that $G \in Z_2$. From (3.3) we get $\|G - g\|_p \leq c\delta^{1/p}(|\alpha| + |\beta|)$ and

$$G''(x) = g''(x) + 6\delta^{-2} \left[\alpha \left(1 - \frac{x+1}{\delta}\right)_+ + \beta \left(1 - \frac{1-x}{\delta}\right)_+ \right].$$

Hence $\|\varphi^2 G''\|_p \leq \|\varphi^2 g''\|_p + c\delta^{-1+1/p}(|\alpha| + |\beta|)$, and

$$\begin{aligned} K(F, t; L_p, Z_2, D_2) &\leq \|F - G\|_p + t \|D_2 G\|_p \\ &\leq \|F - g\|_p + t \|D_2 g\|_p + c\delta^{1/p}(1 + t\delta^{-1})(|\alpha| + |\beta|). \end{aligned}$$

In order to estimate $|\alpha| + |\beta|$ we calculate

$$\begin{aligned}
\int_0^1 \frac{y}{\sqrt{1-y^2}} \left(1 - \frac{1-y}{\delta}\right)_+^2 dy &= \frac{1}{\delta^2} \int_{1-\delta}^1 \frac{y}{\sqrt{1-y^2}} (\delta - 1 + y)^2 dy \\
&\geq \frac{c}{\delta^2} \int_{1-\delta}^1 \frac{1}{\sqrt{1-y}} (\delta - (1-y))^2 dy \\
&= \frac{c}{\delta^2} \int_0^\delta \frac{(\delta-t)^2}{\sqrt{t}} dt = c\delta^{1/2}.
\end{aligned}$$

In a similar way we get

$$\left| \int_{-1}^0 \frac{y}{\sqrt{1-y^2}} \left(1 - \frac{y+1}{\delta}\right)_+^2 dy \right| \geq c\delta^{1/2}.$$

Then

$$|\alpha| \leq c\sqrt{\delta} \left| \int_{-1}^0 \frac{y}{\sqrt{1-y^2}} g'(y) dy \right|, \quad |\beta| \leq c\sqrt{\delta} \left| \int_0^1 \frac{y}{\sqrt{1-y^2}} g'(y) dy \right|.$$

Using Hölder inequality we estimate

$$\begin{aligned}
|\alpha| &\leq c\sqrt{\delta} \left| \int_{-1}^0 \frac{y}{\sqrt{1-y^2}} g'(y) dy \right| \\
&\leq c\sqrt{\delta} \left\{ \int_{-1}^0 \left| \frac{y}{\sqrt{1-y^2}} \right|^q dy \right\}^{1/q} \left\{ \int_{-1}^0 |g'(y)|^p dy \right\}^{1/p} \\
&\leq c\sqrt{\delta} \left\{ \int_{-1}^0 |g'(y)|^p dy \right\}^{1/p} \\
&= c\sqrt{\delta} \|g'\|_{p[-1,0]} \text{ for } p > 2, \frac{1}{p} + \frac{1}{q} = 1. \text{ Similarly} \\
|\beta| &\leq c\sqrt{\delta} \|g'\|_{p[0,1]} \text{ for } p > 2. \text{ Hence } |\alpha| + |\beta| \leq c\sqrt{\delta} \|g'\|_p \text{ for } p > 2.
\end{aligned}$$

In order to estimate the norm of g' we apply the inequality

$$\|g'\|_p \leq c \left(\|D_2 g\|_p + E_0(g)_p \right),$$

which, for instance, follows from [2, p. 135, assertion (a)]. Then we get

$$|\alpha| + |\beta| \leq c\sqrt{\delta} \left(\|D_2g\|_p + E_0(g)_p \right) \leq c\sqrt{\delta} \left(\|D_2g\|_p + \|F - g\|_p + E_0(F)_p \right).$$

Now we take $\delta = t/2$. Thus

$$K(F, t; L_p, Z_2, D_2) \leq c \left(\|F - g\|_p + t^{1/p+1/2} \|D_2g\|_p + t^{1/p+1/2} E_0(F)_p \right)$$

for every $g \in C^2[-1, 1]$, which proves part a).

In order to prove part b) it is sufficient to show (see Lemma 2 in [3, p. 116]) that for every $g \in C^2[-1, 1]$ and every $\varepsilon > 0$ there exists $G \in Z_2$ such that $\|G - g\|_p < \varepsilon$ and $\|\varphi^2 G''\|_p < \|\varphi^2 g''\|_p + \varepsilon$. For $1 \leq p < 2$ we can define G by (3.3). We have

$$\begin{aligned} \|G - g\|_p &\leq c\delta^{1/p}(|\alpha| + |\beta|) \\ &\leq c\delta^{1/p+1/2} \left(\left| \int_{-1}^0 \frac{y}{\sqrt{1-y^2}} g'(y) dy \right| + \left| \int_0^1 \frac{y}{\sqrt{1-y^2}} g'(y) dy \right| \right) \xrightarrow{\delta \rightarrow 0} 0. \end{aligned}$$

$$\begin{aligned} \|\varphi^2 G''\|_p &\leq \|\varphi^2 g''\|_p + c\delta^{-1+1/p}(|\alpha| + |\beta|) \\ &\leq \|\varphi^2 g''\|_p + c\delta^{1/p-1/2} \left(\left| \int_{-1}^0 \frac{y}{\sqrt{1-y^2}} g'(y) dy \right| + \left| \int_0^1 \frac{y}{\sqrt{1-y^2}} g'(y) dy \right| \right). \end{aligned}$$

When $\frac{1}{p} - \frac{1}{2} > 0$ the last term tends to zero as $\delta \rightarrow 0$, which proves part b) in case $1 \leq p < 2$.

The case $p = 2$ needs special consideration and different definition of G .

Let $\delta \in \left(0, \frac{1}{2}\right)$. We set

$$\psi''_\delta(x) = \begin{cases} 0 & \text{for } x \in [-1, 0]; \\ \frac{x}{(1-x^2)^{3/2}} & \text{for } x \in (0, 1-\delta]; \\ \frac{1-\delta}{(2\delta-\delta^2)^{3/2}} & \text{for } x \in (1-\delta, 1]. \end{cases}$$

By integration we have

$$\psi'_\delta(x) = \begin{cases} 0 & \text{for } x \in [-1, 0]; \\ \frac{1}{\sqrt{1-x^2}} - 1 & \text{for } x \in (0, 1-\delta]; \\ \frac{1}{\sqrt{2\delta-\delta^2}} - 1 + \frac{1-\delta}{(2\delta-\delta^2)^{3/2}} [x - (1-\delta)] & \text{for } x \in (1-\delta, 1]. \end{cases}$$

$$\psi_\delta(x) = \begin{cases} 0 & \text{for } x \in [-1, 0]; \\ \arcsin x - x & \text{for } x \in (0, 1 - \delta); \\ \arcsin(1 - \delta) - (1 - \delta) \\ + \left(\frac{1}{\sqrt{2\delta - \delta^2}} - 1 \right) [x - (1 - \delta)] & \text{for } x \in (1 - \delta, 1]; \\ + \frac{1 - \delta}{2(2\delta - \delta^2)^{3/2}} [x - (1 - \delta)]^2 \end{cases}$$

$\psi_\delta''(x)$, $\psi_\delta'(x)$ and $\psi_\delta(x)$ are continuous and increasing functions.

We set now $\mu(x) = \psi_\delta(x)$. For $g \in C^2[-1, 1]$ we set

$$(3.4) \quad G(x) = g(x) + \alpha\mu(x) + \beta\mu(-x).$$

Parameters α and β are chosen in such way that $G \in Z_2$:

$$0 = \int_0^1 \frac{xG'(x)}{\sqrt{1-x^2}} dx = \int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}} dx + \alpha \int_0^1 \frac{x\psi_\delta'(x)}{\sqrt{1-x^2}} dx. \quad \text{Hence}$$

$$\alpha = -\frac{\int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}} dx}{\int_0^1 \frac{x\psi_\delta'(x)}{\sqrt{1-x^2}} dx}. \quad \text{Similarly } \beta = -\frac{\int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}} dx}{-\int_{-1}^0 \frac{x\psi_\delta'(-x)}{\sqrt{1-x^2}} dx} = -\frac{\int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}} dx}{\int_0^1 \frac{x\psi_\delta'(x)}{\sqrt{1-x^2}} dx}.$$

From (3.4) we get

$$\|G - g\|_2 \leq (|\alpha| + |\beta|) \|\psi_\delta\|_2 \quad \text{and}$$

$$\|\varphi^2 G''\|_2 \leq \|\varphi^2 g''\|_2 + (|\alpha| + |\beta|) \|\varphi^2 \psi_\delta''\|_2.$$

In order to estimate the last expressions we use some properties of ψ_δ given in the following

Assertion 1. Let $\delta \in \left(0, \frac{1}{2}\right)$. Then we have

$$\text{a) } \int_0^1 \frac{x\psi_\delta'(x)}{\sqrt{1-x^2}} dx \sim \ln \frac{1}{\delta}.$$

$$\text{b) } \|\varphi^2 \psi_\delta''\|_2 \sim \sqrt{\ln \frac{1}{\delta}}.$$

$$\text{c) } \|\psi_\delta\|_2 \sim 1.$$

Using Assertion 1 we obtain

$$\begin{aligned} \|G - g\|_2 &\leq (|\alpha| + |\beta|) \|\psi_\delta\|_2 = \frac{\left| \int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| + \left| \int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right|}{\int_0^1 \frac{x\psi'_\delta(x)}{\sqrt{1-x^2}} dx} \|\psi_\delta\|_2 \\ &\leq \frac{c}{\ln \frac{1}{\delta}} \left(\left| \int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| + \left| \int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| \right). \end{aligned}$$

$$\begin{aligned} \|\varphi^2 G''\|_2 &\leq \|\varphi^2 g''\|_2 + (|\alpha| + |\beta|) \|\varphi^2 \psi''_\delta\|_2 \\ &= \|\varphi^2 g''\|_2 + \left(\left| \int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| + \left| \int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| \right) \frac{\|\varphi^2 \psi''_\delta\|_2}{\int_0^1 \frac{x\psi'_\delta(x)}{\sqrt{1-x^2}} dx} \\ &\leq \|\varphi^2 g''\|_2 + \frac{c}{\sqrt{\ln \frac{1}{\delta}}} \left(\left| \int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| + \left| \int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| \right). \end{aligned}$$

Let $g \in C^2[-1, 1]$, $\varepsilon > 0$ is a small number. For a given function g and $\varepsilon > 0$ we may choose $\delta > 0$ such that

$$\frac{c}{\sqrt{\ln \frac{1}{\delta}}} \left(\left| \int_{-1}^0 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| + \left| \int_0^1 \frac{xg'(x)}{\sqrt{1-x^2}} dx \right| \right) \leq \varepsilon,$$

which proves b) in case $p = 2$ in view of [3, Lemma 2, p.116]). \square

Lemma 4. For every $t \in (0, 1]$ and $f \in L_p[-1, 1]$, $2 < p < \infty$, we have

$$tE_0(f)_p \leq cK(f, t; L_p[-1, 1], C^2, D_1).$$

Proof. For every $g \in C^2[-1, 1]$ we have

$$|g(x) - g(0)| \leq |\arcsin x| \|\varphi g'\|_\infty.$$

Hence $\|g - g(0)\|_p \leq c \|\varphi g'\|_\infty$. Using that $\varphi(1)g'(1) = \varphi(-1)g'(-1) = 0$ and

Hölder inequality we get for every $x \in [-1, 1]$

$$\begin{aligned} \left| \sqrt{1-x^2} g'(x) \right| &= \left| \int_{-1}^x \left(\sqrt{1-t^2} g'(t) \right)' dt \right| \\ &= \left| \int_{-1}^x \frac{1}{\sqrt{1-t^2}} \sqrt{1-t^2} \left(\sqrt{1-t^2} g'(t) \right)' dt \right| \\ &\leq \left\{ \int_{-1}^x \left(\frac{1}{\sqrt{1-t^2}} \right)^q dt \right\}^{1/q} \left\{ \int_{-1}^x \left| \sqrt{1-t^2} \left(\sqrt{1-t^2} g'(t) \right)' \right|^p dt \right\}^{1/p} \\ &\leq c \|D_1 g\|_p \text{ for } p > 2 \text{ and } \frac{1}{p} + \frac{1}{q} = 1. \text{ Thus,} \end{aligned}$$

$$tE_0(f)_p \leq t \|f - g(0)\|_p \leq t \|f - g\|_p + t \|g - g(0)\|_p \leq c \left[\|f - g\|_p + t \|D_1 g\|_p \right],$$

which proves the lemma. \square

Proof of Theorems 1 and 2. From parts a) and b) of Theorems 3 and 4 we get $E_0(f)_p \sim E_0(Af)_p$. Using Corollary 2 and Lemma 3 part a) with $F = Af$ we get

$$\begin{aligned} &K(f, t; L_p[-1, 1], C^2, D_1) \\ &\leq c \left[K(Af, t^{\frac{1}{p} + \frac{1}{2}}; L_p[-1, 1], C^2, D_2) + t^{\frac{1}{p} + \frac{1}{2}} E_0(f)_p \right], \text{ for } 2 < p < \infty. \end{aligned}$$

From Corollary 2 and Lemma 3 part b) we obtain

$$K(f, t; L_p[-1, 1], C^2, D_1) \sim K(Af, t; L_p[-1, 1], C^2, D_2), \quad \text{for } 1 \leq p \leq 2.$$

From Corollary 2 and Lemma 4 we obtain for $2 < p < \infty$

$$\begin{aligned} K(Af, t; L_p[-1, 1], C^2, D_2) + tE_0(f)_p &\leq K(Af, t; L_p[-1, 1], Z_2, D_2) + tE_0(f)_p \\ &\leq cK(f, t; L_p[-1, 1], C^2, D_1), \end{aligned}$$

which proves the theorems. \square

4. Generalization. The results can be dealt with in a generalized case as in the K -functional (1.1) $D_1 g := \varphi^{2-2\lambda}(\varphi^{2\lambda} g)'$ for $\lambda \in (0, 1)$, while in the K -functional (1.2) D_2 remains the same. The corresponding linear operators A and A^{-1} for the general case are:

$$(Af)(x) := f(x) + \int_0^x f(y) \left[(y-x) \frac{\theta'(y)}{\theta(y)} \right]' dy$$

$$(A^{-1}f) := f(x) + \int_0^x f(y) \left[\theta''(y) \int_y^x \frac{dt}{\theta(t)} - \frac{\theta'(y)}{\theta(y)} \right] dy,$$

where $\theta(y) = \varphi^{2\lambda}(y) = (1 - y^2)^\lambda$.

Then the analogue of Theorem 1 is

Theorem 1'. *Let $\lambda \in (0, 1)$. Then for every $t \in (0, 1]$ and $f(x) \in L_p[-1, 1]$, $1 \leq p \leq \frac{1}{\lambda}$ we have*

$$K(f, t; L_p[-1, 1], C^2, D_1) \sim K(Af, t; L_p[-1, 1], C^2, D_2).$$

The proof of Theorem 1' follows the same pattern. The analogues of Theorems 3 and 4 are the same to the value of absolute constant in the inequality of the norm. In the analogue of Theorem 5 the space Z_1 is the same, while the space

$$Z_2 = \left\{ f \in C^2[-1, 1] : \int_{-1}^0 (\varphi^{2\lambda}(x))' f'(x) dx = 0, \int_0^1 (\varphi^{2\lambda}(x))' f'(x) dx = 0 \right\}.$$

Lemmas 1 and 2 and Corollary 2 remain the same. The conclusion of Lemma 3 b) is the same under assumption $1 \leq p \leq \frac{1}{\lambda}$.

Theorem 1' is not true for $\lambda = 1$ – see Theorem B. To make it true on the right hand side of the relation we have to add the term $tE_0(f)_1$, what is exactly the result in [3] for $p = 1$. That is not strange, because for $\lambda = 1$ after we integrate by parts the integral conditions (describing the space Z_2 in the general case) we obtain the conditions considered by Ivanov in [3, p. 120] and for $\lambda = 1$ the differential operator $(D_1g)(x) = (1 - x^2)g'' - 2xg'(x)$ what is exactly the analogue of D_3 in the interval $[-1, 1]$.

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T. D. Zapryanova
Varna University of Economics
77, Kniaz Boris I Blvd.
9002 Varna, Bulgaria
e-mail: teodorazap@abv.bg

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