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AN ANALYTIC POINT OF VIEW AT TORIC VARIETIES

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Abstract. The goal of this paper is to give an elementary and analytic introduction to toric varieties, a first view at these objects that will prepare the readers, especially analysts, for a more advanced study. These notes could serve a wider audience from many, mainly analytic fields, who want to learn basics about toric varieties. Also they can be used to introduce this subject in graduate or advanced undergraduate courses.

Contents

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1. Introduction. The goal of this paper is to give an elementary and analytic introduction to toric varieties, a first view at these objects that will prepare the readers, especially analysts, for a more advanced study. These notes could serve a wider audience from many, mainly analytic fields, who want to learn basics about toric varieties and also introduce this subject in graduate or advanced undergraduate courses.

There are excellent books (G. Ewald [8], W. Fulton [9], T. Oda [12], [13]) and review papers (D. A. Cox [5], V. I. Danilov [7]) about this subject, but they require sometimes an advanced knowledge of algebra and algebraic geometry. For instance, an affine toric variety (a particular case) is defined as the spectrum of a ring, that is, the set of prime ideals. Algebraists have a good intuition about such notions but I suspect that many analysts not.

In complex analysis it is a little bit hard to construct examples to illustrate some properties. Usually the projective spaces, vector bundles over projective spaces or sometimes special algebraic varieties are used. These classes are limited. Toric varieties are also, of course, very particular, but what is nice about them, their properties can be translated to simple combinatorics (at least in low dimensions). Because of that they are good candidates for consideration. In the literature there are not many analytic papers that use toric varieties extensively. Maybe one of the reasons is that a first look at these objects is very algebraic. Of course there are several "analytic papers" on toric varieties (e.g., [15], [16]) and we hope that more will appear soon.

Toric varieties can be defined over any field K, but to simplify matters and for our purposes it is enough to take the field of complex numbers $\mathbb{C}$. Very roughly speaking, a fan in $\mathbb{R}^n$ is a family of cones with vertices at the origin. In $\mathbb{R}^n$ we have the natural grid $\mathbb{Z}^n$. We look at a cone not only from the point of view of linear algebra, but also from the lattice point of view, i.e., which lattice points are included in the cone and what semigroup they generate. There is a natural way to associate an affine variety for each cone and a complex variety (by gluing affine varieties), called *toric variety*, to a fan. Here in this paper we describe this process in elementary way and prove some properties from an analytic point of view.

Of course this paper does not pretend to be complete. The literature on toric varieties since the beginning of 1970's contains hundreds or even thousands of research papers. These notes base on several textbooks (mainly [13], [9], [8]), papers ([5], [7]), notes ([3], [17]). This paper can be much longer, but it was not an intention of the author to write a book, but only a first analytic introduction to this subject.

We analyze precisely strongly convex polyhedral cones and their dual, their spanning over nonnegative real numbers and over nonnegative integers. Then we look from different point of view at affine toric varieties and how to glue them to get toric varieties. The latter is also related to orbits of torus action. Next, we consider in detail polytopes and their relation to toric varieties and the so-called support functions. We stop at a criterion for embedding of toric varieties into projective spaces. At the very end we include a homogeneous point of view at toric varieties, similar to the standard definition of projective spaces.

All the material is illustrated on dozens of examples and dozens of figures to better understand the geometry and combinatorics behind the concepts.

2. Basic definitions and notation. We denote integer numbers by $\mathbb{Z}$, rational numbers by $\mathbb{Q}$, real numbers by $\mathbb{R}$, and complex numbers by $\mathbb{C}$. Later on we use the notation $\mathbb{R}_{\geqslant 0} = \{x \in \mathbb{R} \mid x \geqslant 0\}$ and $\mathbb{Z}_{\geqslant 0} = \{k \in \mathbb{Z} \mid k \geqslant 0\}$. We will work in $\mathbb{C}^r$, complex r dimensional space, $\mathbb{R}^r$ real r dimensional space.

Lattices. It will be convenient to distinguish between the initial space $N_{\mathbb{R}} = \mathbb{R}^r$ and its dual space $M_{\mathbb{R}} := N_{\mathbb{R}}^* = \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R}) \simeq \mathbb{R}^r$, however, if there is no confusion, we keep $M_{\mathbb{R}} = \mathbb{R}^r$.

Let $N_{\mathbb{Z}} = \mathbb{Z}^r$ with standard addition "+" and multiplication by integers (free module). The dual $\mathbb{Z}$-module is $M_{\mathbb{Z}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r, \mathbb{Z})$. We have a canonical $\mathbb{Z}$-bilinear pairing given by the standard dot product on $\mathbb{R}^r$.

$$\langle \cdot\,, \cdot \rangle \;:\; M_{\mathbb{Z}} \times N_{\mathbb{Z}} \longrightarrow \mathbb{Z}, \quad \langle \mathbf{m}, \mathbf{n} \rangle = m_1 n_1 + \ldots + m_r n_r,$$
$$\mathbf{m} = (m_1, \ldots, m_r) \in M_{\mathbb{Z}}, \;\; \mathbf{n} = (n_1, \ldots, n_r) \in N_{\mathbb{Z}}.$$

Affine varieties in $\mathbb{C}^r$. If $f_1, \ldots, f_k$ is a sequence of complex polynomials, then

$$V = V(f_1, \ldots, f_k) = \{ z \in \mathbb{C}^r \mid f_1(z) = \ldots = f_k(z) = 0\}$$

is called an *affine variety*. Given varieties W and V, $W \subset V$, we call the complement $V \setminus W = \{v \in V \mid v \notin W\}$ a *Zariski open subset of* V.

Projective varieties. If $F_1, \ldots, F_k$ is a sequence of homogeneous polynomials on $\mathbb{C}^{r+1}$, then a *projective variety* in $\mathbb{CP}^r$ is

$$V = V(F_1, \ldots, F_k) = \{z = (z_0, \ldots, z_r) \mid F_1(z) = \ldots = F_k(z) = 0\}/_{\sim} \;\subset \mathbb{CP}^r,$$
$$(z_0, \ldots, z_r) \sim (\lambda z_0, \ldots, \lambda z_r), \quad \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},$$

where " $\sim$ " is the relation that produces the projective space.

Complex torus. By a complex torus of dimension r we mean $T_r = (\mathbb{C}^*)^r$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The complex torus can be naturally considered as a group, namely the group action is

$$(s_1, \ldots, s_r) \bullet (t_1, \ldots, t_r) = (s_1 t_1, \ldots, s_r t_r).$$

Example 2.1. $(\mathbb{C}^*)^r \subset \mathbb{C}^r$ is an affine variety since the map

$$(\mathbb{C}^*)^r \ni (t_1, \ldots, t_r) \longrightarrow (1/(t_1 \ldots t_r), t_1, \ldots, t_r) \in \mathbb{C}^{r+1}$$

gives an isomorphism

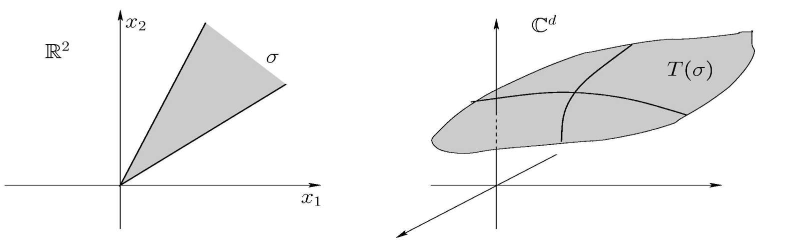
$$(\mathbb{C}^*)^r \simeq V(z_0 z_1 \dots z_r - 1) = \{(z_0, \dots, z_r) \mid z_0 \dots z_r - 1 = 0\} \subset \mathbb{C}^{r+1}.$$

Laurent monomials. If U is an open subset in $\mathbb{C}^r$, then by a monomial (or Laurent monomial) we mean $z^{\mathbf{m}} = z_1^{m_1} \dots z_r^{m_r}$, where $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$.

3. A first rough description of toric varieties. We give few points of view at toric varieties.

1. For a cone in $\mathbb{R}^r$ we define an *affine toric variety* of complex dimension r, which is, roughly speaking, an affine variety in $\mathbb{C}^d$ for some d; see Fig. 1. If we have a family of cones which is called a *fan* (some conditions are imposed on this family), then there is a method to "glue" together affine toric varieties that correspond to each cone from the fan; see Fig. 2.

   To describe how an affine toric variety is constructed from a cone in $\mathbb{R}^r$ requires some work. But very roughly we can say that a face of the cone of dimension l corresponds to a subvariety of complex codimension l. In particular, the cone $\{0\}$ corresponds to an open dense set in the affine toric variety.

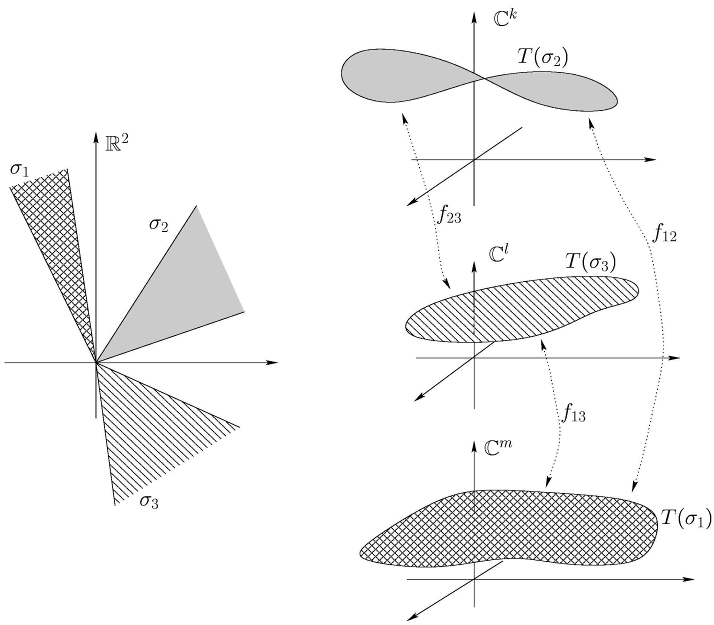


Two-dimensional cone σ in $\mathbb{R}^2$ Two-dimensional affine toric variety $T(\sigma)$ in $\mathbb{C}^d$

Fig. 1. From a cone to an affine toric variety

2. A toric variety of dimension r is a compactification of the r-dimensional complex torus T_r.

   It appears that a complex torus can be compactified in many not equivalent ways. The theory of toric varieties describes nicely some of these ways.



Fan in $\mathbb{R}^2$ f_{jk} are transition maps between open dense set

Fig. 2. From a fan to a toric variety

Topologically, all toric varieties of dimension r differ by a small set only: $\overline{T_r} \setminus T_r$, where the closure is in a corresponding toric variety and actually $\overline{T_r}$ is the entire toric variety.

3. A toric variety is an irreducible variety T such that

   (a) $T_r = (\mathbb{C}^*)^r$ is an open and dense subset of T, and

   (b) The action of $T_r = (\mathbb{C}^*)^r$ on itself extends to an action of $T_r = (\mathbb{C}^*)^r$ on T.

4. Another view at toric varieties is via looking at the coordinate transition functions. Consider here a particular case, namely a non-singular toric variety (look at Fig. 2). In this case, there exists a covering by coordinate charts $\{U_\alpha\}_{\alpha \in A}$ with $U_\alpha \simeq \mathbb{C}^r$

$$\{U_\alpha,\ z_\alpha = (z_{\alpha 1}, \dots, z_{\alpha r})\}_{\alpha \in A}$$

and such that the transition functions are Laurent monomials, i.e.,

$$z_\alpha = (z_{\alpha 1}, \dots, z_{\alpha r}) = f_{\alpha\beta}(z_\beta) = (f_{\alpha\beta 1}(z_\beta), \dots, f_{\alpha\beta r}(z_\beta)),$$

where

$$f_{\alpha\beta j}(z_\beta) = z_\beta^{\mathbf{m}_j} = z_{\beta 1}^{m_{j1}} \cdot \ldots \cdot z_{\beta r}^{m_{jr}} \quad \text{for} \quad \mathbf{m}_j = (m_{j1}, \dots, m_{jr}),$$

where $\mathbf{m}_j$ depends on α and β only.

4. Cones and fans.

4.1. Convex rational polyhedral cones. We use the following notation (recall, $M_\mathbb{R} = N_\mathbb{R}^* = \mathrm{Hom}(N_\mathbb{R}, \mathbb{R})$):

$\sigma_\mathbb{R}$ denotes a cone in $N_\mathbb{R} = N_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^r$.

$\sigma_\mathbb{Z}$ denotes the intersection $\sigma_\mathbb{R} \cap N_\mathbb{Z}$ in the lattice $N_\mathbb{Z} = \mathbb{Z}^r$.

$\check{\sigma}$ denotes the dual cone in $M_\mathbb{R} = M_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^r$ to the cone $\sigma_\mathbb{R}$.

$\check{\sigma}_\mathbb{Z}$ denotes the intersection $\check{\sigma}_\mathbb{R} \cap M_\mathbb{Z}$ in the lattice $M_\mathbb{Z} = \mathbb{Z}^r$.

Sometimes we drop the index $\mathbb{R}$ at $\sigma_\mathbb{R}$ or $\check{\sigma}_\mathbb{R}$ if it does not lead to a confusion.

Definition 4.1 (Definition of a strongly convex rational polyhedral cone). *A subset $\sigma_\mathbb{R}$ of $N_\mathbb{R}$ is called a* convex rational polyhedral cone *if there exists a finite number of elements $\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_s$ in $N_\mathbb{Z}$ such that*

$$\sigma_\mathbb{R} = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \ldots + \mathbb{R}_{\geqslant 0}\mathbf{n}_s = \{a_1\mathbf{n}_1 + \ldots + a_s\mathbf{n}_s \mid a_i \in \mathbb{R}_{\geqslant 0} \text{ for all } i\} \subset N_\mathbb{R}$$

$$\sigma_\mathbb{R} \cap (-\sigma_\mathbb{R}) = \{0\}.$$

The dimension *of $\sigma_\mathbb{R}$ is the dimension of the smallest $\mathbb{R}$-subspace that contains $\sigma_\mathbb{R}$.*

Definition 4.2. (Definition of the dual cone). *The dual cone to $\sigma_\mathbb{R} \subset N_\mathbb{R}$ is defined as*

$$\check{\sigma}_\mathbb{R} := \{y \in M_\mathbb{R} \mid \langle y, x\rangle \geqslant 0 \text{ for all } x \in \sigma_\mathbb{R}\}.$$

We note that in the definition of a cone, the cone is spanned by vectors with integer components, but over the real numbers. On the right-hand side picture in Fig. 3 we draw the cone σ and its dual $\check{\sigma}$ on the same grid. It is not

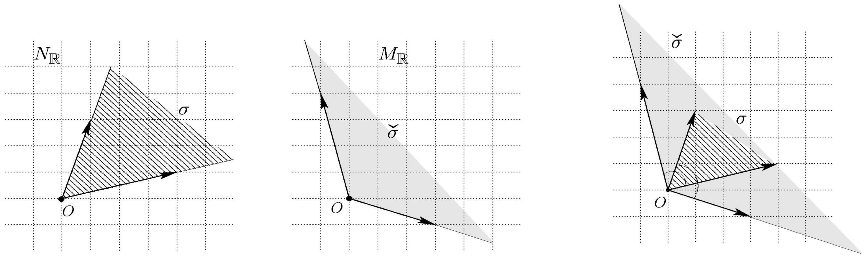


Fig. 3. Cone and dual cone

quite correct since $\sigma \subset N_{\mathbb{R}}$ and $\check{\sigma} \subset M_{\mathbb{R}}$. However for intuition purposes it is convenient to have a geometric relation between these two cones.

Definition 4.3 (Definition of the face of a cone)**.** *A subset $\tau_{\mathbb{R}}$ of $\sigma_{\mathbb{R}}$ is called a face and denoted $\tau_{\mathbb{R}} \prec \sigma_{\mathbb{R}}$ if*

$$\tau_{\mathbb{R}} := \sigma_{\mathbb{R}} \cap \{\mathbf{m}_0\}^{\perp} = \{x \in \sigma_{\mathbb{R}} \mid \langle \mathbf{m}_0, x\rangle = 0\} \quad \text{for an} \quad \mathbf{m}_0 \in \check{\sigma}_{\mathbb{R}}.$$

4.2. To generate over $\mathbb{R}_{\geqslant 0}$ or over $\mathbb{Z}_{\geqslant 0}$. We should clearly distinguish between the notions: to be generated over $\mathbb{R}_{\geqslant 0}$ or $\mathbb{Z}_{\geqslant 0}$.

Let $\sigma_{\mathbb{R}}$ be a strictly convex rational polyhedral cone in $N_{\mathbb{R}} \simeq \mathbb{R}^r$ and $\sigma_{\mathbb{Z}}$ the corresponding lattice cone. We can consider that both are contained in $\mathbb{R}^r$, but the second one contains only the lattice points from $\mathbb{Z}^r$. Assume that $\mathbf{n}_1, \ldots, \mathbf{n}_k \in \sigma_{\mathbb{Z}}$. We say that they generate $\sigma_{\mathbb{R}}$ if

$$\sigma_{\mathbb{R}} = \operatorname{Span}_{\mathbb{R}_{\geqslant 0}}\{\mathbf{n}_1, \ldots, \mathbf{n}_k\} = \{a_1\mathbf{n}_1 + \ldots + a_k\mathbf{n}_k \mid a_1, \ldots, a_k \in \mathbb{R}_{\geqslant 0}\}.$$

However, in general it is not true that the same vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$ generate $\sigma_{\mathbb{Z}}$, i.e., usually we have

$$\sigma_{\mathbb{Z}} \supset \operatorname{Span}_{\mathbb{Z}_{\geqslant 0}}\{\mathbf{n}_1, \ldots, \mathbf{n}_k\} = \{a_1\mathbf{n}_1 + \ldots + a_k\mathbf{n}_k \mid a_1, \ldots, a_k \in \mathbb{Z}_{\geqslant 0}\}$$

and more vectors are needed that the equality holds. Here is an example.

Example 4.4. Let $\sigma_{\mathbb{R}}$ be the strongly convex rational polyhedral cone in $\mathbb{R}^2$ generated over $\mathbb{R}_{\geqslant 0}$ by $\mathbf{n}_1 = [1,\ 3]$ and $\mathbf{n}_2 = [4, 1]$ as in Fig. 4. The lattice point $[1, 1]$ lies inside the cone, but it cannot be obtained as a combination of these two vectors with non-negative integer coefficients:

$$a_1\mathbf{n}_1 + a_2\mathbf{n}_2 = [1, 1] \quad \Longrightarrow \quad \begin{cases} a_1 + 4a_2 = 1 \\ 3a_1 + a_2 = 1 \end{cases} \quad \Longrightarrow \quad a_1 = \frac{3}{11},\ a_2 = \frac{2}{11}.$$

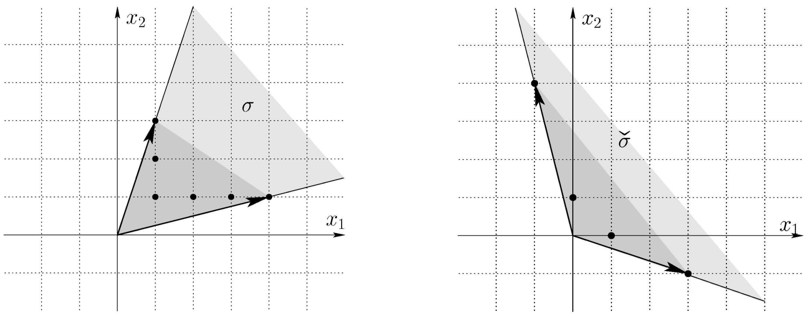


Fig. 4. Cone and its dual spanned and generated

Even more, the lattice points $[1,2]$, $[3,1]$ and $[2,1]$ cannot be obtained by a non-negative integer linear combination of $[1,3]$, $[4,1]$, $[1,1]$ and so on. It is elementary to verify that the lattice vectors that generate the cone are $[1,1]$, $[1,2]$, $[1,3]$, $[2,1]$, $[3,1]$, $[4,1]$.

The dual cone $\check{\sigma}_{\mathbb{R}}$ is spanned over $\mathbb{R}$ by vectors $\mathbf{m}_1 = [-1,4]$ and $\mathbf{m}_2 = [3,-1]$. Additional two vectors are needed to generate the cone $\check{\sigma}_{\mathbb{Z}}$, namely $[1,0]$ and $[0,1]$. So the lattice vectors that generate the dual cone are $[-1,4]$, $[3,-1]$, $[1,0]$, $[0,1]$.

4.3. Gordon's lemma and other properties. Later on we need the following proposition about some properties of the dual cones. These properties are proved precisely, e.g., in [13], Prop. 1.1 and Prop. 1.3.

Proposition 4.5. *Let $\sigma_{\mathbb{R}}$ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Then the following properties hold:*

(a) *The dual cone $\check{\sigma}_{\mathbb{R}}$ is a rational polyhedral cone.*

(b) (Gordon's Lemma) *The cone $\check{\sigma}_{\mathbb{Z}} = \check{\sigma}_{\mathbb{R}} \cap \mathbb{Z}^r$ is finitely generated as an additive semigroup, i.e., there exist $\mathbf{m}_1, \ldots, \mathbf{m}_k \in \check{\sigma}_{\mathbb{R}} \cap \mathbb{Z}^r$ such that*

$$\check{\sigma}_{\mathbb{Z}} = \mathbb{Z}_{\geqslant 0}\mathbf{m}_1 + \ldots + \mathbb{Z}_{\geqslant 0}\mathbf{m}_k := \{a_1\mathbf{m}_1 + \ldots + a_k\mathbf{m}_k \mid a_i \in \mathbb{Z}_{\geqslant 0} \text{ for all } i\}.$$

The proof of Gordon's Lemma can be found in any textbook on toric varieties. However, it is very useful to see the property from part (b) how it works practically, i.e., how to find generators (over $\mathbb{Z}$) of $\check{\sigma}_{\mathbb{Z}}$. Because of that we give a sketch of proof.

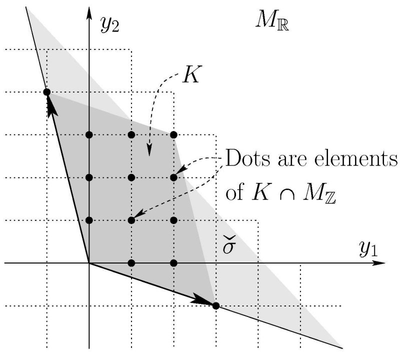


Fig. 5. Illustration of Gordon's lemma

We know (from part (a)) that always $\check{\sigma}_{\mathbb{R}}$ is finitely generated over $\mathbb{R}$ by vectors from $M_{\mathbb{Z}}$. Denote the finite set of such generators by A and we have $\check{\sigma}_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}_{\geqslant 0}}(A)$. The set

$$K = \left\{ \sum_{\mathbf{m} \in A} b_{\mathbf{m}}\, \mathbf{m} \in M_{\mathbb{R}} \mid \quad 0 \leqslant b_{\mathbf{m}} \leqslant 1 \right\}$$

is compact, therefore the set $K \cap M_{\mathbb{Z}}$ is finite and we note that $K \cap M_{\mathbb{Z}} \subset \check{\sigma}_{\mathbb{Z}}$. We claim that $K \cap M_{\mathbb{Z}}$ generates $\check{\sigma}_{\mathbb{Z}}$. To prove this, take $\mathbf{w} \in \check{\sigma}_{\mathbb{Z}}$ and write $\mathbf{w} = \sum_{\mathbf{m} \in A} a_{\mathbf{m}}\, \mathbf{m}$, where $a_{\mathbf{m}} \in \mathbb{R}_{\geqslant 0}$. Since $a_{\mathbf{m}} = [\![a_{\mathbf{m}}]\!] + b_{\mathbf{m}}$, where $[\![a_{\mathbf{m}}]\!] \in \mathbb{Z}_{\geqslant 0}$, $0 \leqslant b_{\mathbf{m}} < 1$, we obtain

$$\check{\sigma}_{\mathbb{Z}} \ni \mathbf{w} = \underbrace{\sum_{\mathbf{m} \in A} [\![a_{\mathbf{m}}]\!]\, \mathbf{m}}_{\in \check{\sigma}_{\mathbb{Z}}} \quad + \quad \underbrace{\sum_{\mathbf{m} \in A} b_{\mathbf{m}}\, \mathbf{m}}_{\in \check{\sigma}_{\mathbb{R}}} \quad \Longrightarrow \quad \sum_{\mathbf{m} \in A} b_{\mathbf{m}}\, \mathbf{m} \in K \cap M_{\mathbb{Z}} \subset \check{\sigma}_{\mathbb{Z}}$$

therefore $\mathbf{w}$ is a nonnegative integer combination of elements of $K \cap M_{\mathbb{Z}}$.

4.4. Regular (smooth) cones.

Definition 4.6. *A* cone is regular *if it is generated by a subset of a basis of* $\mathbb{Z}^r$*.*

Definition 4.7. *A* cone is simplicial *if it is generated by a subset of a basis of* $\mathbb{R}^r$*.*

Remark 4.8. Any cone in $\mathbb{R}^2$ is simplicial.

Definition 4.9. *We say that a* lattice vector is primitive or simple *if its coordinates are relatively prime. If generating lattice vectors are primitive, then we say that they are* primitive generators *of* σ*.*

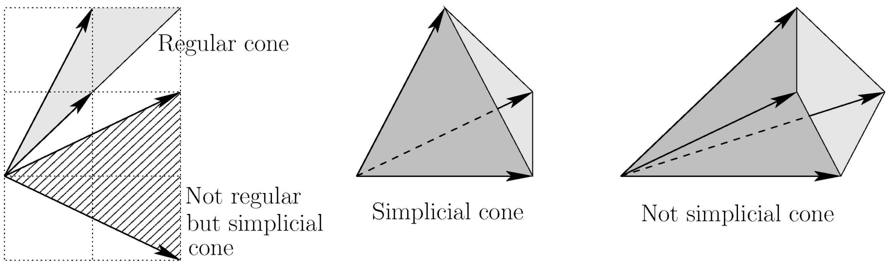


Fig. 6. Regular, non-regular, simplicial and non-simplicial cones

Regular cones will be of interest later on. Because of that we list some of their properties in the following elementary lemma.

Lemma 4.10 (compare Lemma 1.11, p. 147 in [8]). *The following conditions for an r-dimensional cone $\sigma_{\mathbb{R}}$ generated by $\mathbf{n}_1, \ldots, \mathbf{n}_r \in \mathbb{Z}^r$ are equivalent:*

(a) *σ is regular.*

(b) *$\mathbf{n}_1, \ldots, \mathbf{n}_r$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^r$.*

(c) *Any element of $\sigma_{\mathbb{Z}}$ is a nonnegative, integral, linear combination of $\mathbf{n}_1, \ldots, \mathbf{n}_r$.*

(d) $\det(\mathbf{n}_1, \ldots, \mathbf{n}_r) = \pm 1$.

(e) *There exists a linear transformation $L : \mathbb{R}^r \longrightarrow \mathbb{R}^r$, $L(\mathbb{Z}^r) \subset \mathbb{Z}^r$, that maps the canonical basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$ onto $\mathbf{n}_1, \ldots, \mathbf{n}_r$, respectively.*

4.5. Fans.

Definition 4.11 (Definition of fan). *A* fan *in $N_{\mathbb{R}}$ is a nonempty collection Δ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:*

(a) *Every face of any $\sigma \in \Delta$ is contained in Δ.*

(b) *For any σ, $\sigma' \in \Delta$ the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.*

The union $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ is called the support *of Δ.*

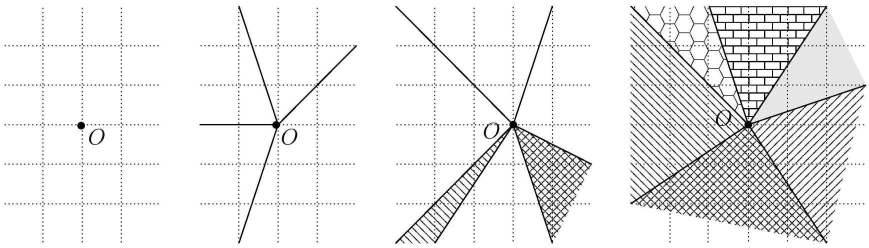


Fig. 7. Examples of fans in $\mathbb{R}^2$

We note that the half lines in the fans on Fig. 7 are passing through the lattice points.

Definition 4.12. *A fan Δ in $N_{\mathbb{R}} \simeq \mathbb{R}^r$ is regular (or smooth) if each nonzero cone is regular. That is, if for each $\sigma \in \Delta$ there exist a basis $\{\mathbf{n}_1, \ldots, \mathbf{n}_r\}$ of $\mathbb{Z}^r$ and a number $s \leqslant r$ such that $\sigma_{\mathbb{Z}}$ is generated by the first s vectors of the basis.*

Definition 4.13. *A fan Δ in $N_{\mathbb{R}} \simeq \mathbb{R}^r$ is complete if $|\Delta| = N_{\mathbb{R}}$.*

Later on we will use the notation: $\Delta(j)$ denotes the set of j-dimensional cones in Δ.

5. Affine toric varieties. First we define *affine toric varieties* which are particular cases of toric varieties. An affine toric variety corresponds to a single cone (together with its faces). A toric variety corresponds to a fan. Because a fan is a collection of cones with some relations between them, the same holds for a toric variety which is glued from affine toric varieties.

5.1. Abstract definitions of affine toric varieties. The definition of an affine toric variety can be given rigorously and abstractly in terms of algebraic geometry. We give a couple of abstract definitions, which are actually equivalent. In the next subsection we clarify them in more analytic terms.

First abstract definition. We start with a strictly convex rational polyhedral cone $\sigma_{\mathbb{R}}$. We know that the dual cone $\check{\sigma}_{\mathbb{Z}}$ is a finitely generated semigroup (Gordon's lemma). This semigroup determines $\mathbb{C}[\check{\sigma}_{\mathbb{Z}}]$ a commutative $\mathbb{C}$-algebra. Finally the *affine toric variety* is the spectrum of this $\mathbb{C}$-algebra, i.e., $\mathrm{Spec}(\mathbb{C}[\check{\sigma}_{\mathbb{Z}}])$. Formally the spectrum of an algebra A is the set of prime ideals equipped with Zariski topology.

Second abstract definition. Let

$$U_\sigma := \{u : \check{\sigma}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid u(0) = 1,\ u(\mathbf{m} + \mathbf{m}') = u(\mathbf{m})u(\mathbf{m}'),\ \forall \mathbf{m}, \mathbf{m}' \in \check{\sigma}_{\mathbb{Z}}\}.$$

We call the set U_σ an *affine toric variety.*

From this definition it is not so clear how U_σ looks like. Moreover, U_σ is like a dual to $\check{\sigma}_{\mathbb{Z}}$, but actually not quite. A general procedure from a cone to the affine toric variety can be described in the following sketchy diagram:

$$\sigma_{\mathbb{R}} \quad \rightsquigarrow \quad \check{\sigma}_{\mathbb{R}} \quad \rightsquigarrow \quad \check{\sigma}_{\mathbb{Z}} \quad \rightsquigarrow \quad U_\sigma \quad \text{“}\sim\text{”} \quad \text{dual to } \check{\sigma}_{\mathbb{Z}}.$$

5.2. Less abstract definitions of affine toric varieties. We give two slightly different but equivalent approaches. Actually we explain the structure of U_σ defined above.

Let $\sigma_{\mathbb{R}}$ be a cone in $N_{\mathbb{R}}$. Then take the dual cone $\check{\sigma}_{\mathbb{R}}$ and assume that $\mathbf{m}_1, \ldots, \mathbf{m}_k$ generate the cone $\check{\sigma}_{\mathbb{Z}}$, i.e.,

$$\check{\sigma}_{\mathbb{Z}} = \mathbb{Z}_{\geqslant 0}\{\mathbf{m}_1, \ldots, \mathbf{m}_k\} = \mathbb{Z}_{\geqslant 0}\mathbf{m}_1 + \ldots + \mathbb{Z}_{\geqslant 0}\mathbf{m}_k. \tag{1}$$

Since the cone $\sigma_{\mathbb{R}}$ is strictly convex, therefore the dimension of its dual is r. We note that the number of generators k is at least r, $k \geqslant r$, the dimension of the ambient space.

I. First explanation – identification of U_σ with the image on generators. If $\mathbf{m}_1, \ldots, \mathbf{m}_k$ are generators of $\check{\sigma}_{\mathbb{Z}}$ over $\mathbb{Z}$, then each $u \in U_\sigma$ is uniquely determined by the values of u at the generators:

$$\begin{array}{ccc} U_\sigma \ni u & \longleftrightarrow & (u(\mathbf{m}_1), \ldots, u(\mathbf{m}_k)) \in \mathbb{C}^k \\ U_\sigma & \sim & V_\sigma = \{(u(\mathbf{m}_1), \ldots, u(\mathbf{m}_k)) \mid u \in U_\sigma\} \subset \mathbb{C}^k \end{array} \tag{2}$$

II. Second explanation – monomial equations for U_σ. Again, as in I., $u \in U_\sigma$ is identified with its value at the generators $u(\mathbf{m}_1), \ldots, u(\mathbf{m}_k) \in \mathbb{C}^k$. Since $k \geqslant r$, the generators are $\mathbb{Z}$ linearly dependent, and they satisfy conditions of the form

$$c_1\mathbf{m}_1 + \ldots + c_k\mathbf{m}_k = 0, \quad \text{where} \quad c_1, \ldots, c_k \in \mathbb{Z}. \tag{3}$$

The coefficients can be positive, negative or zero. However we can modify equation (3) in such a way that the terms with negative coefficients are moved to the other side, so the equation becomes

$$a_1\mathbf{m}_1 + \ldots + a_k\mathbf{m}_k = b_1\mathbf{m}_1 + \ldots + b_k\mathbf{m}_k, \ \text{ where } \ a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Z}_{\geqslant 0}. \tag{4}$$

We can cancel the excessive terms if they appear on both sides of the equation, of course keeping the coefficients nonnegative. Without loss of generality we can assume that if $\mathbf{m}_j$ appears on one side of the equation, then it does not appear on the other side.

The left and the right-hand sides of (4) are elements of $\check{\sigma}_{\mathbb{Z}}$ so u can be applied and we get

$$\begin{aligned}(u(\mathbf{m}_1))^{a_1} \cdot \ldots \cdot (u(\mathbf{m}_k))^{a_k} &= u(a_1\mathbf{m}_1 + \ldots + a_k\mathbf{m}_k) \\ &= u(b_1\mathbf{m}_1 + \ldots + b_k\mathbf{m}_k) \\ &= (u(\mathbf{m}_1))^{b_1} \cdot \ldots \cdot (u(\mathbf{m}_k))^{b_k}.\end{aligned}$$

Denoting $z_1 = u(\mathbf{m}_1), \ldots, z_k = u(\mathbf{m}_k)$, the above considerations give a next explanation of U_σ, namely
The *affine toric variety* determined by a cone σ is an algebraic affine variety of $\mathbb{C}^k$ given by *irreducible equations*

$$z_1^{a_1} z_2^{a_2} \ldots z_k^{a_k} - z_1^{b_1} z_2^{b_2} \ldots z_k^{b_k} = 0 \tag{5}$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Z}_{\geqslant 0}$ and satisfying

$$a_1\mathbf{m}_1 + \ldots + a_k\mathbf{m}_k = b_1\mathbf{m}_1 + \ldots + b_k\mathbf{m}_k. \tag{6}$$

The number of essentially different equations is finite, namely $k - r$, where r is the dimension of $\check{\sigma}_{\mathbb{R}}$.

III. Third explanation – parametrization of U_σ. We shall show that for any $u \in U_\sigma$, which takes nonzero values (denote this subset by U_σ^*), there exists (in general not uniquely) $t^{\mathbf{m}}$ such that

$$\begin{aligned}&U_\sigma^* \ni u = u(\mathbf{m}) = t^{\mathbf{m}} = t_1^{m_1} t_2^{m_2} \cdot \ldots \cdot t_r^{m_r} \quad \forall\, \mathbf{m} = (m_1, \ldots, m_r) \in \check{\sigma}_{\mathbb{Z}} \\ &t = (t_1, \ldots, t_r) \in (\mathbb{C}^*)^r.\end{aligned}$$

To see this, let $\mathbf{m}_j = (m_{j1}, \ldots, m_{jr})$, $j = 1, \ldots, k$, be generators of $\check{\sigma}_{\mathbb{Z}}$ over $\mathbb{Z}$. To find t_j, we have to solve the system of equations

$$\begin{cases} t^{\mathbf{m}_1} = t_1^{m_{11}} \cdot \ldots \cdot t_r^{m_{1r}} = u(\mathbf{m}_1) \\ \ldots \qquad \ldots \qquad \ldots \\ t^{\mathbf{m}_k} = t_1^{m_{k1}} \cdot \ldots \cdot t_r^{m_{kr}} = u(\mathbf{m}_k) \end{cases}$$

or, taking natural logarithm, we get a system of linear equations with respect to $\log t_j$:

$$(7)\qquad \begin{cases} m_{11}\log t_1 + \ldots + m_{1r}\log t_r = \log(u(\mathbf{m}_1)) \\ \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ m_{k1}\log t_1 + \ldots + m_{kr}\log t_r = \log(u(\mathbf{m}_k))\ . \end{cases}$$

We have to be careful with taking the branch of the logarithm on the right-hand side of the equations in order not to make the system inconsistent. As we have seen in the second explanation, the rank of the matrix $(m_{\alpha\beta})_{\alpha=1,\ldots,k;\,\beta=1,\ldots,r}$ is r since, as we mentioned, the dimension of the cone $\check{\sigma}_{\mathbb{R}}$ is r. So in the system (7) we have more or the same number of equations as the number of unknowns. Without going into elementary linear algebra technical details, this system is consistent if we choose suitably the branches of the logarithms on the right-hand side. Consequently there is a solution of this system.

Taking into account the first explanation, if $\mathbf{m}_1, \ldots, \mathbf{m}_k$ are generators of $\check{\sigma}_{\mathbb{Z}}$, then

$$(8)\qquad \begin{array}{lcl} U_\sigma & \sim & V_\sigma = \{(u(\mathbf{m}_1), \ldots, u(\mathbf{m}_k)) \mid \quad u \in U_\sigma\} \subset \mathbb{C}^k \\ U_\sigma^* & \sim & V_\sigma^* = \{(t^{\mathbf{m}_1}, \ldots, t^{\mathbf{m}_k}) \mid \quad t = (t_1, \ldots, t_r) \in (\mathbb{C}^*)^r\}. \end{array}$$

Consequently we have the mapping

$$(9)\qquad \varphi \ :\ (\mathbb{C}^*)^r \longrightarrow V_\sigma \subset \mathbb{C}^k$$

defined by

$$(10)\qquad \varphi(t_1, \ldots, t_r) = (t^{\mathbf{m_1}}, \ldots, t^{\mathbf{m_k}}) = (t_1^{m_{11}} \cdot \ldots \cdot t_r^{m_{1r}}, \ldots, t_1^{m_{k1}} \cdot \ldots \cdot t_r^{m_{kr}})\ .$$

We have seen that $U_\sigma \sim V_\sigma$ can be considered as a subset of $\mathbb{C}^k$. If so, then we can write $U_\sigma \subset \mathbb{C}^k$ and U_σ is the *(Zariski) closure* of the image of the map φ. This means that U_σ is the smallest affine variety containing the image of φ. We note that the dimension k, of the space into which U_σ is mapped, depends on the number of generators of the dual cone $\check{\sigma}_{\mathbb{Z}}$ we choose. It is not necessary to choose the minimal number of generators.

IV. Relation between the first and second explanations. The components of parametrization of U_σ as in (10) satisfy equations (5):

$$(t^{\mathbf{m_1}})^{a_1}(t^{\mathbf{m_2}})^{a_2}\ldots(t^{\mathbf{m_k}})^{a_k} - (t^{\mathbf{m_1}})^{b_1}(t^{\mathbf{m_2}})^{b_2}\ldots(t^{\mathbf{m_k}})^{b_k} =$$
$$= t^{a_1\mathbf{m_1}+\ldots+a_k\mathbf{m_k}} - t^{b_1\mathbf{m_1}+\ldots+b_k\mathbf{m_k}} = 0$$

by (6).

5.3. How an affine toric variety can be found practically? The question is how such equations in (5) can be found practically. Here we describe the procedure. Since $\sigma_{\mathbb{R}}$ is a strictly convex polyhedral cone in $N_{\mathbb{R}} \simeq \mathbb{R}^r$, the dimension of $\check{\sigma}_{\mathbb{R}}$ is r.

Let $\mathbf{m}_1, \dots, \mathbf{m}_k$ be generators of $\check{\sigma}_{\mathbb{Z}}$. We can find $k-r$ linear relations between $\mathbf{m}_1, \dots, \mathbf{m}_k$. These relations are with integer coefficients $c_{\alpha\beta}$, where the coefficients are allowed to be positive, negative or zero:

$$\begin{cases} c_{11}\mathbf{m}_1 + \ldots + c_{1k}\mathbf{m}_k & = 0 \\ \ldots \quad \ldots \quad \ldots & \\ c_{k-r\,1}\mathbf{m}_1 + \ldots + c_{k-r\,k}\mathbf{m}_k & = 0 \end{cases}$$

But now we move to the right-hand side the terms with negative coefficients, so on both sides we have only non-negative integer coefficients of the vectors $\mathbf{m}_\alpha$. Moreover, by cancelling excessive vectors, we can assume that in each equation on both sides we have different vectors $\mathbf{m}_\alpha$, i.e., if $\mathbf{m}_\alpha$ appears on the left-hand side of an equation, then does not on the right-hand side, and vice-versa. Assume that these equations are

$$\begin{cases} a_{11}\mathbf{m}_1 + \ldots + a_{1k}\mathbf{m}_k & = & b_{11}\mathbf{m}_1 + \ldots + b_{1k}\mathbf{m}_k \\ \ldots \quad \ldots & \ldots & \ldots \quad \ldots \\ a_{k-r\,1}\mathbf{m}_1 + \ldots + a_{k-r\,k}\mathbf{m}_k & = & b_{k-r\,1}\mathbf{m}_1 + \ldots + b_{k-r\,k}\mathbf{m}_k \end{cases}$$

Therefore

$$a_{\alpha\beta}\, b_{\alpha\beta} = 0 \qquad \text{for} \quad \alpha = 1, \dots, k-r, \quad \beta = 1, \dots, k.$$

If we assign a variable z_α to each $u(\mathbf{m}_\alpha)$, i.e., $u(\mathbf{m}_\alpha) = z_\alpha$, then the affine toric variety is given by

$$\begin{cases} z_1^{a_{11}} \cdot \ldots \cdot z_k^{a_{1k}} & = & z_1^{b_{11}} \cdot \ldots \cdot z_k^{b_{1k}} \\ \ldots \quad \ldots & \ldots & \ldots \quad \ldots \\ z_1^{a_{k-r\,1}} \cdot \ldots \cdot z_k^{a_{k-r\,k}} & = & z_1^{b_{k-r\,1}} \cdot \ldots \cdot z_k^{b_{k-r\,k}} \end{cases}$$

5.4. Examples of affine toric varieties.

Example 5.1 ($(\mathbb{C}^*)^r$ complex torus)**.** Let take the cone $\sigma_{\mathbb{R}} = \{0\} \subset \mathbb{R}^r$. The dual cone is the entire space $\mathbb{R}^r$ and is generated by $2r$ vectors

$$\mathbf{m}_1 = \mathbf{e}_1,\ \mathbf{m}_2 = -\mathbf{e}_1, \dots, \mathbf{m}_{2r-1} = \mathbf{e}_r,\ \mathbf{m}_{2r} = -\mathbf{e}_r,$$

with the relations

$$\mathbf{m}_1 + \mathbf{m}_2 = 0, \ldots, \mathbf{m}_{2r-1} + \mathbf{m}_{2r} = 0.$$

Consequently,

$$\begin{cases} u(\mathbf{m}_1) = z_1 \\ u(\mathbf{m}_2) = z_2 \\ \ldots \quad \ldots \\ u(\mathbf{m}_{2r-1}) = z_{2r-1} \\ u(\mathbf{m}_{2r}) = z_{2r} \end{cases} \Longrightarrow \begin{cases} u(\mathbf{m}_1 + \mathbf{m}_2) = u(0) = 1 \\ \ldots \quad \ldots \quad \ldots \\ u(\mathbf{m}_{2r-1} + \mathbf{m}_{2r}) = u(0) = 1 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} z_1 \neq 0, \ z_2 \neq 0 \\ \ldots \quad \ldots \\ z_{2r-1} \neq 0, \ z_{2r} \neq 0 \end{cases} \Longleftarrow \quad \mathbb{C}^{2r} \supset V_\sigma \ : \ \begin{cases} z_1 z_2 = 1 \\ \ldots \quad \ldots \\ z_{2r-1} z_{2r} = 1 \end{cases}$$

or the parameterized form is

$$\varphi \ : \ (C^*)^r \longrightarrow \mathbb{C}^{2r}, \quad \varphi(t) = (t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}).$$

Example 5.2 $(\mathbb{C}^r)$. Let $\sigma_{\mathbb{R}} = \mathbb{R}_{\geqslant 0}\mathbf{e}_1 + \ldots + \mathbb{R}_{\geqslant 0}\mathbf{e}_r$. Then its dual cone is actually the same (formally in the dual space)

$$\check{\sigma}_{\mathbb{R}} = \mathbb{R}_{\geqslant 0}\mathbf{e}_1 + \ldots + \mathbb{R}_{\geqslant 0}\mathbf{e}_r, \qquad \check{\sigma}_{\mathbb{Z}} = \mathbb{Z}_{\geqslant 0}\mathbf{e}_1 + \ldots + \mathbb{Z}_{\geqslant 0}\mathbf{e}_r.$$

The generators of $\check{\sigma}_{\mathbb{Z}}$ are $\mathbf{m}_1 = \mathbf{e}_1, \ldots, \mathbf{m}_r = \mathbf{e}_r$, are linearly independent, and they do not satisfy any linear (nontrivial) conditions, therefore the mapping is

$$\begin{cases} u(\mathbf{m}_1) = u(\mathbf{e}_1) = z_1 \\ \ldots \quad \ldots \\ u(\mathbf{m}_r) = u(\mathbf{e}_r) = z_r \end{cases} \qquad V_\sigma = \mathbb{C}^r$$

or the parametric form

$$\varphi \ : \ (\mathbb{C}^*)^r \longrightarrow \mathbb{C}^r, \quad \varphi(t) = (t^{\mathbf{m}_1}, \ldots, t^{\mathbf{m}_r}) = (t_1, \ldots, t_r).$$

The Zariski closure of $(\mathbb{C}^*)^r$ in $\mathbb{C}^r$ is just $\mathbb{C}^r$.

Example 5.3. Here we construct the affine toric variety for the cone from Example 4.4. The generators of the dual cone $\check{\sigma}_{\mathbb{Z}}$ are

$$\begin{array}{ll} \mathbf{m}_1 = -\mathbf{e}_1 + 4\mathbf{e}_2 & u(\mathbf{m}_1) = z_1 \\ \mathbf{m}_2 = \mathbf{e}_2 & u(\mathbf{m}_2) = z_2 \\ \mathbf{m}_3 = \mathbf{e}_1 & u(\mathbf{m}_3) = z_3 \\ \mathbf{m}_4 = 3\mathbf{e}_1 - \mathbf{e}_2 & u(\mathbf{m}_4) = z_4 \end{array} \quad \begin{cases} \mathbf{m}_1 + \mathbf{m}_3 = 4\mathbf{m}_2 \\ \mathbf{m}_2 + \mathbf{m}_4 = 3\mathbf{m}_3 \end{cases} \Longrightarrow \begin{cases} z_1 z_3 = z_2^4 \\ z_2 z_4 = z_3^3 \end{cases} \text{ in } \mathbb{C}^4$$

In parametric form, the mapping is

$$\varphi \,:\, (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^4, \quad \varphi(t_1, t_2) = (t^{\mathbf{m}_1},\, t^{\mathbf{m}_2},\, t^{\mathbf{m}_3},\, t^{\mathbf{m}_4}) = (t_1^{-1} t_2^4,\ t_2,\ t_1,\ t_1^3 t_2^{-1}),$$

5.5. Complex torus and affine toric varieties. As we have seen in Example 5.1, the complex torus can be identified with the group

$$T = T(N_{\mathbb{Z}}) = \{t : \mathbb{Z}^r \longrightarrow \mathbb{C} \mid t(0) = 1,\ t(\mathbf{m} + \mathbf{m}') = t(\mathbf{m})\, u(\mathbf{m}'),\ \forall \mathbf{m}, \mathbf{m}' \in \mathbb{Z}^r\}.$$

Recall the definition of the affine toric variety

$$U_\sigma := \{u : \check{\sigma}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid u(0) = 1,\ u(\mathbf{m} + \mathbf{m}') = u(\mathbf{m})\, u(\mathbf{m}'),\ \forall \mathbf{m}, \mathbf{m}' \in \check{\sigma}_{\mathbb{Z}}\}.$$

Since $\sigma_{\mathbb{Z}} \subset \mathbb{Z}^r$, we have naturally $T \subset U_\sigma$.

Also T acts naturally on U_σ, namely

$$(tu)(\mathbf{m}) = t(\mathbf{m})\, u(\mathbf{m}) \qquad \text{for} \quad t \in T, \quad u \in U_\sigma.$$

5.6. Affine toric varieties generated by faces of a cone. Let $\tau_{\mathbb{R}}$ be a face of a cone $\sigma_{\mathbb{R}}$. From the definition of a face, there exists an element $\mathbf{m}_0 \in \check{\sigma}_{\mathbb{Z}}$ such that

$$\tau_{\mathbb{R}} = \sigma_{\mathbb{R}} \cap \mathbf{m}_0^{\perp} = \{\mathbf{n} \in \sigma_{\mathbb{R}} \mid \ \langle \mathbf{m}_0,\ \mathbf{n}\rangle = 0\}.$$

Using the property that for closed convex cones ρ_1 and ρ_2 we have $(\rho_1 \cap \rho_2)^{\vee} = \check{\rho}_1 + \check{\rho}_2$, we get

$$\begin{aligned} &\check{\tau}_{\mathbb{R}} = (\sigma_{\mathbb{R}} \cap \mathbf{m}_0^{\perp})^{\vee} = \check{\sigma}_{\mathbb{R}} + (\mathbf{m}_0^{\perp})^{\vee} = \check{\sigma}_{\mathbb{R}} + \mathbb{R}\,\mathbf{m}_0 = \check{\sigma}_{\mathbb{R}} + \mathbb{R}_{\geqslant 0}(-\mathbf{m}_0) \\ &\check{\tau}_{\mathbb{Z}} = \check{\sigma}_{\mathbb{Z}} + \mathbb{Z}_{\geqslant 0}(-\mathbf{m}_0)\ . \end{aligned} \tag{11}$$

Take a look at the definition of affine toric varieties for σ and τ:

$$U_\sigma = \{u : \check{\sigma}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid \ u(0) = 1,\ u(\mathbf{m} + \mathbf{m}') = u(\mathbf{m}) u(\mathbf{m}'),\ \ \forall \mathbf{m}, \mathbf{m}' \in \check{\sigma}_{\mathbb{Z}}\}$$

$$U_\tau = \{v : \check{\tau}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid \ v(0) = 1,\ v(\mathbf{m} + \mathbf{m}') = v(\mathbf{m}) v(\mathbf{m}'),\ \ \forall \mathbf{m}, \mathbf{m}' \in \check{\tau}_{\mathbb{Z}}\}.$$

Since $\check{\sigma}_{\mathbb{Z}} \subset \check{\tau}_{\mathbb{Z}}$ therefore $U_\sigma \supset U_\tau$. Both $\mathbf{m}_0$ and $-\mathbf{m}_0$ are in $\check{\tau}_{\mathbb{Z}}$ therefore for $u \in U_\tau$ we have $u(\mathbf{m}_0)u(-\mathbf{m}_0) = u(0) = 1$ which implies that $u(\mathbf{m}_0) \neq 0$. Consequently, $U_\tau = \{u \in U_\sigma \mid u(\mathbf{m}_0) \neq 0\} \subset U_\sigma$, which, in particular, implies that U_τ is a dense open subset of U_σ.

How to see this when parameterizing U_σ and U_τ? We choose generators:

$$\begin{aligned} \check{\sigma}_{\mathbb{Z}} &: \quad \mathbf{m}_0,\, \mathbf{m}_1, \ldots, \mathbf{m}_k \\ \check{\tau}_{\mathbb{Z}} &: \quad \mathbf{m}_0,\, \mathbf{m}_1, \ldots, \mathbf{m}_k,\, \mathbf{m}_{k+1} = -\mathbf{m}_0 \end{aligned}$$

and then we have the following identifications:

$$U_\sigma \ni u \longleftrightarrow (u(\mathbf{m}_0), u(\mathbf{m}_1), \ldots, u(\mathbf{m}_k)) \in \mathbb{C}^{k+1} \tag{12}$$

$$U_\tau \ni v \longleftrightarrow (v(\mathbf{m}_0), v(\mathbf{m}_1), \ldots, v(\mathbf{m}_k), v(-\mathbf{m}_0)) \in \mathbb{C}^{k+2} \tag{13}$$

Looking at the first and the last component of the parametrization in (13) we clearly see that $v(\mathbf{m}_0)v(-\mathbf{m}_0) = 1$ which implies that $v(\mathbf{m_0}) \neq 0$. Because $U_\tau \subset U_\sigma$, projecting U_τ under the parametrization (13) onto $\mathbb{C}^{k+1}$ we get the open subset of U_σ, where $z_0 \neq 0$.

Example 5.4. Here we consider the cone with faces from Example 4.4. In the cone $\sigma_{\mathbb{R}}$ take the face τ through $4\mathbf{e}_1 + \mathbf{e}_2 = [4,\, 1]$. The dual cone $\check{\tau}_{\mathbb{R}}$ is a

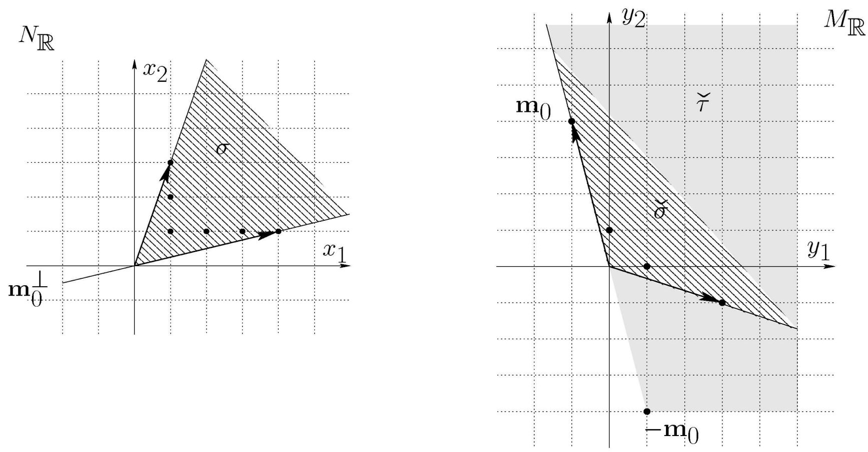


Fig. 8. Cone, its face and their duals

half plane perpendicular to $[4,1]$. The generators of $\check{\tau}_{\mathbb{Z}}$ and $\check{\sigma}_{\mathbb{Z}}$ can be chosen

$$\check{\sigma}_{\mathbb{Z}}: \begin{cases} \mathbf{m}_0 = -\mathbf{e}_1 + 4\mathbf{e}_2 & u(\mathbf{m}_0) = z_0 \\ \mathbf{m}_1 = \mathbf{e}_2 & u(\mathbf{m}_1) = z_1 \\ \mathbf{m}_2 = \mathbf{e}_1 & u(\mathbf{m}_2) = z_2 \\ \mathbf{m}_3 = 3\mathbf{e}_1 - \mathbf{e}_2 & u(\mathbf{m}_3) = z_3 \end{cases} \quad \begin{cases} \mathbf{m}_0 + \mathbf{m_2} = 4\mathbf{m}_1 \\ \mathbf{m}_3 + \mathbf{m}_1 = 3\mathbf{m}_2 \end{cases} \quad \begin{cases} z_0 z_2 = z_1^4 \\ z_3 z_1 = z_2^3 \end{cases}$$

$$\check{\tau}_{\mathbb{Z}}: \begin{cases} \mathbf{m}_0 = -\mathbf{e}_1 + 4\mathbf{e}_2 & v(\mathbf{m}_0) = z_0 \\ \mathbf{m}_1 = \mathbf{e}_2 & v(\mathbf{m}_1) = z_1 \\ \mathbf{m}_2 = \mathbf{e}_1 & v(\mathbf{m}_2) = z_2 \\ \mathbf{m}_3 = 3\mathbf{e}_1 - \mathbf{e}_2 & v(\mathbf{m}_3) = z_3 \\ \mathbf{m}_4 = \mathbf{e}_1 - 4\mathbf{e}_2 & v(\mathbf{m}_4) = z_4 \end{cases} \quad \begin{cases} \mathbf{m}_0 + \mathbf{m_2} = 4\mathbf{m}_1 \\ \mathbf{m}_3 + \mathbf{m}_1 = 3\mathbf{m}_2 \\ \mathbf{m}_0 + \mathbf{m}_4 = 0 \end{cases} \quad \begin{cases} z_0 z_2 = z_1^4 \\ z_3 z_1 = z_2^3 \\ z_0 z_4 = 1 \end{cases}$$

Using parametrization, we get

$$U_\tau \sim \{(t_1^{-1}t_2^4,\ t_2,\ t_1,\ t_1^3 t_2^{-1},\ t_1 t_2^{-4})\} \subset \mathbb{C}^5$$

$$\downarrow$$

$$\{(t_1^{-1}t_2^4,\ t_2,\ t_1,\ t_1^3 t_2^{-1}) \ \mid\ t_1^{-1}t_2^4 \neq 0\} \subset \mathbb{C}^4$$

$$\cap$$

$$U_\sigma \sim \{(t_1^{-1}t_2^4,\ t_2,\ t_1,\ t_1^3 t_2^{-1})\} \subset \mathbb{C}^4$$

6. Definition and examples of toric varieties.

6.1. Definition of toric varieties and change of coordinates. A more interesting and more complicated case than affine toric varieties is just *toric varieties*. The difference can be described as: An affine toric variety corresponds to a single cone, but a toric variety corresponds to a fan, which is a collection of cones. Let Δ be a fan in $\mathbb{R}^r$. For each cone $\sigma \in \Delta$ the corresponding affine toric variety is denoted by U_σ or $T(\sigma)$ and the toric variety corresponding to the fan Δ will be denoted by $T(\Delta)$.

Definition 6.1 (Toric variety; see e.g., [9], p. 20–21)**.** *The toric variety $T(\Delta)$ is constructed by taking the disjoint union of the affine toric varieties U_σ, one for each σ in Δ, and gluing as follows: for cones σ and τ, the intersection $\sigma \cap \tau$ is a face of each, so $U_{\sigma\cap\tau}$ is identified as a principal open subvariety of U_σ and of U_τ. We glue U_σ to U_τ by this identification on these open subvarieties.*

Of course this definition requires some explanation. Intuitively it is clear, but we give a concrete way how to find the transition mappings between charts.

Here in this subsection we take a closer look at the change of variables between two charts on a toric variety. Assume that we have two cones σ and τ that have a common face $\sigma \cap \tau$. We consider two cases, one simpler and one more complicated:

1^0 Dual cones $\check{\sigma}_{\mathbb{R}}$ and $\check{\tau}_{\mathbb{R}}$ are r-dimensional and regular, i.e., the number of generators of $\check{\sigma}_{\mathbb{Z}}$ is r and the same for $\check{\tau}_{\mathbb{R}}$.

2^0 The number of generators of $\check{\sigma}_{\mathbb{Z}}$ is greater than r and the same for $\check{\tau}_{\mathbb{R}}$.

6.2. Regular fans. Let σ and τ in $\mathbb{R}^r$ be two regular cones of dimension r with a common face $\sigma \cap \tau$. We note that always the common face is not empty, because in the worst case it is just the origin. We take $\check{\sigma}_{\mathbb{Z}}$ with generators $\mathbf{m}_1, \ldots, \mathbf{m}_r$ and $\check{\tau}_{\mathbb{Z}}$ with generators $\mathbf{g}_1, \ldots, \mathbf{g}_r$.

From the first definition of an affine toric variety we have

$$U_\sigma = \{u : \check{\sigma}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid u(0) = 1,\ u(\mathbf{m} + \mathbf{m}') = u(\mathbf{m})\, u(\mathbf{m}'),\ \forall \mathbf{m}, \mathbf{m}' \in \check{\sigma}_{\mathbb{Z}}\}.$$

$$U_\tau = \{u : \check{\tau}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid u(0) = 1,\ u(\mathbf{g} + \mathbf{g}') = u(\mathbf{g})\, u(\mathbf{g}'),\ \forall \mathbf{g}, \mathbf{g}' \in \check{\tau}_{\mathbb{Z}}\}.$$

Since $\mathbf{m}_1, \ldots, \mathbf{m}_r$ are linearly independent, therefore there are no linear relations between them and consequently $V_\sigma = \mathbb{C}^r$. Similarly $V_\tau = \mathbb{C}^r$.

If we assign coordinates for u in both U_σ and U_τ as in (8) then we obtain

$$\begin{array}{ccc} & U_\sigma \cap U_\tau & \\ & & \\ & u & \\ & \swarrow \quad \searrow & \\ (u(\mathbf{m}_1), \ldots, u(\mathbf{m}_r)) & & (u(\mathbf{g}_1), \ldots, u(\mathbf{g}_r)) \\ (z_{\sigma 1}, \ldots, z_{\sigma r}) & & (z_{\tau 1}, \ldots, z_{\tau r}) \\ V_\sigma & & V_\tau \\ \| & & \| \\ \mathbb{C}^r & & \mathbb{C}^r \end{array} \tag{14}$$

Since both cones σ and τ are regular, their generators $\mathbf{m}_1, \ldots, \mathbf{m}_r$ and $\mathbf{g}_1, \ldots, \mathbf{g}_r$ also span $\mathbb{Z}^r$. So we have relations between these two sets of generators

$$\left\{ \begin{array}{lcl} \mathbf{g}_1 & = & a_{11}\mathbf{m}_1 + \ldots + a_{1r}\mathbf{m}_r \\ \ldots & \ldots & \qquad\ldots \\ \mathbf{g}_r & = & a_{r1}\mathbf{m}_1 + \ldots + a_{rr}\mathbf{m}_r \end{array} \right. \qquad a_{\alpha\beta} \in \mathbb{Z}$$

and from here

$$\begin{cases} u(\mathbf{g}_1) & = & [u(\mathbf{m}_1)]^{a_{11}} \dots [u(\mathbf{m}_r)]^{a_{1r}} \\ \dots & \dots & \dots \qquad \dots \\ u(\mathbf{g}_r) & = & [u(\mathbf{m}_1)]^{a_{r1}} \dots u[(\mathbf{m}_r)]^{a_{rr}} \end{cases}$$

and finally

$$\begin{cases} z_{\tau 1} & = & z_{\sigma 1}^{a_{11}} \cdot \ldots \cdot z_{\sigma r}^{a_{1r}} \\ \dots & \dots & \dots \\ z_{\tau r} & = & z_{\sigma 1}^{a_{r1}} \cdot \ldots \cdot z_{\sigma r}^{a_{rr}} \end{cases} \tag{15}$$

From the above procedure we see explicitly the change of variables between two charts. Moreover they are Laurent monomials.

6.2.1. $\mathbb{CP}^2$ as a toric variety. We take the fan composed of the 2-dimensional cones $\rho_{\mathbb{R}}$, $\sigma_{\mathbb{R}}$ and $\tau_{\mathbb{R}}$, and of their faces. The cones and their dual are generated by the vectors:

$$\sigma_{\mathbb{Z}} : \mathbf{n}_{\sigma 1} = [1,\ 0], \quad \mathbf{n}_{\sigma 2} = [0,\ 1]; \qquad \check{\sigma}_{\mathbb{Z}} : \mathbf{m}_{\sigma 1} = [1,\ 0], \quad \mathbf{m}_{\sigma 2} = [0,\ 1];$$

$$\tau_{\mathbb{Z}} : \mathbf{n}_{\tau 1} = [0,\ 1], \quad \mathbf{n}_{\tau 2} = [-1,\ -1]; \qquad \check{\tau}_{\mathbb{Z}} : \mathbf{m}_{\tau 1} = [-1,\ 0], \quad \mathbf{m}_{\tau 2} = [-1,\ 1];$$

$$\rho_{\mathbb{Z}} : \mathbf{n}_{\rho 1} = [1,\ 0], \quad \mathbf{n}_{\rho 2} = [-1,\ -1]; \qquad \check{\rho}_{\mathbb{Z}} : \mathbf{m}_{\rho 1} = [0,\ -1], \quad \mathbf{m}_{\rho 2} = [1,\ -1].$$

They are reflected on the figure Fig. 9. Using the procedure described above, we

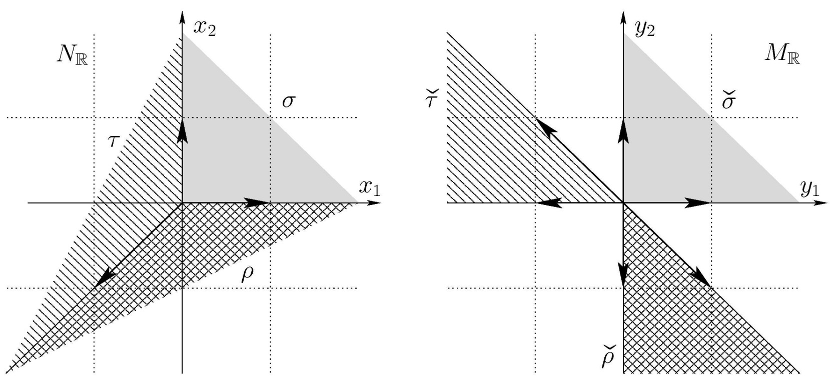


Fig. 9. Fan and dual cones for $\mathbb{CP}^2$

have the following relations between the generators of the dual cones:

$$\tau/\sigma : \begin{cases} \mathbf{m}_{\tau 1} = -\mathbf{m}_{\sigma 1} \\ \mathbf{m}_{\tau 2} = -\mathbf{m}_{\sigma 1} + \mathbf{m}_{\sigma 2} \end{cases} \Longrightarrow \begin{cases} z_{\tau 1} = z_{\sigma 1}^{-1} \\ z_{\tau 2} = z_{\sigma 1}^{-1} z_{\sigma 2} \end{cases}$$

$$\rho/\tau : \begin{cases} \mathbf{m}_{\rho 1} = \mathbf{m}_{\tau 1} - \mathbf{m}_{\tau 2} \\ \mathbf{m}_{\rho 2} = \qquad -\mathbf{m}_{\tau 2} \end{cases} \Longrightarrow \begin{cases} z_{\rho 1} = z_{\tau 1} z_{\tau 2}^{-1} \\ z_{\rho 2} = z_{\tau 2}^{-1} \end{cases}$$

$$\rho/\sigma : \begin{cases} \mathbf{m}_{\rho 1} = \qquad -\mathbf{m}_{\sigma 2} \\ \mathbf{m}_{\rho 2} = \mathbf{m}_{\sigma 1} - \mathbf{m}_{\sigma 2} \end{cases} \Longrightarrow \begin{cases} z_{\rho 1} = z_{\sigma 2}^{-1} \\ z_{\rho 2} = z_{\sigma 1} z_{\sigma 2}^{-1} \end{cases}$$

From the transition functions clearly we see that the variety is $\mathbb{CP}^2$.

6.2.2. Hirzebruch surfaces. For an integer a let the fan Δ consist of four 2-dimensional cones with their faces

$$\sigma_1 : \ = \mathbb{R}_{\geqslant 0}\mathbf{e}_1 + \mathbb{R}_{\geqslant 0}\mathbf{e}_2 \qquad \sigma_2 : \ = \mathbb{R}_{\geqslant 0}\mathbf{e}_1 + \mathbb{R}_{\geqslant 0}(-\mathbf{e}_2)$$

$$\sigma_3 : \ = \mathbb{R}_{\geqslant 0}(-\mathbf{e}_1 + a\mathbf{e}_2) + \mathbb{R}_{\geqslant 0}(-\mathbf{e}_2) \quad \sigma_4 : \ = \mathbb{R}_{\geqslant 0}(-\mathbf{e}_1 + a\mathbf{e}_2) + \mathbb{R}_{\geqslant 0}\mathbf{e}_2$$

The fan Δ determines a toric variety $T(\Delta)$, which is called *Hirzebruch surface*, usually denoted by F_a or Σ_a. Since F_a and F_{-a} are easily seen to be isomorphic, we usually assume that $a \geqslant 0$. We examine more precisely this toric variety and calculate the transition functions between charts.

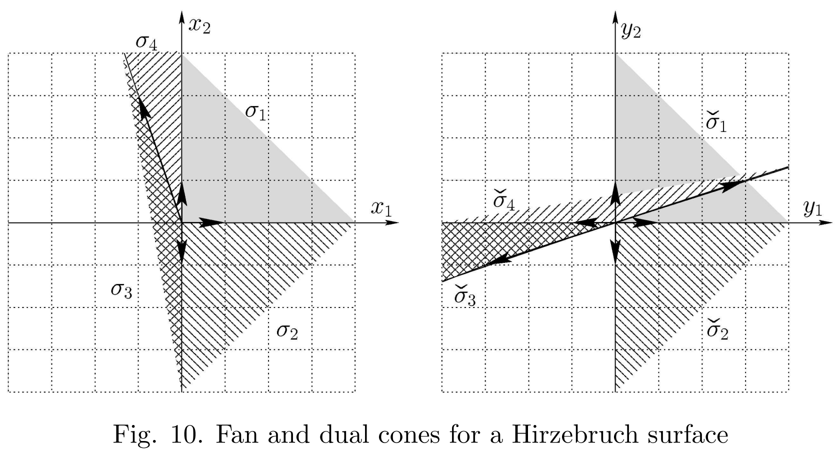


Fig. 10. Fan and dual cones for a Hirzebruch surface

First we list generators of the original cones and the corresponding dual cones.

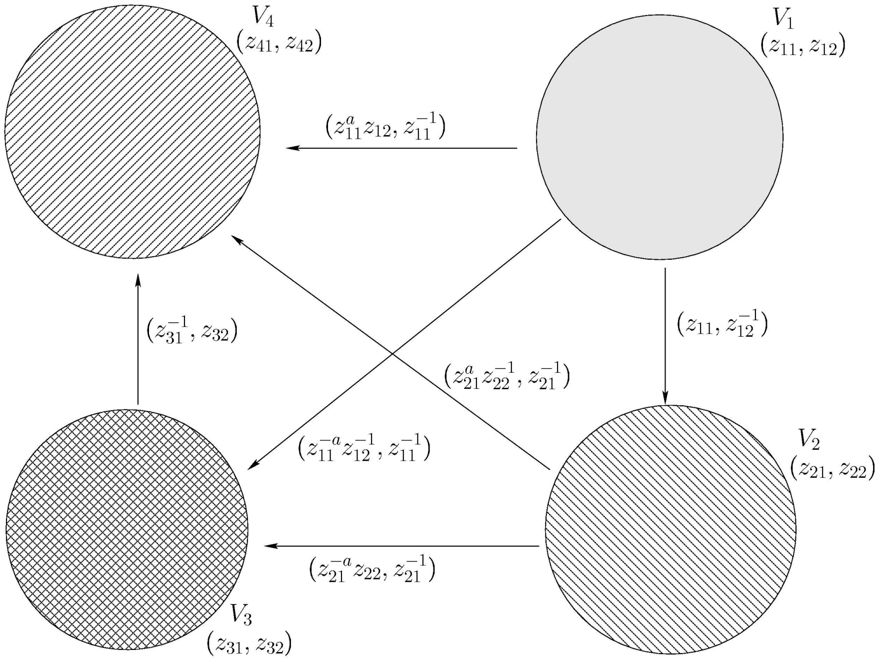


Fig. 11. Transition functions for a Hirzebruch surface

$$
\begin{array}{llllll}
\sigma_1: & \mathbf{e_1}, & \mathbf{e_2} & \check{\sigma}_1: & \mathbf{m}_{11} = \mathbf{e}_1, & \mathbf{m}_{12} = \mathbf{e}_2 \\
\sigma_2: & \mathbf{e_1}, & -\mathbf{e_2} & \check{\sigma}_2: & \mathbf{m}_{21} = \mathbf{e_1}, & \mathbf{m}_{22} = -\mathbf{e_2} \\
\sigma_3: & -\mathbf{e_1} + a\mathbf{e_2}, & -\mathbf{e_2} & \check{\sigma}_3: & \mathbf{m}_{31} = -a\mathbf{e_1} - \mathbf{e_2}, & \mathbf{m}_{32} = -\mathbf{e_1} \\
\sigma_4: & -\mathbf{e_1} + a\mathbf{e_2}, & \mathbf{e_2} & \check{\sigma}_4: & \mathbf{m}_{41} = a\mathbf{e_1} + \mathbf{e_2}, & \mathbf{m}_{42} = -\mathbf{e_1}
\end{array}
$$

To calculate the transition functions, we take into account the generators of the dual cones.

$$
f_{21}: \begin{cases} \mathbf{m}_{21} = \mathbf{m}_{11} \\ \mathbf{m}_{22} = -\mathbf{m}_{12} \end{cases} \Longrightarrow \begin{cases} z_{21} = z_{11} \\ z_{22} = z_{12}^{-1} \end{cases}
$$

$$
f_{31}: \begin{cases} \mathbf{m}_{31} = -a\mathbf{m}_{11} - \mathbf{m}_{12} \\ \mathbf{m}_{32} = -\mathbf{m}_{11} \end{cases} \Longrightarrow \begin{cases} z_{31} = z_{11}^{-a} z_{12}^{-1} \\ z_{32} = z_{11}^{-1} \end{cases}
$$

$$f_{41}: \begin{cases} \mathbf{m}_{41} = a\mathbf{m}_{11} + \mathbf{m}_{12} \\ \mathbf{m}_{42} = -\mathbf{m}_{11} \end{cases} \Longrightarrow \begin{cases} z_{41} = z_{11}^{a} z_{12} \\ z_{42} = z_{11}^{-1} \end{cases}$$

$$f_{32}: \begin{cases} \mathbf{m}_{31} = -a\mathbf{m}_{21} + \mathbf{m}_{22} \\ \mathbf{m}_{32} = -\mathbf{m}_{21} \end{cases} \Longrightarrow \begin{cases} z_{31} = z_{21}^{-a} z_{22} \\ z_{32} = z_{21}^{-1} \end{cases}$$

$$f_{42}: \begin{cases} \mathbf{m}_{41} = a\mathbf{m}_{21} - \mathbf{m}_{22} \\ \mathbf{m}_{42} = -\mathbf{m}_{21} \end{cases} \Longrightarrow \begin{cases} z_{41} = z_{21}^{a} z_{22}^{-1} \\ z_{42} = z_{21}^{-1} \end{cases}$$

$$f_{43}: \begin{cases} \mathbf{m}_{41} = -\mathbf{m}_{31} \\ \mathbf{m}_{42} = \mathbf{m}_{32} \end{cases} \Longrightarrow \begin{cases} z_{41} = z_{31}^{-1} \\ z_{42} = z_{32} \end{cases}$$

6.3. More complicated fans.

Affine toric varieties with many generators. If the number of generators of each cone in a fan is not equal to the dimension of the ambient space, then the situation is a little bit more complicated. In such case, the local model (affine toric variety) is not an open subset of $\mathbb{C}^r$, but an affine variety in $\mathbb{C}^k$, where k is the number of generators of the corresponding dual cone. Below we see what can we get in such situation.

Let $\sigma_{\mathbb{R}}$ and $\tau_{\mathbb{R}}$ in $N_{\mathbb{R}} = \mathbb{R}^r$ be two r-dimensional cones with the common face $\sigma_{\mathbb{R}} \cap \tau_{\mathbb{R}}$. Assume that $\check{\sigma}_{\mathbb{Z}}$ has generators $\mathbf{m}_1, \ldots, \mathbf{m}_k$ and $\check{\tau}_{\mathbb{Z}}$ has generators $\mathbf{g}_1, \ldots, \mathbf{g}_l$. For the generators of the cone $\check{\sigma}_{\mathbb{Z}}$ we have $k-r$ independent relations with nonnegative coefficients $a_{\alpha\beta}$, $b_{\alpha\beta} \in \mathbb{Z}_{\geqslant 0}$:

$$\begin{cases} a_{11}\mathbf{m}_1 + \ldots + a_{1k}\mathbf{m}_k = b_{11}\mathbf{m}_1 + \ldots + b_{1k}\mathbf{m}_k \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{(k-r)1}\mathbf{m}_1 + \ldots + a_{(k-r)k}\mathbf{m}_k = b_{(k-r)1}\mathbf{m}_1 + \ldots + b_{(k-r)k}\mathbf{m}_k \end{cases}$$

$$\Downarrow$$

$$V_\sigma : \begin{cases} z_1^{a_{11}} \cdot \ldots \cdot z_k^{a_{1k}} = z_1^{b_{11}} \cdot \ldots \cdot z_k^{b_{1k}} \\ \ldots \quad \ldots \quad \ldots \\ z_1^{a_{(k-r)1}} \cdot \ldots \cdot z_k^{a_{(k-r)k}} = z_1^{b_{(k-r)1}} \cdot \ldots \cdot z_k^{b_{(k-r)k}} \end{cases}$$

Similarly for V_τ, which is given by $l-r$ equations in $\mathbb{C}^l$.

Transition mappings between affine charts. Remind again,

$$U_\sigma = \{u : \check{\sigma}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid u(0) = 1,\ u(\mathbf{m} + \mathbf{m}') = u(\mathbf{m})u(\mathbf{m}'),\ \forall \mathbf{m}, \mathbf{m}' \in \check{\sigma}_{\mathbb{Z}}\}.$$

$$U_\tau = \{u : \check{\tau}_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid u(0) = 1,\ u(\mathbf{g}+\mathbf{g}') = u(\mathbf{g})u(\mathbf{g}'),\ \forall \mathbf{g}, \mathbf{g}' \in \check{\tau}_{\mathbb{Z}}\}.$$

Using identification of u in terms of coordinates as in (2), we obtain

$$\begin{array}{ccc} & U_\sigma \cap U_\tau & \\ & & \\ & u & \\ & \swarrow \quad \searrow & \\ & & \\ (u(\mathbf{m}_1), \dots, u(\mathbf{m}_k)) & & (u(\mathbf{g}_1), \dots, u(\mathbf{g}_l)) \\ & & \\ (z_{\sigma 1}, \dots, z_{\sigma k}) & & (z_{\tau 1}, \dots, z_{\tau l}) \\ & & \\ V_\sigma & & V_\tau \\ \cap & & \cap \\ \mathbb{C}^k & & \mathbb{C}^l \end{array} \tag{16}$$

Now we compute the transition functions, using the fact that the both sets of generators $\mathbf{m_1}, \dots, \mathbf{m_k}$ and $\mathbf{g_1}, \dots, \mathbf{g_l}$ span the entire $\mathbb{Z}^r$ over $\mathbb{Z}$. But we should clearly note that these relations are *not* unique. For instance, choose some relations between these two sets of generators

$$\begin{cases} \mathbf{g}_1 & = & a_{11}\mathbf{m}_1 + \dots + a_{1k}\mathbf{m}_k \\ \dots & \dots & \dots \\ \mathbf{g_l} & = & a_{l1}\mathbf{m}_1 + \dots + a_{lk}\mathbf{m}_k \end{cases}$$

and from here

$$\begin{cases} u(\mathbf{g}_1) & = & [u(\mathbf{m}_1)]^{a_{11}} \dots [u(\mathbf{m}_k)]^{a_{1k}} \\ \dots & \dots & \dots \quad \dots \quad \dots \\ u(\mathbf{g}_l) & = & [u(\mathbf{m}_1)]^{a_{l1}} \dots u[(\mathbf{m}_k)]^{a_{lk}} \end{cases}$$

which produces

$$\begin{cases} z_{\tau 1} & = & z_{\sigma 1}^{a_{11}} \cdot \ldots \cdot z_{\sigma k}^{a_{1k}} \\ \dots & \dots & \dots \\ z_{\tau l} & = & z_{\sigma 1}^{a_{l1}} \cdot \ldots \cdot z_{\sigma k}^{a_{lk}} \end{cases} \tag{17}$$

Even the change of variables in (17) is similar as in the regular case, the main difference is that this change of variables is not defined between an open set of $\mathbb{C}^k$ and an open set of $\mathbb{C}^l$ but is restricted to affine algebraic varieties in these spaces respectively.

Example 6.2. In this example we consider a fan that contains the cone from Example 4.4. In order to calculate the transition functions, we need only to consider the dual cones of the original fan. Here are generators of the dual cones:

$$\check{\sigma}_1 : \quad \mathbf{m}_{11} = \mathbf{e}_1, \quad \mathbf{m}_{12} = \mathbf{e}_2, \quad \mathbf{m}_{13} = 3\mathbf{e}_1 - \mathbf{e}_2, \quad \mathbf{m}_{14} = -\mathbf{e}_1 + 4\mathbf{e}_2$$

$$(\mathbb{C}^*)^2 \ni (t_1, t_2) \longrightarrow (t_1,\, t_2,\, t_1^3 t_2^{-1},\, t_1^{-1} t_2^4) \in \mathbb{C}^4$$

$$\begin{cases} \mathbf{m}_{12} + \mathbf{m}_{13} = 3\mathbf{m}_{11} \\ \mathbf{m}_{11} + \mathbf{m}_{14} = 4\mathbf{m}_{12} \end{cases} \Longrightarrow \quad \mathbb{C}^4 \supset V_1 : \begin{cases} z_{12} z_{13} = z_{11}^3 \\ z_{11} z_{14} = z_{12}^4 \end{cases}$$

$$\check{\sigma}_2 : \quad \begin{array}{l} \mathbf{m}_{21} = \mathbf{e}_1, \quad \mathbf{m}_{22} = \mathbf{e}_1 - \mathbf{e}_2, \quad \mathbf{m}_{23} = \mathbf{e}_1 - 2\mathbf{e}_2, \\ \mathbf{m}_{24} = \mathbf{e}_1 - 3\mathbf{e}_2, \quad \mathbf{m}_{25} = \mathbf{e}_1 - 4\mathbf{e}_2 \end{array}$$

$$(\mathbb{C}^*)^2 \ni (t_1, t_2) \longrightarrow (t_1,\, t_1 t_2^{-1},\, t_1 t_2^{-2},\, t_1 t_2^{-3},\, t_1 t_2^{-4}) \in \mathbb{C}^5$$

$$\begin{cases} \mathbf{m}_{21} + \mathbf{m}_{23} = 2\mathbf{m}_{22} \\ \mathbf{m}_{21} + \mathbf{m}_{24} = \mathbf{m}_{22} + \mathbf{m}_{23} \\ \mathbf{m}_{21} + \mathbf{m}_{25} = \mathbf{m}_{22} + \mathbf{m}_{24} \end{cases} \Longrightarrow \quad \mathbb{C}^5 \supset V_2 : \begin{cases} z_{21} z_{23} = z_{22}^2 \\ z_{21} z_{24} = z_{22} z_{23} \\ z_{21} z_{25} = z_{22} z_{24} \end{cases}$$

$$\check{\sigma}_3 : \quad \mathbf{m}_{31} = -\mathbf{e}_1, \quad \mathbf{m}_{32} = -\mathbf{e}_2$$

$$(\mathbb{C}^*)^2 \ni (t_1, t_2) \longrightarrow (t_1^{-1},\, t_2^{-1}) \in \mathbb{C}^2$$

$$\text{no relations} \quad \Longrightarrow \quad \mathbb{C}^2 = V_3$$

$$\check{\sigma}_4 : \quad \mathbf{m}_{41} = \mathbf{e}_2, \quad \mathbf{m}_{42} = -\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{m}_{43} = -2\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{m}_{44} = -3\mathbf{e}_1 + \mathbf{e}_2$$

$$(\mathbb{C}^*)^2 \ni (t_1, t_2) \longrightarrow (t_2,\, t_1^{-1} t_2,\, t_1^{-2} t_2,\, t_1^{-3} t_2) \in \mathbb{C}^4$$

$$\begin{cases} \mathbf{m}_{41} + \mathbf{m}_{44} = \mathbf{m}_{42} + \mathbf{m}_{43} \\ \mathbf{m}_{41} + \mathbf{m}_{43} = 2\mathbf{m}_{42} \end{cases} \Longrightarrow \quad \mathbb{C}^4 \supset V_4 : \begin{cases} z_{41} z_{44} = z_{42} z_{43} \\ z_{41} z_{43} = z_{42}^2 \end{cases}$$

Transition functions between charts in $T(\Delta)$, where Δ as in Fig. 12. We can compute the transition functions between charts, but actually, as we will see below, it would be better to identify points parameterized by (t_1, t_2). However first we compute the transition function $z_2 = f_{21}(z_1)$, where $z_1 = (z_{11}, z_{12}, z_{13}, z_{14})$ and $z_2 = (z_{21}, z_{22}, z_{23}, z_{24}, z_{25})$. As we already noted, the transition functions are not determined uniquely as Laurent monomials. One possible

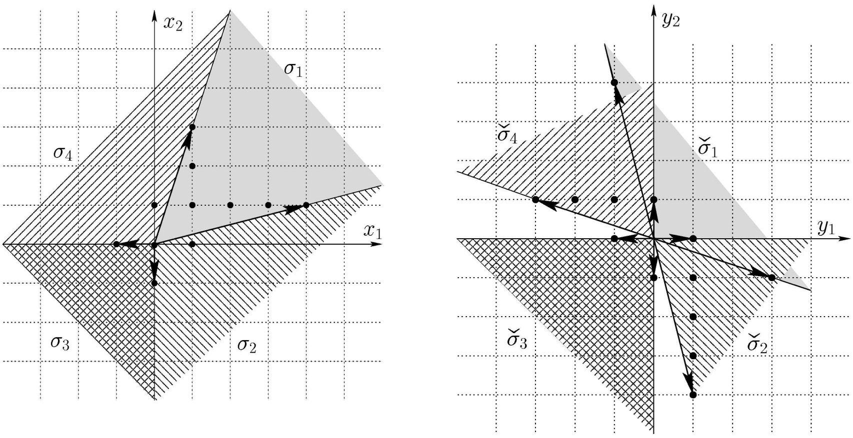


Fig. 12. Fan and dual cones

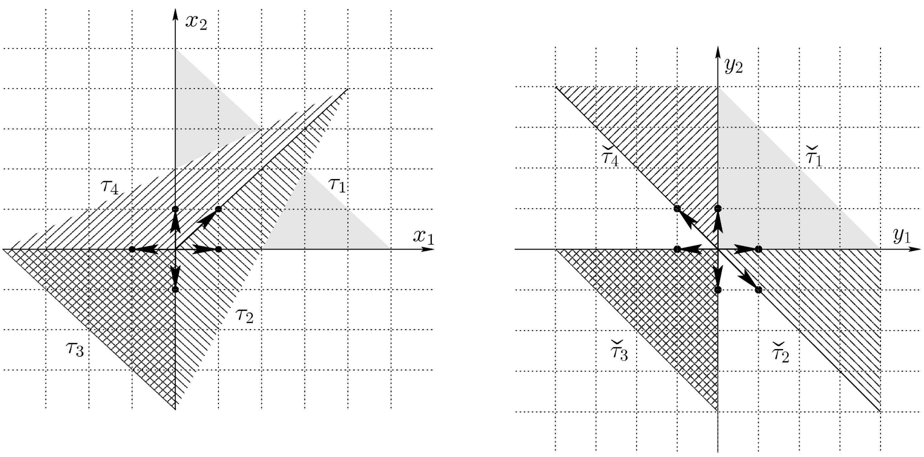
choice of relations between the generators of $\check{\sigma}_1$ and $\check{\sigma}_2$ is

$$\begin{cases} \mathbf{m}_{21} &= \mathbf{m}_{11} \\ \mathbf{m}_{22} &= \mathbf{m}_{11} - \mathbf{m}_{12} \\ \mathbf{m}_{23} &= \mathbf{m}_{11} - 2\mathbf{m}_{12} \\ \mathbf{m}_{24} &= \mathbf{m}_{11} - 3\mathbf{m}_{12} \\ \mathbf{m}_{25} &= \mathbf{m}_{11} - 4\mathbf{m}_{12} \end{cases} \qquad \Longrightarrow \qquad \begin{cases} z_{21} &= z_{11} \\ z_{22} &= z_{11} z_{12}^{-1} \\ z_{23} &= z_{11} z_{12}^{-2} \\ z_{24} &= z_{11} z_{12}^{-3} \\ z_{25} &= z_{11} z_{12}^{-4} \end{cases}$$

In this example, the transition functions actually identify the points parameterized by (t_1, t_2), namely if $u(e_1) = t_1$, $u(e_2) = t_2$, then we have the following (not very useful in practice) identifications:

$$\begin{tikzcd} \mathbb{C}^4 \supset V_4 \ni (t_2,\, t_1^{-1}t_2,\, t_1^{-2}t_2,\, t_1^{-3}t_2) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & (t_1,\, t_2,\, t_1^3 t_2^{-1},\, t_1^{-1}t_2^4) \in V_1 \subset \mathbb{C}^4 \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ \mathbb{C}^2 \supset V_3 \ni (t_1^{-1}, t_2^{-1}) \arrow[r, leftrightarrow] & (t_1,\, t_1 t_2^{-1},\, t_1 t_2^{-2},\, t_1 t_2^{-3},\, t_1 t_2^{-4}) \in V_2 \subset \mathbb{C}^5 \end{tikzcd}$$

Example 6.3. In this example we take a closer look at the transition functions from the previous example. We note that in each chart, the projection



Collection of cones which is not a fan — Dual cones

Fig. 13. Dual cones and a collection of cones which is not a fan

on the first two coordinates uniquely determines the point.

$$
\begin{array}{llllrl}
\pi_1 : \mathbb{C}^4 \longrightarrow \mathbb{C}^2, & (t_1,\, t_2), & \{\mathbf{e_1}, & & \mathbf{e_2}\} & : \check{\tau}_1 \\
\pi_2 : \mathbb{C}^5 \longrightarrow \mathbb{C}^2, & (t_1,\, t_1 t_2^{-1}), & \{\mathbf{e_1}, & & \mathbf{e_1} - \mathbf{e_2}\} & : \check{\tau}_2 \\
\pi_3 : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, & (t_1^{-1},\, t_2^{-1}), & \{-\mathbf{e_1}, & & -\mathbf{e_2}\} & : \check{\tau}_3 \\
\pi_4 : \mathbb{C}^4 \longrightarrow \mathbb{C}^2, & (t_2,\, t_1^{-1} t_2), & \{\mathbf{e_2}, & & -\mathbf{e_1} + \mathbf{e_2}\} & : \check{\tau}_4
\end{array}
$$

But the set of cones $\{\check{\tau}_1,\ \check{\tau}_2,\ \check{\tau}_3,\ \check{\tau}_4\}$ in $\mathbb{R}^2$ does not come from a fan in $\mathbb{R}^2$, what can be seen on Figure 13. So taking projections on $\mathbb{C}^2$ and then transition functions is not a good idea.

6.4. Equivariant holomorphic maps. In Subsection 5.5 we have seen that torus acts naturally on the affine toric variety, namely

$$(tu)(\mathbf{m}) = t(\mathbf{m})u(\mathbf{m}).$$

It is easy to check that the action of torus can be extended to a toric variety, because the actions on affine parts are compatible.

Let Δ and Δ' be two fans in $N_{\mathbb{R}} \simeq \mathbb{R}^r$ and $N'_{\mathbb{R}} \simeq \mathbb{R}^{r'}$, and $T(\Delta)$ and $T(\Delta')$ be the corresponding toric varieties.

Definition 6.4. *A map $f : T(\Delta) \longrightarrow T(\Delta')$ is called equivariant if it commutes with the torus actions on $T(\Delta)$ and $T(\Delta')$, i.e.,*

$$f(tu) = f(t)f(u), \qquad t \in T_r, \quad u \in T(\Delta).$$

Now we define a map between toric varieties that is generated by a map of fans.

Definition 6.5. *A map of fans $\varphi : (N_{\mathbb{R}}, \Delta) \longrightarrow (N'_{\mathbb{R}}, \Delta')$ is a linear map (denoted for simplicity by the same φ) $\varphi : N_{\mathbb{R}} \longrightarrow N'_{\mathbb{R}}$ such that $\varphi(N_{\mathbb{Z}}) \subset N'_{\mathbb{Z}}$ and for each $\sigma \in \Delta$ there exists $\sigma' \in \Delta'$ with $\varphi(\sigma) \subset \sigma'$. For short, we also denote such map as $\varphi : \Delta \longrightarrow \Delta'$.*

It appears that there is a one-to-one correspondence between equivariant maps between toric varieties and maps of the fans. This can be found in [13], Theorem 1.13. Here we give a procedure how to obtain $\varphi_* : T(\Delta) \longrightarrow T(\Delta')$ from $\varphi : \Delta \longrightarrow \Delta'$.

$$\begin{array}{lcl}
\varphi : N_{\mathbb{R}} \longrightarrow N'_{\mathbb{R}} & \rightsquigarrow & \varphi^* : M'_{\mathbb{R}} \longrightarrow M_{\mathbb{R}} \\
\varphi : N_{\mathbb{Z}} \longrightarrow N'_{\mathbb{Z}} & \rightsquigarrow & \varphi^* : M'_{\mathbb{Z}} \longrightarrow M_{\mathbb{Z}} \\
\varphi(\sigma_{\mathbb{R}}) \subset \sigma'_{\mathbb{R}}, \quad \varphi(\sigma_{\mathbb{Z}}) \subset \sigma'_{\mathbb{Z}} & \rightsquigarrow & \varphi^*(\check{\sigma}'_{\mathbb{R}}) \subset \check{\sigma}_{\mathbb{R}}, \quad \varphi^*(\check{\sigma}'_{\mathbb{Z}}) \subset \check{\sigma}_{\mathbb{Z}}
\end{array}$$

Now we can define $\varphi_* : U_\sigma \longrightarrow U'_{\sigma'}$ as follows

$$u \in U_\sigma \quad \rightsquigarrow \quad u : \check{\sigma}_{\mathbb{Z}} \longrightarrow \mathbb{C} \quad \rightsquigarrow \quad (\varphi_* u)(\mathbf{m}') := u(\varphi^*(\mathbf{m}')) \quad \rightsquigarrow \quad \begin{tikzcd} \check{\sigma}'_{\mathbb{Z}} \arrow[r, "\varphi^*"] \arrow[d, "\varphi_* u"'] & \check{\sigma}_{\mathbb{Z}} \arrow[dl, "u"] \\ \mathbb{C} & \end{tikzcd}$$

Compatibility with another cone τ follows from the very definition of φ_* if we apply it to $u \in U_\sigma \cap U_\tau$ because φ^* is a global mapping defined on $M'_{\mathbb{Z}}$, independently of cones. Therefore we can glue the affine toric varieties U_σ, $\sigma \in \Delta$, to get the global map $\varphi_* : T(\Delta) \longrightarrow T(\Delta')$.

Finally to prove that φ_* is equivariant we have

$$\varphi_*(tu)(\mathbf{m}') = (tu)(\varphi^*(\mathbf{m}')) = t(\varphi^*(\mathbf{m}'))\, u(\varphi^*(\mathbf{m}')) = (\varphi_* t)(\mathbf{m}')\, (\varphi_* u)(\mathbf{m}')$$

or in short

$$\varphi_*(tu) = (\varphi_* t)(\varphi_* u).$$

7. Nonsingular and compact toric varieties.

7.1. Nonsingular toric varieties. There is a beautiful theorem that characterizes toric complex manifolds, i.e., regular (nonsingular) toric varieties (for regular fans, see Definition 4.12).

Theorem 7.1. *The toric variety $T(\Delta)$ associated to a fan Δ is regular if and only if the fan Δ is regular.*

The proof of this theorem can be found in basic books on toric varieties, e.g., [13], p. 15. Since the sufficiency is simple, we give a proof here.

Proof (sufficiency). For each $\sigma \in \Delta$ choose a $\mathbb{Z}$-basis $\{\mathbf{n}_1, \ldots, \mathbf{n}_r\}$ of $N_{\mathbb{R}} \simeq \mathbb{R}^r$ and such that $\sigma_{\mathbb{R}}$ is spanned by a part of that basis, say $\mathbf{n}_1, \ldots, \mathbf{n}_s$, where $s \leqslant r$, i.e.,

$$\sigma_{\mathbb{R}} = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \ldots + \mathbb{R}_{\geqslant 0}\mathbf{n}_s.$$

We denote the dual basis in $M_{\mathbb{R}}$ by $\{\mathbf{n}_1^*, \ldots, \mathbf{n}_r^*\}$. Using the property of the dual basis, the dual cone is

$$\check{\sigma}_{\mathbb{R}} = \sum_{1\leqslant j\leqslant s} \mathbb{R}_{\geqslant 0}\, \mathbf{n}_j^* + \sum_{s+1\leqslant j\leqslant r} \mathbb{R}\, \mathbf{n}_j^* \quad\Longrightarrow\quad \check{\sigma}_{\mathbb{Z}} = \sum_{1\leqslant j\leqslant s} \mathbb{Z}_{\geqslant 0}\, \mathbf{n}_j^* + \sum_{s+1\leqslant j\leqslant r} \mathbb{Z}\, \mathbf{n}_j^*.$$

Therefore the generators of the dual cone are

$$\mathbf{n}_1^*, \ldots, \mathbf{n}_s^*, \mathbf{n}_{s+1}^*, (-\mathbf{n}_{s+1}^*), \ldots, \mathbf{n}_r^*, (-\mathbf{n}_r^*)$$

and consequently U_σ can be identified with

$$\big\{\big(u(\mathbf{n}_1^*), \ldots, u(\mathbf{n}_s^*), u(\mathbf{n}_{s+1}^*), u(-\mathbf{n}_{s+1}^*), \ldots, u(\mathbf{n}_r^*), u(-\mathbf{n}_r^*)\big) \in \mathbb{C}^s \times \mathbb{C}^{2r-2s}\big\}$$

or

$$\begin{aligned}&\big\{(z_1, \ldots, z_s, z_{s+1}, w_{s+1}, \ldots, z_r, w_r) \in \mathbb{C}^s \times \mathbb{C}^{2r-2s} \mid z_j w_j = 1,\ j = s+1 \ldots, r\big\}\\ &\sim \mathbb{C}^s \times (\mathbb{C}^*)^{r-s}.\end{aligned}$$

Thus U_σ is nonsingular. In particular, if the dimension of the cone σ is r, then $U_\sigma \simeq \mathbb{C}^r$.

7.2. Compact toric varieties. There is a nice criterion for compactness of toric varieties, namely if the support of the fan Δ is the entire $\mathbb{R}^r$ (see Definition 4.13). Such *fans are called complete.*

Theorem 7.2. *The toric variety $T(\Delta)$ associated to a fan Δ is compact if and only if Δ is finite and complete.*

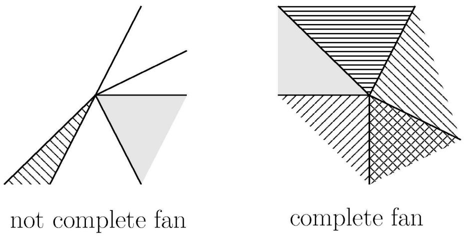


Fig. 14. Complete and not complete fans

Again the proof of this theorem can be found in any basic textbook on toric varieties, e.g., [13], p. 16 or [8], p. 257. The proof is not difficult, but it requires some preparations, because of that we do not include it here.

7.3. Examples of nonsingular and compact toric varieties.

7.3.1. Again $\mathbb{CP}^2$. As a nice and simple exercise we will show that any regular and complete fan composed of three vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ in $\mathbb{R}^2$ determines $\mathbb{CP}^2$.

Let

$$\mathbf{n}_1 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2, \qquad \mathbf{n}_2 = b_1\mathbf{e}_1 + b_2\mathbf{e}_2, \qquad \mathbf{n}_3 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$$

be three vectors in counterclockwise order in $N_{\mathbb{Z}} \simeq \mathbb{Z}^2$, as in Fig. 15. Without loss of generality we can assume that $a_1 > 0$ and $a_2 = 0$ and completeness will imply (again, no loss of generality) that $b_2 > 0$ and $c_2 < 0$. Since the fan is regular, therefore all determinants consisting of two consecutive vectors as columns (or rows) are $+1$, because the angle between two consecutive vectors is smaller than π:

$$\begin{vmatrix} a_1 & b_1 \\ 0 & b_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} c_1 & a_1 \\ c_2 & 0 \end{vmatrix} = 1,$$

$$a_1 > 0, \qquad b_2 > 0, \qquad c_2 < 0,$$

which implies

$$a_1 = 1, \quad b_2 = 1, \quad c_2 = -1, \quad b_1 + c_1 = -1.$$

So we have

$$\mathbf{n}_1 = \mathbf{e}_1, \qquad \mathbf{n}_2 = b_1\mathbf{e}_1 + \mathbf{e}_2, \qquad \mathbf{n}_3 = (-b_1 - 1)\mathbf{e}_1 - \mathbf{e}_2.$$

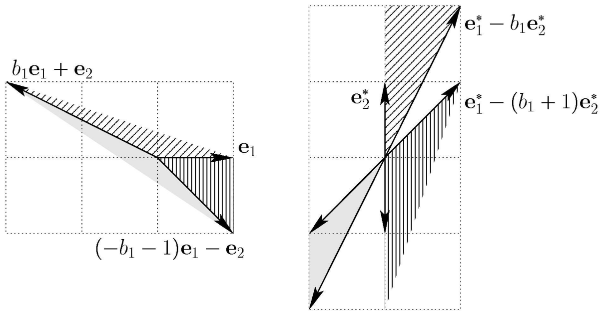


Fig. 15. Vectors for $\mathbb{CP}^2$

The dual cones are generated by:

$$\check{\sigma}_1:\begin{cases}\mathbf{m}_{11}=\mathbf{e}_2\\ \mathbf{m}_{12}=\mathbf{e}_1-b_1\mathbf{e}_2\end{cases}\qquad\begin{cases}\mathbf{m}_{11}=-\mathbf{m}_{21}+\mathbf{m}_{22}\\ \mathbf{m}_{12}=-\mathbf{m}_{21}\end{cases}\qquad\begin{cases}z_{11}=z_{21}^{-1}z_{22}\\ z_{12}=z_{21}^{-1}\end{cases}$$

$$\check{\sigma}_2:\begin{cases}\mathbf{m}_{21}=-\mathbf{e}_1+b_1\mathbf{e}_2\\ \mathbf{m}_{22}=-\mathbf{e}_1+(b_1+1)\mathbf{e}_2\end{cases}\qquad\begin{cases}\mathbf{m}_{21}=\mathbf{m}_{31}-\mathbf{m}_{32}\\ \mathbf{m}_{22}=-\mathbf{m}_{32}\end{cases}\qquad\begin{cases}z_{21}=z_{32}^{-1}z_{31}\\ z_{22}=z_{32}^{-1}\end{cases}$$

$$\check{\sigma}_3:\begin{cases}\mathbf{m}_{31}=-\mathbf{e}_2\\ \mathbf{m}_{32}=\mathbf{e}_1-(b_1+1)\mathbf{e}_2\end{cases}\qquad\begin{cases}\mathbf{m}_{31}=-\mathbf{m}_{11}\\ \mathbf{m}_{32}=-\mathbf{m}_{11}+\mathbf{m}_{12}\end{cases}\qquad\begin{cases}z_{31}=z_{11}^{-1}\\ z_{32}=z_{11}^{-1}z_{12}\end{cases}$$

We see that the transition functions are clearly for $\mathbb{CP}^2$.

7.3.2. Again Hirzebruch surfaces. We shall see that the Hirzebruch surfaces are the only complete, regular toric varieties determined by four vectors (cones) in the plane.

Let

$$\mathbf{n}_1=a_1\mathbf{e}_1+a_2\mathbf{e}_2,\ \ \mathbf{n}_2=b_1\mathbf{e}_1+b_2\mathbf{e}_2,\ \ \mathbf{n}_3=c_1\mathbf{e}_1+c_2\mathbf{e}_2,\ \ \mathbf{n}_4=d_1\mathbf{e}_1+d_2\mathbf{e}_2$$

be four vectors in counterclockwise order in $N_{\mathbb{Z}}\simeq\mathbb{Z}^2$ as in Fig. 16.

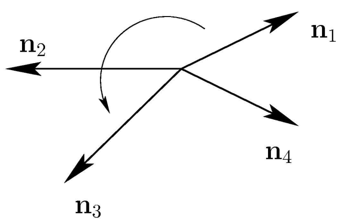


Fig. 16. Vectors in a fan

Before we consider the determinants, we look at each consecutive pair of vectors as a basis of $\mathbb{Z}^2$. Take for instance the pairs $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_2$, $\mathbf{n}_3$. We then have

$$\begin{cases} \mathbf{n}_3 = k_1\mathbf{n}_1 + k_2\mathbf{n}_2 \\ \mathbf{n}_1 = l_2\mathbf{n}_2 + l_3\mathbf{n}_3 \end{cases} \Longrightarrow \mathbf{n}_3 = (k_1l_2 + k_2)\mathbf{n}_2 + k_1l_3\mathbf{n}_3$$

$$\Longrightarrow \begin{cases} k_1l_2 + k_2 = 0 \\ k_1l_3 = 1 \end{cases} \Longrightarrow \begin{cases} k_1 = 1 \quad l_3 = 1 \quad l_2 = -k_2 \\ \text{or} \\ k_1 = -1 \quad l_3 = -1 \quad l_2 = k_2 \end{cases}$$

$$\Longrightarrow \begin{cases} \mathbf{n}_3 = \mathbf{n}_1 + k_2\mathbf{n}_2 \\ \text{or} \\ \mathbf{n}_3 = -\mathbf{n}_1 + k_2\mathbf{n}_2 \end{cases} \Longrightarrow \begin{cases} \mathbf{n}_3 - \mathbf{n}_1 = k_2\mathbf{n}_2 \\ \text{or} \\ \mathbf{n}_3 + \mathbf{n}_1 = k_2\mathbf{n}_2 \end{cases}$$

In the last link of implications, the first option $\mathbf{n}_3 - \mathbf{n}_1 = k_2\mathbf{n}_2$ is not possible because $\mathbf{n}_2$ lies between $\mathbf{n}_1$ and $\mathbf{n}_3$ and moreover, the angle between two consecutive vectors is strictly less than π. Consequently we have $\mathbf{n}_3 + \mathbf{n}_1 = k_2\mathbf{n}_2$. The same happens to other consecutive triples, so summarizing we get (after changing the constants to a, b, c, and d):

$$\mathbf{n}_1 + \mathbf{n}_3 = b\mathbf{n}_2, \quad \mathbf{n}_2 + \mathbf{n}_4 = c\mathbf{n}_3, \quad \mathbf{n}_3 + \mathbf{n}_1 = d\mathbf{n}_4, \quad \mathbf{n}_4 + \mathbf{n}_2 = a\mathbf{n}_1$$

and from here we get

$$b\mathbf{n}_2 = d\mathbf{n}_4, \qquad c\mathbf{n}_3 = a\mathbf{n}_1.$$

We have three options:

1. All a, b, c, and d are zero, then $\mathbf{n}_3 = -\mathbf{n}_1$ and $\mathbf{n}_4 = -\mathbf{n}_2$.
2. A pair of numbers a, b, c, and d is zero, for instance, a and c are not zero but $b = d = 0$, then $\mathbf{n}_3 = -\mathbf{n}_1$ and $c = -a$.
3. If all a, b, c, and d are not zero, then $\mathbf{n}_2$ has the opposite direction to $\mathbf{n}_4$ and $\mathbf{n}_1$ has the opposite direction to $\mathbf{n}_3$.

Now look at the determinants of consecutive pairs of vectors. Without loss of generality we can assume that $a_1 > 0$ and $a_2 = 0$ and completeness implies (again, no loss of generality) that $b_2 > 0$ and $d_2 < 0$. Since the fan is regular, therefore all determinants consisting of two consecutive vectors as columns (or rows) are $+1$.

$$\begin{vmatrix} a_1 & b_1 \\ 0 & b_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = 1, \qquad \begin{vmatrix} d_1 & a_1 \\ d_2 & 0 \end{vmatrix} = 1,$$

$$a_1 > 0, \qquad b_2 > 0, \qquad\qquad d_2 < 0.$$

This implies

$$a_1 = 1, \quad b_2 = 1, \quad d_2 = -1, \quad c_1 = b_1c_2 - 1, \quad c_1 = -d_1c_2 - 1.$$

So we got

$$\begin{aligned} &\mathbf{n}_1 = \mathbf{e}_1, \quad \mathbf{n}_2 = b_1\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{n}_3 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2, \quad \mathbf{n}_4 = d_1\mathbf{e}_1 - \mathbf{e}_2 \\ &\text{where} \quad c_1 = b_1c_2 - 1, \quad c_1 = -d_1c_2 - 1. \end{aligned} \tag{18}$$

Combining (18) with the three options listed above, we get two cases:

1^0 Two pairs of vectors consist opposite vectors (Fig. 17, left figure):

$$\mathbf{n}_1 = \mathbf{e}_1, \quad \mathbf{n}_2 = b_1\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{n}_3 = -\mathbf{e}_1, \quad \mathbf{n}_4 = -b_1\mathbf{e}_1 - \mathbf{e}_2$$

2^0 Exactly one pair of vectors consists opposite vectors:

(a) If $\mathbf{n}_3$ is opposite to $\mathbf{n}_1$ (Fig. 17, middle figure)

$$\mathbf{n}_1 = \mathbf{e}_1, \quad \mathbf{n}_2 = b_1\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{n}_3 = -\mathbf{e}_1, \quad \mathbf{n}_4 = d_1\mathbf{e}_1 - \mathbf{e}_2$$

(b) If $\mathbf{n}_4$ is opposite to $\mathbf{n}_2$ (Fig. 17, right figure)

$$\mathbf{n}_1 = \mathbf{e}_1,\; \mathbf{n}_2 = b_1\mathbf{e}_1 + \mathbf{e}_2,\; \mathbf{n}_3 = (b_1c_2 - 1)\mathbf{e}_1 + c_2\mathbf{e}_2,\; \mathbf{n}_4 = -b_1\mathbf{e}_1 - \mathbf{e}_2.$$

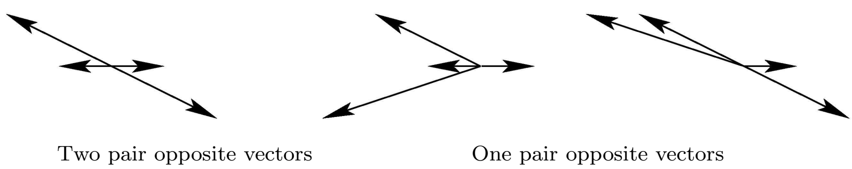


Fig. 17. Two cases of positioning of vectors

Finally how to see that the toric variety determined by the above four vectors is equivalent to a Hirzebruch surface? The answer is simple, in each case it is easy to indicate a map of fans that determines the above equivalence. Take for instance the case 2^0(a). Define

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \varphi \sim \begin{bmatrix} 1 & -b_1 \\ 0 & 1 \end{bmatrix}.$$

Under this mapping we have

$$\mathbf{e}_1 \mapsto \mathbf{e}_1, \quad b_1\mathbf{e}_1 + \mathbf{e}_2 \mapsto \mathbf{e}_2, \quad -\mathbf{e}_1 \mapsto -\mathbf{e}_1, \quad d_1\mathbf{e}_1 - \mathbf{e}_2 \mapsto (d_1 - b_1)\mathbf{e}_1 - \mathbf{e}_2.$$

The fan determined by the images clearly is equivalent to the fan from Subsection 6.2.2.

8. Orbits in toric varieties.

8.1. Definition and basic properties of orbits. From the definition of toric varieties $T(\Delta)$, $\dim_{\mathbb{C}} T(\Delta) = r$, we know that the complex torus T_r acts on $T(\Delta)$. It is well-known that if a group is acting on any set, then its action determines orbits. Any point belongs to one orbit only.

Let $u \in T(\Delta)$. The orbit of u is

$$\mathrm{orb}(u) = \{tu \mid t \in T_r\}.$$

From the definition of affine toric varieties, it is easy to see that U_σ, $\sigma \in \Delta$, is closed with respect to the torus action, this means that if $u \in U_\sigma$, then $\mathrm{orb}\,(u) \subset U_\sigma$. In this section we will see that there is a one-to-one correspondence between the cones τ in the fan Δ and orbits in $T(\Delta)$:

$$\begin{array}{rcl} \mathrm{orb}(u) & \longleftrightarrow & \tau \in \Delta \\ \dim_{\mathbb{C}} \mathrm{orb}(u) & = & r - \dim_{\mathbb{R}}(\tau) \\ \mathrm{orb}(u) & \longleftrightarrow & \{u : M_{\mathbb{Z}} \cap \tau^{\perp} \longrightarrow \mathbb{C}^* \mid \text{ group homomorphisms}\} \end{array} \tag{19}$$

The basic properties we need are:

(a) There is one-to-one correspondence between faces of σ and faces of $\check{\sigma}$ (Proposition A.6 of [13]), namely

$$\{\text{faces of } \sigma\} \ni \tau \quad \longleftrightarrow \quad \check{\sigma} \cap \tau^{\perp} \in \{\text{faces of } \check{\sigma}\}\,;$$
$$\mathbb{Z}(\check{\sigma}_{\mathbb{Z}} \cap \tau^{\perp}) = \mathbb{Z}\tau^{\perp}_{\mathbb{Z}}, \text{ where } \tau^{\perp}_{\mathbb{Z}} = M_{\mathbb{Z}} \cap \tau^{\perp}, \quad \dim \tau = r - \dim(\check{\sigma} \cap \tau^{\perp});$$

(b) For any $u \in U_\sigma$ we have (see [13], p. 11 and Proposition A.9)

$$\{\mathbf{m} \in \check{\sigma}_{\mathbb{Z}} \mid u(\mathbf{m}) \neq 0\} \quad \text{is the intersection of } M_{\mathbb{Z}} \text{ with a face of } \check{\sigma}.$$

Now we are prepared to justify the properties of orbits listed in (19). Two elements $u, v \in U_\sigma$ belong to the same orbit if and only if

$$\{\mathbf{m} \in \check{\sigma}_{\mathbb{Z}} \mid u(\mathbf{m}) \neq 0\} = \{\mathbf{m} \in \check{\sigma}_{\mathbb{Z}} \mid v(\mathbf{m}) \neq 0\}.$$

Therefore the orbit of u is determined by the set

$$\{\mathbf{m} \in \check{\sigma}_{\mathbb{Z}} \mid u(\mathbf{m}) \neq 0\},$$

which, from part (b) above, is the intersection of $M_{\mathbb{Z}}$ with a face of $\check{\sigma}$, and next, from (a), is of the form $\check{\sigma}_{\mathbb{Z}} \cap \tau^{\perp}$ for a unique face τ of σ. Again using (a), since u is not zero on $\check{\sigma}_{\mathbb{Z}} \cap \tau^{\perp}$, therefore u is well defined on the entire $\tau_{\mathbb{Z}}^{\perp} = M_{\mathbb{Z}} \cap \tau^{\perp}$, and finally, u is a homomorphism on $\tau_{\mathbb{Z}}^{\perp}$. Summarizing all the above from this paragraph we get:

$$\begin{aligned}
\mathrm{orb}(u) \quad &\longleftrightarrow \quad \{\mathbf{m} \in \check{\sigma}_{\mathbb{Z}} \mid u(\mathbf{m}) \neq 0\} \\
&\longleftrightarrow \quad \check{\sigma}_{\mathbb{Z}} \cap \tau^{\perp} \text{ (for a unique face } \tau \text{ of } \sigma) \\
&\longleftrightarrow \quad \{v \in U_\sigma \mid \; v(\check{\sigma}_{\mathbb{Z}} \cap \tau^{\perp}) \subset \mathbb{C}^*, \;\; v(\check{\sigma}_{\mathbb{Z}} \setminus \tau^{\perp}) = \{0\}\} \\
&\longleftrightarrow \quad \{v : M_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid \; v(\tau_{\mathbb{Z}}^{\perp}) \subset \mathbb{C}^*, \;\; v(M_{\mathbb{Z}} \setminus \tau^{\perp}) = \{0\}\} \subset U_\sigma.
\end{aligned}$$

Actually this correspondence is not a surprise when we look at the relation between U_σ and U_τ, where τ is a face of σ (see Section 5.6).

8.2. Examples of orbits in toric varieties.

8.2.1. Regular case. First we consider two simple examples: $\mathbb{C}^n$ and $\mathbb{CP}^2$.

Example 8.1. We know from Example 5.2 that $\mathbb{C}^n$ is determined by the cone $\sigma_{\mathbb{R}} = \mathbb{R}_{\geqslant 0}\mathbf{e}_1 + \ldots + \mathbb{R}_{\geqslant 0}\mathbf{e}_n$. Any point $(z_1, \ldots, z_n) \in \mathbb{C}^n$ can be identified with a homomorphism u on the dual cone:

$$\mathbb{C}^n \ni z = (z_1, \ldots, z_n) = (u(\mathbf{e}_1^*), \ldots, u(\mathbf{e}_n^*)).$$

The torus $T_n = (\mathbb{C}^*)^n$ action is

$$tz = (t_1, \ldots, t_n)(z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n).$$

It is clear that two points $z,\ w \in \mathbb{C}^n$ can be obtained from one another if and only if they have zero components on the same places, say $I = \{i_1, \ldots, i_k\}$, $1 \leqslant i_1 < \ldots < i_k \leqslant n$. The set index I determines the cone $\tau_{\mathbb{Z}I}$ generated

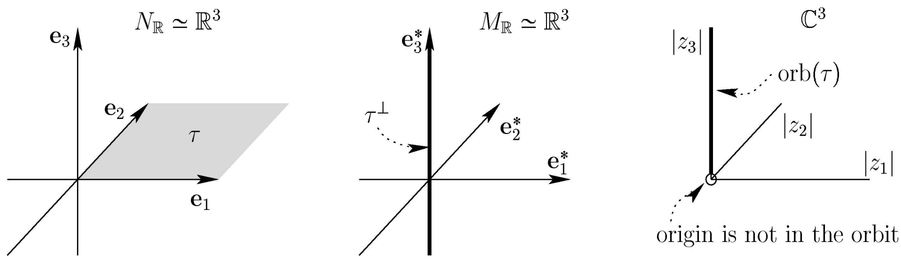


Fig. 18. Orbit in $\mathbb{C}^n$ generated by a cone

by $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$. The orthogonal cone $\tau_{\mathbb{Z}I}^{\perp}$ is generated by $\mathbf{e}_{j_1}^*, \ldots, \mathbf{e}_{j_{n-k}}^*$, where $J = \{j_1, \ldots, j_{n-k}\}$ is the set of remaining indices. Therefore

$$\begin{aligned} \operatorname{orb}(u) \quad &\sim \quad \{v : M_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid \ v : \tau_{\mathbb{Z}I}^{\perp} \longrightarrow \mathbb{C}^*, \ v(\mathbf{m}) = 0 \ \text{ for } \ \mathbf{m} \notin \tau_{\mathbb{Z}I}^{\perp}\} \\ &\sim \quad \{(z_1, \ldots, z_n) \mid \ z_{i_1} = \ldots = z_{i_k} = 0\}. \end{aligned}$$

Example 8.2. Here we consider $\mathbb{CP}^2$ as in Example 6.2.1. The projective space is covered by three affine toric varieties: U_σ, U_ρ, and U_τ. Since each affine toric variety is closed with respect to the torus action and all they are similar, it is enough to consider one of them, say U_ρ.

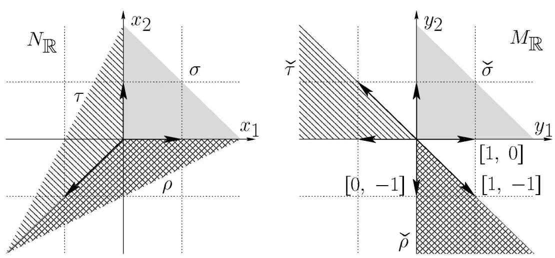


Fig. 19. Cones for $\mathbb{CP}^2$

We take any $u \in U_\rho$, which can be identified with $(u([1,-1]), u([0,-1]))$. There are three possibilities:

1. *Both coordinates are not zero.* In this case, the orbit of u consists all $u \in U_\rho$ such that $u([0,-1]) \neq 0$ and $u([1,-1]) \neq 0$, i.e., is determined by $\{0\}$ face of ρ:

$$\begin{aligned} u &\rightsquigarrow \operatorname{orb}(u) = \{tu \mid t \in T_2\} \\ &\rightsquigarrow \left\{(t_1 u([1,-1]),\, t_2 u([0,-1])) \mid \quad (t_1,t_2) \in (\mathbb{C}^*)^2\right\} \\ &\rightsquigarrow (\mathbb{C}^*)^2 \end{aligned}$$

or more formally

$$\begin{aligned} \rho \supset \{0\} &\longleftrightarrow \check{\rho} \cap \{0\}^\perp = \check{\rho} \\ &\longleftrightarrow \{v \in U_\rho \mid \; v(\mathbf{m}) \neq 0 \;\forall \mathbf{m} \in \check{\rho}_{\mathbb{Z}}\} \\ &\longleftrightarrow \left\{v \in U_\rho \mid v \text{ can be extended to } \{0\}^\perp = M_{\mathbb{Z}}\right\} \\ &\longleftrightarrow \{v : M_{\mathbb{Z}} \longrightarrow \mathbb{C}^*, \text{ group homomorphism}\} \\ &\longleftrightarrow (\mathbb{C}^*)^2. \end{aligned}$$

2. *Exactly one coordinate is zero, say* $u([0,-1]) = 0$. In this case, the orbit of u consists all $u \in U_\rho$ such that $u([0,-1]) = 0$ but $u([1,-1]) \neq 0$.

$$\begin{aligned} u &\rightsquigarrow \operatorname{orb}(u) = \{tu \mid t \in T_2\} \\ &\rightsquigarrow \{(t_1 u([1,-1]),\; 0) \mid (t_1,t_2) \in (\mathbb{C}^*)^2\} \\ &\rightsquigarrow \mathbb{C}^* \times \{0\} \end{aligned}$$

or

$$\begin{aligned} \tau = \mathbb{R}_{\geqslant 0}[-1,\, -1] &\longleftrightarrow \check{\rho} \cap \tau^\perp = \mathbb{R}_{\geqslant 0}[1,-1] \\ &\longleftrightarrow \{v \in U_\rho \mid \; v \neq 0 \text{ on } \mathbb{Z}_{\geqslant 0}[1,-1], \;\; 0 \text{ otherwise}\} \\ &\longleftrightarrow \left\{v : M_{\mathbb{Z}} \longrightarrow \mathbb{C} \mid \begin{array}{lll} v(\mathbf{m}) \neq 0 & \text{for} & \mathbf{m} \in \mathbb{Z}[1,\, -1] \\ v(\mathbf{m}) = 0 & \text{for} & \mathbf{m} \notin \mathbb{Z}[1,\, -1] \end{array}\right\} \\ &\longleftrightarrow \mathbb{C}^* \times \{0\}. \end{aligned}$$

3. *Both coordinates are zero:* $u([0,\, -1]) = 0,\; u([1,\, -1]) = 0$. In this case, the orbit of u consists only the zero point, i.e., is determined by the cone ρ.

$$u \rightsquigarrow \operatorname{orb}(u) = \{tu \mid t \in T_2\} \rightsquigarrow \{(0,\; 0)\} \rightsquigarrow \{0\} \times \{0\}$$

or more formally

$$\begin{aligned} \rho \quad &\longleftrightarrow \quad \check{\rho} \cap \rho^{\perp} = \{0\} \\ &\longleftrightarrow \quad \{v \in U_\rho \mid \ v(0) = 1,\ v(\mathbf{m}) = 0 \text{ for } \mathbf{m} \neq 0\} \\ &\longleftrightarrow \quad \{0\}. \end{aligned}$$

This example can be easily generalized to $\mathbb{CP}^n$: Each orbit is determined by a cone of the fan Δ that describes $\mathbb{CP}^n$.

From these examples we see how the orbits can be defined in the general regular situation, i.e., for $u \in U_\sigma$, where σ is a regular cone. We know that u is determined by its values at the generators $\mathbf{m}_1, \ldots, \mathbf{m}_n$ of $\check{\sigma}_{\mathbb{Z}}$, $\dim \check{\sigma} = n$, namely $u \mapsto (u(\mathbf{m}_1), \ldots, u(\mathbf{m}_n))$. The orbit of u is determined by the positions of zeros of its components, say $1 \leqslant i_1 < \ldots < i_k \leqslant n$. The set of indices $I = \{i_1, \ldots, i_k\}$ uniquely determines the face τ_I of the cone σ.

$$u \rightsquigarrow (0_{i_1}, u(\mathbf{m}_{j_1}), 0_{i_2}, \ldots, u(\mathbf{m}_{j_{n-k}}), 0_{i_k})) \rightsquigarrow \{0\} \times \mathbb{C}^* \times \{0\} \times \ldots \times \mathbb{C}^* \times \{0\}.$$

8.2.2. Non-regular case.

Example 8.3. Consider Example 5.3 that produces a singular affine toric variety. The cone σ (see Fig. 20) is spanned (over $\mathbb{R}$) by $\mathbf{n}_1 = [4,\ 1]$ and $\mathbf{n}_2 = [1,\ 3]$ and its dual by $\mathbf{m}_1 = [-1,\ 4]$ and $\mathbf{m}_4 = [3,\ -1]$, however two more vectors are needed as generators of $\check{\sigma}_{\mathbb{Z}}$. One possible choice is

$$\begin{gathered} \mathbf{m}_1 = [-1,\ 4], \quad \mathbf{m}_2 = [0,\ 1], \quad \mathbf{m}_3 = [1,\ 0], \quad \mathbf{m}_4 = [3,\ -1] \\ u(\mathbf{m}_1) = z_1, \quad u(\mathbf{m}_2) = z_2 \quad u(\mathbf{m}_3) = z_3, \quad u(\mathbf{m}_4) = z_4 \\ \text{with the relations :} \quad \mathbf{m}_1 + \mathbf{m}_3 = 4\mathbf{m}_2, \quad \mathbf{m}_2 + \mathbf{m}_4 = 3\mathbf{m}_3 \end{gathered}$$

$$\text{the corresponding equations :} \quad z_1 z_3 = z_2^4, \quad z_2 z_4 = z_3^3. \tag{20}$$

According to general procedure, described in Section 8.1, there are four orbits corresponding to four faces: 0, τ_1, τ_2 and σ as in Fig. 20. In this example, it is not possible to assign up-front that some values of u at the generators are zero and the other non-zero, because we have relations (20).

$$u([a, b]) = 0, \text{ for some } a \geqslant 0,\ b \geqslant 0 \quad \Longrightarrow \quad z_2 = 0 \text{ or } z_3 = 0 \quad \overset{\text{from(20)}}{\Longrightarrow}$$

$$z_2 = 0 \text{ and } z_3 = 0 \quad \overset{\mathbf{m}_1 + \mathbf{m}_4 = [2,\,3]}{\Longrightarrow} \quad z_2 = 0 \text{ and } z_3 = 0 \text{ and } (z_1 = 0 \text{ or } z_4 = 0).$$

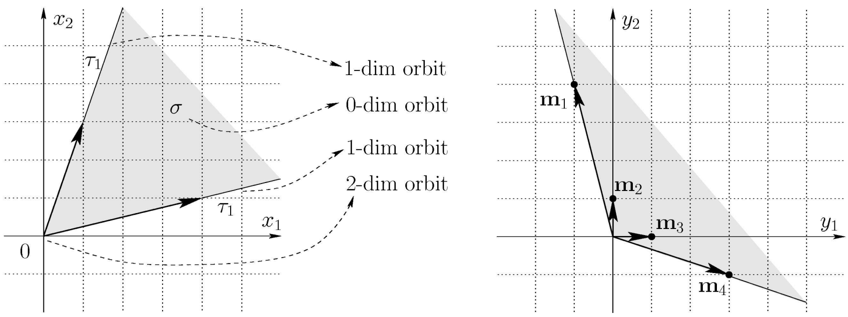


Fig. 20. Faces, generators, orbits

Another case

$$u([a,b]) = 0, \text{ for some } a < 0,\ b \geqslant 0 \quad \Longrightarrow \quad z_1 = 0 \text{ or } z_2 = 0 \quad \overset{\text{from}(20)}{\Longrightarrow}$$
$$\Longrightarrow \quad z_1 = 0 \text{ and } z_2 = 0 \text{ and } z_3 = 0.$$

Consequently, the orbit of u corresponds either to the cone τ_1 or τ_2.

8.3. Relations between orbits, orbit closures and affine toric varieties. Let Δ be a fan in $N_{\mathbb{R}}$ and $T(\Delta)$ the corresponding toric variety, which is glued together from affine toric varieties U_σ, $\sigma \in \Delta$. We define

$$V(\tau) = \overline{\mathrm{orb}(\tau)} \qquad (\text{closure in} \quad T(\Delta)).$$

There are nice properties of orbits and their closures, precise proofs can be found in [9], p. 54 or [13], p. 10. Here we formulate the following

Lemma 8.4. *There are the following relations among orbits* $\mathrm{orb}(\tau)$*, orbit closures* $V(\tau)$ *and the affine toric varieties* U_σ*:*

(a) $U_\sigma = \bigcup_{\tau < \sigma} \mathrm{orb}(\tau)$*;*

(b) $V(\tau) = \bigcup_{\gamma > \tau} \mathrm{orb}(\gamma)$*;*

(c) $\mathrm{orb}(\tau) = V(\tau) \setminus \bigcup_{\gamma > \tau, \gamma \neq \tau} V(\gamma)$.

9. Δ-linear support functions.

9.1. Definition and basic properties of support functions. Let Δ be a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^r$ and recall that $|\Delta| := \bigcup_{\sigma\in\Delta} \sigma$. Following [13] we give the following definition:

Definition 9.1. *A real valued function $h : |\Delta| \longrightarrow \mathbb{R}$ is said to be a Δ-linear support function if it is $\mathbb{Z}$-valued on $N_{\mathbb{Z}} \cap |\Delta|$ and is linear on each $\sigma \in \Delta$. We denote by* $\mathrm{SF}(\Delta)$ *the additive group consisting of Δ-linear support functions.*

Some explanation of the definition. Since h is linear on each cone $\sigma \in \Delta$ and $\mathbb{Z}$-valued on $N_{\mathbb{Z}}$ there exists $l_\sigma \in M_{\mathbb{Z}}$ such that $h(\mathbf{n}) = \langle l_\sigma, \mathbf{n}\rangle$ for $\mathbf{n} \in \sigma$ and that $\langle l_\sigma, \mathbf{n}\rangle = \langle l_\tau, \mathbf{n}\rangle$ holds whenever $\mathbf{n} \in \tau \prec \sigma$. So we see that

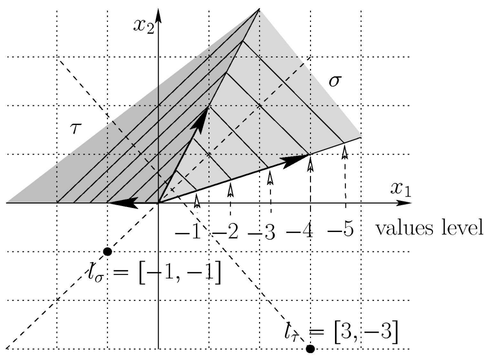


Fig. 21. Support function and l_σ, l_τ elements

$$\mathrm{SF}(\Delta) \ni h \quad \longmapsto \quad \{l_\sigma \mid \sigma \in \Delta\} \subset M_{\mathbb{Z}}$$

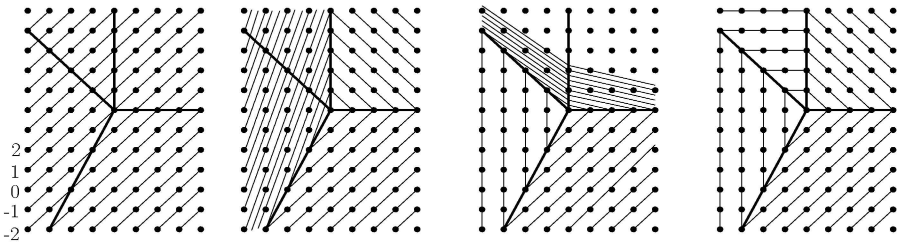
but usually the set is not determined uniquely by h. If we have two l_σ and l'_σ giving the same h on σ,

$$\langle l_\sigma, \mathbf{n}\rangle = \langle l'_\sigma, \mathbf{n}\rangle \quad \text{for} \quad \mathbf{n} \in \sigma \quad \Longrightarrow \quad l'_\sigma - l_\sigma \in M_{\mathbb{Z}} \cap \sigma^\perp.$$

If the dimension of σ is r, then l_σ is determined uniquely since $\sigma^\perp = 0$.

On Fig. 21, l_σ and l_τ are determined uniquely by the values of the function h, but for the cone $\sigma \cap \tau$ we can choose $l_{\sigma\cap\tau} = l_\sigma$ or $= l_\tau$ or any other integral vector on the line through l_σ and l_τ. Formally, l_σ and l_τ belong to $M_{\mathbb{Z}}$, but on Fig. 21 they appear in $N_{\mathbb{Z}}$ to illustrate the situation, which should not lead to any confusion. We note that (after identification of $N_{\mathbb{R}}$ with $M_{\mathbb{R}}$) the vector l_σ is perpendicular to the constant level lines (hyperplanes, in general) as in Fig. 21.

Elements of $M_{\mathbb{Z}}$ as Δ-linear support functions. We may regard each element $\mathbf{m} \in M_{\mathbb{Z}}$ as a Δ-linear support function because it determines a linear



One possibility is: 0 at the origin and then 1, 2, 3, and so on

Another possibility: 0 at the origin and then −1, −2, −3, and so on

Fig. 22. Different support functions and their level sets

functional on $N_{\mathbb{R}}$ with $\mathbb{Z}$-values on $N_{\mathbb{Z}}$, like on the first left picture in Fig. 22. Hence we have a natural homomorphism $M_{\mathbb{Z}} \longrightarrow \mathrm{SF}(\Delta)$. It is injective if $|\Delta|$ generates $N_{\mathbb{R}}$ over $\mathbb{R}$. In this case, we usually identify $M_{\mathbb{Z}}$ with its image in $\mathrm{SF}(\Delta)$.

Support function on 1-dimensional cones. Another useful observation. Recall the notation

$$\Delta(1) := \{\rho \in \Delta \mid \dim \rho = 1\}.$$

For each $\rho \in \Delta(1)$ there exists a unique element $\mathbf{n}_\rho$ in $N_{\mathbb{Z}} \cap \rho$ such that $\rho = \mathbb{R}_{\geqslant 0}\mathbf{n}_\rho$ and $\rho_{\mathbb{Z}} = \mathbb{Z}_{\geqslant 0}\mathbf{n}_\rho$. Such element is called primitive (Definition 4.9). Any $\sigma \in \Delta$ can be written as

$$\sigma = \sum_{\rho \in \Delta(1),\, \rho \prec \sigma} \mathbb{R}_{\geqslant 0}\, \mathbf{n}_\rho.$$

Thus $h \in \mathrm{SF}(\Delta)$ is determined by the integers $h(\mathbf{n}_\rho)$ for all $\rho \in \Delta(1)$, and we obtain an injective homomorphism

$$\mathrm{SF}(\Delta) \hookrightarrow \mathbb{Z}^{\Delta(1)}, \qquad \mathrm{SF}(\Delta) \ni h \longmapsto \{h(\mathbf{n}_\rho) \mid \rho \in \Delta(1)\} \in \mathbb{Z}^{\Delta(1)}.$$

Note that $l_\sigma \in M_{\mathbb{Z}}$ is a solution in $M_{\mathbb{Z}}$ of the system of equations

$$\{\langle l_\sigma, \mathbf{n}_\rho\rangle = h(\mathbf{n}_\rho)\ ;\ \rho \in \Delta(1),\ \rho \prec \sigma\}.$$

Since such a solution may not exist in general, the above homomorphism need not be surjective. To see this, as on Fig. 21, for the cone σ the one-dimensional faces are generated by the vectors $[1,2]$ and $[3,1]$. If we choose $h([1,2]) = 1$ and $h([3,1]) = 2$, then there is no l_σ with integer components:

$$h(x,y) = ax + by \quad \Longrightarrow \quad \left\{ \begin{array}{l} a + 2b = 1 \\ 3a + b = 2 \end{array} \right. \quad \Longrightarrow \quad \left\{ \begin{array}{l} a = 3/5 \\ b = 1/5 \end{array} \right. .$$

If σ is nonsingular, however, $\{\mathbf{n}_\rho\ ;\ \rho \in \Delta(1),\ \rho \prec \sigma\}$ is a part of a $\mathbb{Z}$-basis of $N_\mathbb{Z}$, hence such a solution $l_\sigma \in M_\mathbb{Z}$ exists. In particular, if the toric variety $T(\Delta)$ is nonsingular, we have an isomorphism $\mathrm{SF}(\Delta) \longrightarrow \mathbb{Z}^{\Delta(1)}$.

9.2. Strictly upper convex support functions. Let Δ be a complete fan in $N_\mathbb{R}$ (i.e., $|\Delta| = N_\mathbb{R}$) and let $h \in \mathrm{SF}(\Delta)$.

Definition 9.2. *The function h is called* upper convex *if $h(\mathbf{n}+\mathbf{n}') \geqslant h(\mathbf{n}) + h(\mathbf{n}')$. It is called* strictly upper convex *if additionally for r-dimensional cones σ and τ, $\sigma \neq \tau$, we have $l_\sigma \neq l_\tau$.*

Example 9.3. On Fig. 22 the first function is linear on the entire $N_\mathbb{R}$, so is upper convex. The second function is upper convex if $h(\mathbf{e}_1) = -1$ but not strictly upper convex. The function is not upper convex if $h(\mathbf{e}_1) = 1$. The third function is not upper convex in either case:

$$h(\mathbf{e}_1) = 1 \quad \Longrightarrow \quad 0 = h(\mathbf{e}_1 + (-\mathbf{e}_1)) \not\geqslant h(\mathbf{e}_1) + h(-\mathbf{e}_1) = 2.$$

$$h(\mathbf{e}_1) = -1 \Longrightarrow -4 = h(\mathbf{e}_2) = h(\mathbf{e}_1 + (\mathbf{e}_2 - \mathbf{e}_1)) \not\geqslant h(\mathbf{e}_1) + h(\mathbf{e}_2 - \mathbf{e}_1) = -2.$$

The fourth function is strictly upper convex if $h(\mathbf{e}_1) = -1$ and not upper convex if $h(\mathbf{e}_1) = 1$.

In [13], Lemma 2.12, several equivalent conditions are given for strictly upper convex support function, some conditions are in terms of polytopes, which we consider in the next section. Here we mention two conditions equivalent to strict upper convexity:

A Δ-linear support function h is strictly upper convex if and only if one of the following conditions is satisfied

1. *h is upper convex, and Δ is the coarsest among the fans Δ' in $N_\mathbb{R}$ for which h is Δ'-linear.*
2. *For any $\sigma \in \Delta(r)$ and any $\mathbf{n} \in N_\mathbb{R}$, we have $\langle l_\sigma, \mathbf{n}\rangle \geqslant h(\mathbf{n})$ with the equality holding only if $\mathbf{n} \in \sigma$.*

10. Polytopes and fans. An alternative way to define a toric variety is, instead of starting with a fan in the lattice $N_\mathbb{Z}$, to start with a polyhedron in the dual lattice $M_\mathbb{Z}$. However, this method produces only complete projective toric varieties, i.e., compact toric varieties which can be embedded into projective spaces. Still this subclass is big enough and worth to be considered.

10.1. Definitions of polyhedral sets and polytopes. Let V be a finite dimensional $\mathbb{R}$-vector space and V^* its dual. An *affine half space* of V is a subset of the form

$$H^+(u;b) := \{v \in V \mid \langle u, v\rangle \geqslant b\} \quad \text{for} \quad u \in V^* \quad \text{and} \quad b \in \mathbb{R}.$$

A *convex polyhedral set* in V is the intersection of a finite number of affine half spaces. A convex polyhedral set K in V is compact if and only if it is the convex hull of a finite subset of V (see Theorem A.12 in [13]). A compact convex polyhedron is called also a *convex polytope*.

A (proper) *face* F of a convex polytope K is the intersection of K with a supporting affine hyperplane, i.e.,

$$F = K \cap H, \qquad H = \{v \in V \mid \langle u,\ v\rangle = b\},$$
$$\text{where } u \in V^* \text{ satisfies } K \subset \{v \in V \mid \langle u,\ v\rangle \geqslant b\}.$$

The polytope K is usually included as an improper face.

Usually we consider polytopes in $M_{\mathbb{R}}$ or in $N_{\mathbb{R}}$ where we have the lattices $M_{\mathbb{Z}}$ and $N_{\mathbb{Z}}$, respectively. If the vertices of the polytope are lattice points, then we call such polytope a *rational* or *integral polytope*.

10.2. From polytopes to fans. Here we describe a way how to obtain a fan in $N_{\mathbb{Z}}$ from a rational convex polytope P in $M_{\mathbb{Z}}$.

Let Q be a face of $P \subset M_{\mathbb{R}}$, then we define

$$\sigma_Q := \{\mathbf{n} \in N_{\mathbb{R}} \mid \ \langle \mathbf{m}',\ \mathbf{n}\rangle \geqslant \langle \mathbf{m},\ \mathbf{n}\rangle \ \text{for all} \ \mathbf{m}' \in P \ \text{and} \ \mathbf{m} \in Q\} \subset N_{\mathbb{R}}. \tag{21}$$

Better we see the structure of σ_Q if first we consider the convex cone

$$Q_P = \mathrm{Span}_{\mathbb{R}_{\geqslant 0}}(P - Q) = \mathrm{Span}_{\mathbb{R}_{\geqslant 0}}\{\mathbf{m}' - \mathbf{m} \mid \ \mathbf{m}' \in P \ \text{and} \ \mathbf{m} \in Q\} \subset M_{\mathbb{R}}.$$

Now if we look at the definition of σ_Q and Q_P we see that σ_Q is dual to Q_P:

$$\sigma_Q = \{\mathbf{n} \in N_{\mathbb{R}} \mid \ \langle \mathbf{q},\ \mathbf{n}\rangle \geqslant 0 \ \text{for all} \ \mathbf{q} \in Q_P\} = \check{Q}_P, \qquad \check{\sigma}_Q = Q_P.$$

On Figures 23 and 24 a graphic method is presented how to construct a fan from a polytope (triangle in these pictures) for vertices (0-dimensional faces) and sides (1-dimensional faces), respectively. The construction for the 2-dimensional face, i.e., the entire polytope is trivial because $P_P = M_{\mathbb{R}}$, so $\sigma_P = \check{P}_P = \{0\}$.

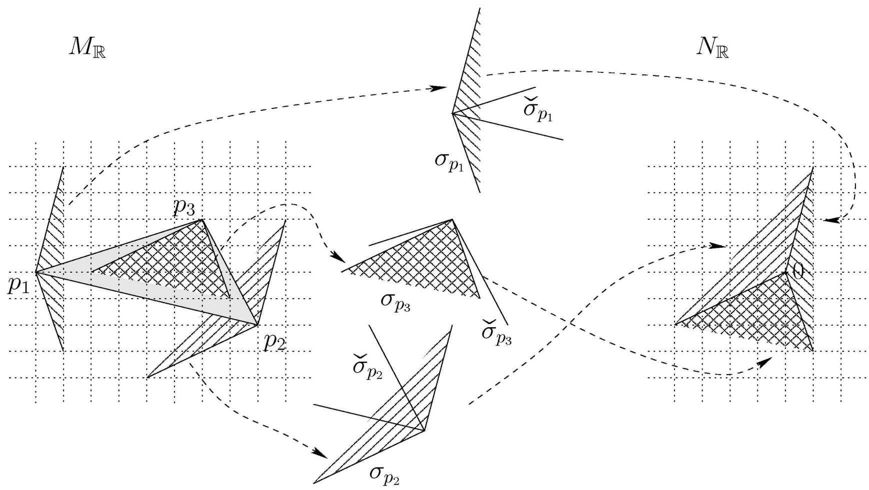


Fig. 23. From 0-dimensional faces p_1, p_2 and p_3 of a polygon to a fan

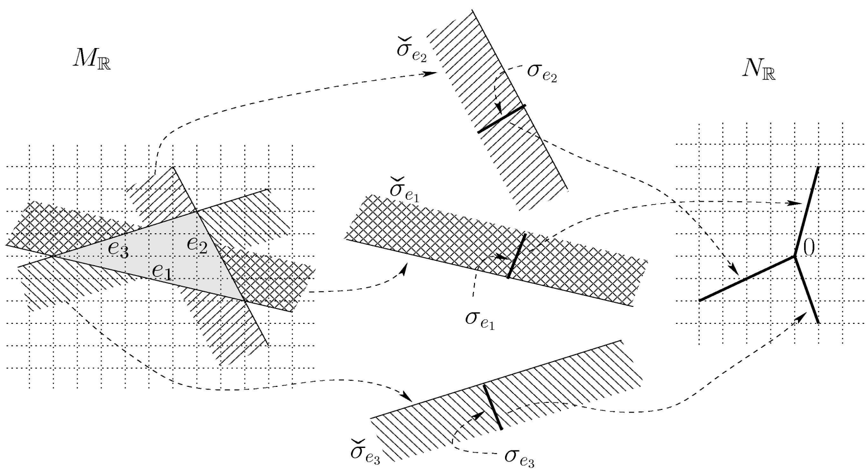


Fig. 24. From 1-dimensional faces e_1, e_2 and e_3 of a polygon to a fan

Especially if $Q = p_\alpha$ is a vertex of the polytope P, then Q_P is the cone "starting at this vertex" $\check{\sigma}_{p_\alpha} + p_\alpha$. So we have

$$P = \bigcap_{\alpha=1}^{k} (\check{\sigma}_{p_\alpha} + p_\alpha), \quad \text{where } k \text{ is the number of vertices in } P. \tag{22}$$

We assemble the fan from the cones σ_{p_α}. Schematically we have

$$\{p_\alpha\} \quad \rightsquigarrow \quad \{\check{\sigma}_{p_\alpha} + p_\alpha\} \quad \rightsquigarrow \quad \{\check{\sigma}_{p_\alpha}\} \quad \rightsquigarrow \quad \{\sigma_{p_\alpha}\} \quad \rightsquigarrow \quad \text{fan } \Delta. \tag{23}$$

Finally we formulate a useful lemma

Lemma 10.1 ([9], p. 26)**.** *The cones σ_Q, as Q varies over the faces of P, form a fan Δ_P.*

10.3. From fans to polytopes. In the previous subsection we have seen that a convex polytope in $M_\mathbb{R}$ uniquely determines a fan in $N_\mathbb{R}$. Here we consider the construction in the opposite direction, i.e., invert the procedure described above in (23). A natural question appears, whether a fan can determine a convex polytope such that the fan itself can be recovered? The answer is no, in general. Even if it determines a polytope, usually not uniquely, as on Fig. 25. It is easy to

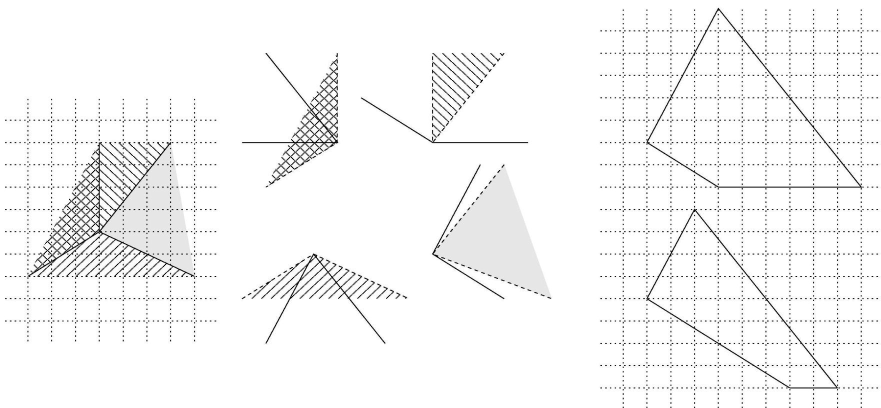


Fig. 25. From a fan to polygons – not unique

describe the procedure. Take a fan Δ and $\sigma_1, \ldots, \sigma_k$ be the r dimensional cones from Δ. Then the polytope (if possible to construct) is of the form

$$\Delta \quad \rightsquigarrow \quad \{\sigma_1, \ldots, \sigma_k\} \quad \rightsquigarrow \quad (\check{\sigma}_1 + \mathbf{m}_1) \cap \ldots \cap (\check{\sigma}_k + \mathbf{m}_k) \tag{24}$$

for some integral vectors $\mathbf{m}_\alpha \in M_\mathbb{Z}$. Even if the intersection (24) forms a polytope, is it possible to recover the original fan? We answer these questions in the following subsections. For instance, if the fan Δ comes together with a Δ-support function, then they both determine a convex integral polytope as we will see in Sections 10.4 and 10.5. If additionally the support function is strictly upper convex, then the fan determines uniquely the polygon and this polygon determines the original fan (Section 10.6).

10.4. From fans with support function to polytopes. We know how to get a polytope from a fan and we know that the polytope in this procedure is not uniquely determined. Assume that we have an additional structure on the fan, namely a support function, then the polytope can be defined directly by the support function.

Suppose that Δ is a complete fan in $N_\mathbb{Z}$ with support function h. Then we define the polytope $P(\Delta, h)$ by

$$P(\Delta, h) = \{\mathbf{m} \in M_\mathbb{R} \mid \langle \mathbf{m}, \mathbf{n} \rangle \geqslant h(\mathbf{n}), \text{ for all } \mathbf{n} \in N_\mathbb{R}\} \tag{25}$$

We take a closer look at this polygon. Let $\Delta(r)$ be the set of r dimensional cones (i.e., maximal dimension) in Δ. We know that for each $\sigma \in \Delta(r)$ there is a unique l_σ such that $h(\mathbf{n}) = \langle l_\sigma, \mathbf{n} \rangle$ for $\mathbf{n} \in \sigma$. Since the fan is complete, to describe $P(\Delta, h)$ it is enough to consider the r-dimensional cones only because the support function h generates its values on the lower dimensional faces. So we have

$$\begin{aligned}
P(\Delta, h) &= \{\mathbf{m} \in M_\mathbb{R} \mid \langle \mathbf{m}, \mathbf{n} \rangle \geqslant \langle l_\sigma, \mathbf{n} \rangle, \ \ \mathbf{n} \in \sigma_\mathbb{R}, \ \ \sigma_\mathbb{R} \in \Delta(r)\} \\
&= \{\mathbf{m} \in M_\mathbb{R} \mid \langle \mathbf{m} - l_\sigma, \mathbf{n} \rangle \geqslant 0, \ \ \mathbf{n} \in \sigma_\mathbb{R}, \ \ \sigma_\mathbb{R} \in \Delta(r)\} \\
&= \{\mathbf{m} \in M_\mathbb{R} \mid \mathbf{m} - l_\sigma \in \check{\sigma}_\mathbb{R}, \ \sigma_\mathbb{R} \in \Delta(r)\} \\
&= \{\mathbf{m} \in M_\mathbb{R} \mid \mathbf{m} \in l_\sigma + \check{\sigma}_\mathbb{R}, \ \sigma_\mathbb{R} \in \Delta(r)\} \\
&= \bigcap_{\sigma \in \Delta(r)} (l_\sigma + \check{\sigma}_\mathbb{R}) \qquad (26)
\end{aligned}$$

The formula (26) gives the procedure to construct the polytope from a fan with a support function: for each r-dimensional cone $\sigma_\mathbb{R}$ calculate the element $l_\sigma \in M_\mathbb{Z}$ then find the dual cone $\check{\sigma}_\mathbb{R}$ and translate the dual cone by l_σ. The intersection of these translated dual cones gives the polyhedron $P(\Delta, h)$:

$$\{\sigma_\alpha, l_{\sigma_\alpha}\} \quad \rightsquigarrow \quad \{\check{\sigma}_\alpha, l_{\sigma_\alpha}\} \quad \rightsquigarrow \quad P(\Delta, h) = \bigcap_\alpha (\check{\sigma}_\alpha + l_{\sigma_\alpha}).$$

In the four examples below, the fan is the same but support functions h are changing. We get different polytopes, which are illustrated on four figures – Figs 26–29. In the remaining part of this subsection we fix the fan generated by the lattice vectors

$$\mathbf{m}_1 = \mathbf{e}_1, \quad \mathbf{m}_2 = \mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{m}_3 = -2\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{m}_4 = -\mathbf{e}_2.$$

Obviously, the fan is complete.

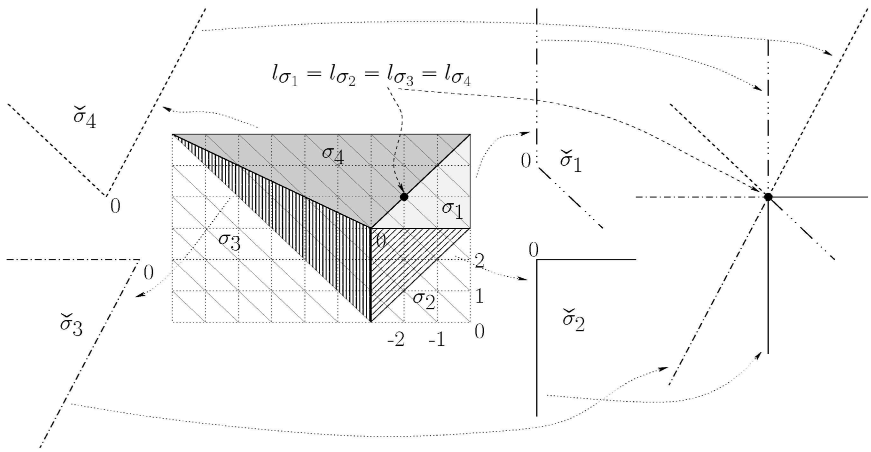


Fig. 26. From a fan with a global linear support function to a polygon

Example 10.2. This example is illustrated on Fig. 26. Let $N_{\mathbb{Z}} \ni \mathbf{n} = [n_1,\ n_2]$. The support function is $h(\mathbf{n}) = n_1 + n_2 = \langle \mathbf{e}_1^* + \mathbf{e}_2^*,\ \mathbf{n}\rangle$ for all $\mathbf{n} \in N_{\mathbb{Z}}$, so we have $l_{\sigma_\alpha} = \mathbf{e}_1^* + \mathbf{e}_2^*$ for all cones of the fan. The construction of the polygon $P(\Delta, h)$ is as follows:

$$\{(\sigma_\alpha,\ l_{\sigma_\alpha} = \mathbf{e}_1^* + \mathbf{e}_2^*)\}_{\alpha=1}^4 \quad \rightsquigarrow \quad \{(\check{\sigma}_\alpha,\ l_\sigma = \mathbf{e}_1^* + \mathbf{e}_2^*)\}_{\alpha=1}^4 \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \bigcap_{\alpha=1}^{4} (\check{\sigma}_\alpha + \mathbf{e}_1^* + \mathbf{e}_2^*) = \{\mathbf{e}_1^* + \mathbf{e}_2^*\} = P(\Delta, h).$$

Consequently, their intersection consists only the vertex $\mathbf{e}_1^* + \mathbf{e}_2^*$, i.e., $P(\Delta, h) = \mathbf{e}_1^* + \mathbf{e}_2^*$. The original fan cannot be reconstructed from this polygon. The support function in this example is linear, so in particular upper convex. Please note that $P(\Delta, h)$ is the convex hull in $M_{\mathbb{R}}$ of $\{l_\sigma \mid \sigma \in \Delta\}$.

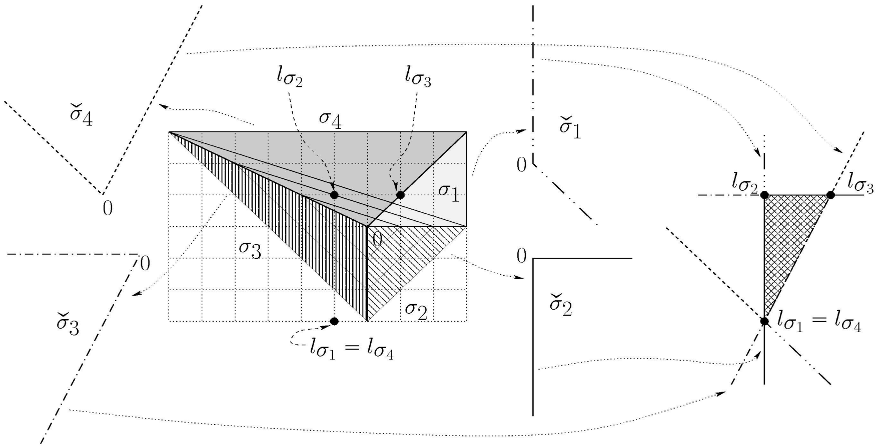


Fig. 27. From a fan with support function, the same on two consecutive cones, to a polygon

Example 10.3. This example is illustrated on Fig. 27. The support function $h(\mathbf{n})$ is defined as follows:

$$h(\mathbf{n}) = \begin{cases} \langle -\mathbf{e}_1^* - 3\mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_1 \quad \rightsquigarrow \quad l_{\sigma_1} = -\mathbf{e}_1^* - 3\mathbf{e}_2^* \\ \langle -\mathbf{e}_1^* + \mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_2 \quad \rightsquigarrow \quad l_{\sigma_2} = -\mathbf{e}_1^* + \mathbf{e}_2^* \\ \langle \mathbf{e}_1^* + \mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_3 \quad \rightsquigarrow \quad l_{\sigma_3} = \mathbf{e}_1^* + \mathbf{e}_2^* \\ \langle -\mathbf{e}_1^* - 3\mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_4 \quad \rightsquigarrow \quad l_{\sigma_4} = -\mathbf{e}_1^* - 3\mathbf{e}_2^* \end{cases}$$

So the construction of the polygon goes as

$$(\sigma_1,\ -\mathbf{e}_1^* - 3\mathbf{e}_2^*) \quad (\sigma_2,\ -\mathbf{e}_1^* + \mathbf{e}_2^*) \quad (\sigma_3,\ \mathbf{e}_1^* + \mathbf{e}_2^*) \quad (\sigma_4,\ -\mathbf{e}_1^* - 3\mathbf{e}_2^*)$$

$$(\check{\sigma}_1,\ -\mathbf{e}_1^* - 3\mathbf{e}_2^*) \quad (\check{\sigma}_2,\ -\mathbf{e}_1^* + \mathbf{e}_2^*) \quad (\check{\sigma}_3,\ \mathbf{e}_1^* + \mathbf{e}_2^*) \quad (\check{\sigma}_4,\ -\mathbf{e}_1^* - 3\mathbf{e}_2^*)$$

$$P(\Delta, h) = (\check{\sigma}_1 - \mathbf{e}_1^* - 3\mathbf{e}_2^*) \cap (\check{\sigma}_2 - \mathbf{e}_1^* + \mathbf{e}_2^*) \cap (\check{\sigma}_3, +\mathbf{e}_1^* + \mathbf{e}_2^*) \cap (\check{\sigma}_4, -\mathbf{e}_1^* - 3\mathbf{e}_2^*)$$

The polygon $P(\Delta, h)$ is a triangle, the convex hull of $\{l_{\sigma_1} = l_{\sigma_4},\ l_{\sigma_2},\ l_{\sigma_3}\}$, but the original fan cannot be reconstructed from it.

Example 10.4. This example is illustrated on Fig. 28. The support

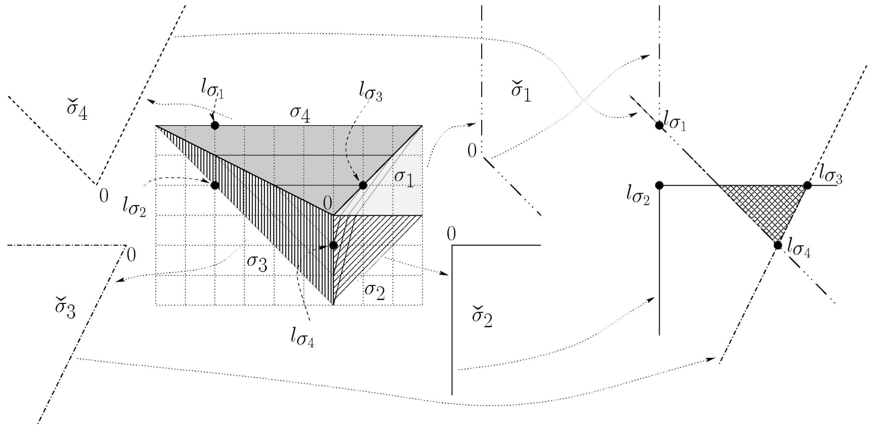


Fig. 28. From a fan with "convex-concave" support function to a polygon

function is defined as follows:

$$h(\mathbf{n}) = \begin{cases} \langle -4\mathbf{e}_1^* + 3\mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_1 \quad \rightsquigarrow \quad l_{\sigma_1} = -4\mathbf{e}_1^* + 3\mathbf{e}_2^* \\ \langle -4\mathbf{e}_1^* + \mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_2 \quad \rightsquigarrow \quad l_{\sigma_2} = -4\mathbf{e}_1^* + \mathbf{e}_2^* \\ \langle \mathbf{e}_1^* + \mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_3 \quad \rightsquigarrow \quad l_{\sigma_3} = \mathbf{e}_1^* + \mathbf{e}_2^* \\ \langle -\mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_4 \quad \rightsquigarrow \quad l_{\sigma_4} = -\mathbf{e}_2^* \end{cases}$$

The vertices of the polygon are actually not l_{σ_1}, l_{σ_2}, l_{σ_3} and l_{σ_4}, but the intersection of the corresponding translated dual cones gives a triangle with vertices $-2\mathbf{e}_1^* + \mathbf{e}_2^*$, $\mathbf{e}_1^* + \mathbf{e}_2^*$, $-\mathbf{e}_2^*$. Again, the fan cannot be reconstructed from the polygon.

Example 10.5. This example is illustrated on Fig. 29. The support function is defined as follows:

$$h(\mathbf{n}) = \begin{cases} \langle -\mathbf{e}_1^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_1 \quad \rightsquigarrow \quad l_{\sigma_1} = -\mathbf{e}_1^* \\ \langle -\mathbf{e}_1^* + \mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_2 \quad \rightsquigarrow \quad l_{\sigma_2} = -\mathbf{e}_1^* + \mathbf{e}_2^* \\ \langle \mathbf{e}_1^* + \mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_3 \quad \rightsquigarrow \quad l_{\sigma_3} = \mathbf{e}_1^* + \mathbf{e}_2^* \\ \langle -\mathbf{e}_2^*, \mathbf{n}\rangle & \text{for} \quad \mathbf{n} \in \sigma_4 \quad \rightsquigarrow \quad l_{\sigma_4} = -\mathbf{e}_2^* \end{cases}$$

The vertices of the polygon are l_{σ_1}, l_{σ_2}, l_{σ_3} and l_{σ_4} and this polygon is equal to the intersection of the corresponding translated dual cones. Here clearly we see that the fan can be fully reconstructed from the polygon.

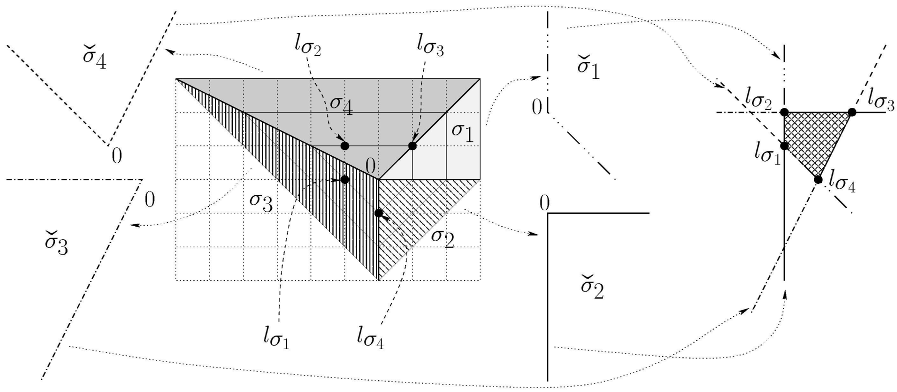


Fig. 29. From a fan with strictly convex support function to a polygon

In the above four examples clearly we have seen that the polytope depends on the support function h. In the first three examples, the fan cannot be recovered from the polytope; only in the last example yes. We will see in the subsequent subsection that this is not accidental: the support function should be strictly upper continuous in order the fan can be recovered.

10.5. From fans with upper convex support function to polytopes. In the preceding subsection we have seen how to construct a polytope from a fan with a support function. The polytope always is convex since it is the intersection of convex dual cones. In Examples 10.2 (Fig. 26), 10.3 (Fig. 27) and 10.5 (Fig. 29) the support function is upper convex. In this case, the polytope is the convex hull of $\{l_\sigma \mid \sigma \in \Delta(r)\}$. We notice that it is not true in Example 10.4 (Fig. 28), where the polytope does not coincide with the convex hull. More formally we formulate this property in the following lemma.

Lemma 10.6 (Theorem 2.7, [13]). *Let $T(\Delta)$ be a compact toric variety. Then the following are equivalent for $h \in SF(\Delta)$:*

(a) *h is upper convex, i.e., $h(\mathbf{n}) + h(\mathbf{n}') \leqslant h(\mathbf{n} + \mathbf{n}')$ for all $\mathbf{n}, \mathbf{n}' \in N_{\mathbb{R}}$;*

(b) *The convex polytope $P(\Delta, h)$ coincides with the convex hull in $M_{\mathbb{R}}$ of the finite set $\{l_\sigma \mid \sigma \in \Delta(r)\}$.*

When these equivalent conditions are satisfied, we have

$$h(\mathbf{n}) = \inf\{\langle \mathbf{m}, \mathbf{n}\rangle \mid \mathbf{m} \in M_{\mathbb{Z}} \cap P(\Delta, h)\} = \inf\{\langle l_\sigma, \mathbf{n}\rangle \mid \sigma \in \Delta(r)\},$$

and h is the support function for $P(\Delta, h)$.

The first part of the lemma (equivalence of (a) and (b)) has been illustrated in examples from the previous subsection. Now we illustrate the second part of the lemma.

Example 10.7. Let us consider Example 10.2 (Fig. 26) again. The polytope is $P(\Delta, h) = \{\mathbf{e}_1^* + \mathbf{e}_2^*\}$ and obviously

$$h(\mathbf{n}) = \langle \mathbf{e}_1^* + \mathbf{e}_2^*, \mathbf{n}\rangle = n_1 + n_2, \quad \text{where} \quad \mathbf{n} = [n_1,\ n_2].$$

Example 10.8. Let consider Example 10.3 (Fig. 27). The vertices of the polytope are

$$l_{\sigma_1} = l_{\sigma_4} = -\mathbf{e}_1^* - 3\mathbf{e}_2^*, \quad l_{\sigma_2} = -\mathbf{e}_1^* + \mathbf{e}_2^*, \quad l_{\sigma_3} = \mathbf{e}_1^* + \mathbf{e}_2^*.$$

Then taking all eight integral points in the polytope (including the boundary), we get

$$\begin{aligned} h(\mathbf{n}) &= \inf\{\langle \mathbf{m}, \mathbf{n}\rangle \mid \ \mathbf{m} \in M_{\mathbb{Z}} \cap P(\Delta, h)\} \\ &= \min\{n_1 + n_2,\ n_2,\ -n_2,\ 0,\ -n_1 + n_2, \\ &\qquad\qquad -n_1,\ -n_1 - n_2,\ -n_1 - 2n_2,\ -n_1 - 3n_2\} \end{aligned}$$

Then it is an elementary calculation to check that we get the original support function h.

10.6. From fans with strictly upper convex support function to polytopes. Let Δ be a complete fan in $N_{\mathbb{R}}$. We recall that a Δ-linear support function $h \in \mathrm{SF}(\Delta)$ is upper convex if $h(\mathbf{n} + \mathbf{n}') \geqslant h(\mathbf{n}) + h(\mathbf{n}')$. The function h is strictly upper convex if additionally is different as a linear function on different r-dimensional cones.

If the support function h is strictly upper convex, then there is a one-to-one correspondence between fans with support function and polytopes and the composition of the operations

$$(\Delta, h) \longrightarrow P(\Delta, h) \longrightarrow (\Delta(P(\Delta, h),\ h)) \quad \text{is identity, i.e.,} \quad \Delta(P(\Delta, h)) = \Delta.$$

Following [13], we have

Lemma 10.9. (Lemma 2.12, [13]). *The following conditions are equivalent:*

1. The support function $h \in \mathrm{SF}(\Delta)$ *is strictly upper convex.*

2. The integral convex polytope

$$P(\Delta, h) = \{\mathbf{m} \in M_{\mathbb{R}} \mid \langle \mathbf{m}, \mathbf{n} \rangle \geqslant h(\mathbf{m}), \forall \mathbf{n} \in N_{\mathbb{R}}\}$$

is r*-dimensional and has exactly* $\{l_\sigma \mid \sigma \in \Delta(r)\}$ *as the set of its vertices. Moreover,* $l_\sigma \neq l_\tau$ *holds for each pair* $\sigma \neq \tau$ *in* $\Delta(r)$*.*

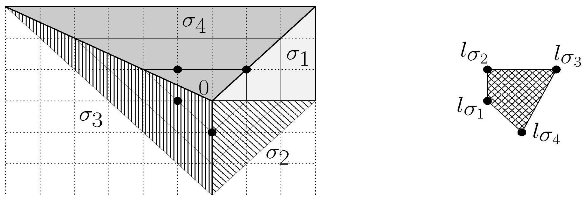


Fig. 30. Correspondence of faces of the polygon and cones

When these equivalent conditions are satisfied, the following correspondence between cones in Δ and of nonempty faces of $P(\Delta, h)$ are inverse to each other:

(27) $P(\Delta, h) > F \rightsquigarrow F^* = \{\mathbf{n} \in N_{\mathbb{R}} \mid \langle \mathbf{m}, \mathbf{n} \rangle = h(\mathbf{n}),\ \forall \mathbf{m} \in F\} \in \Delta,$

(28) $\Delta \ni \sigma \rightsquigarrow \sigma^* = \{\mathbf{m} \in P(\Delta, h) \mid \langle \mathbf{m}, \mathbf{n} \rangle = h(\mathbf{n}),\ \forall \mathbf{n} \in \sigma\}$ face of $P(\Delta, h)$,

(29) $\dim F + \dim F^* = r \quad \text{and} \quad \dim \sigma + \dim \sigma^* = r.$

Example 10.10. We illustrate the properties (27) and (28). Consider the fan and the strictly upper convex support function as in Example 10.5 and Fig. 30.

Take the vertex $l_{\sigma_3} = [1,\ 1]$ (0-dimensional face) of the polygon and calculate $l^*_{\sigma_3}$ according to (27). Just using the definition of the function h, we get

$$\begin{aligned} l^*_{\sigma_3} &= \{\mathbf{n} \in N_{\mathbb{R}} \mid \langle [1,\ 1],\ \mathbf{n} \rangle = h(n_1, n_2)\} \\ &= \{\mathbf{n} \in N_{\mathbb{R}} \mid n_1 + n_2 = h(n_1, n_2)\} \\ &= \{\mathbf{n} \in \sigma_1 \mid n_1 + n_2 = -n_1\} \cup \{\mathbf{n} \in \sigma_2 \mid n_1 + n_2 = -n_1 + n_2\} \cup \\ &\qquad \cup \{\mathbf{n} \in \sigma_3 \mid n_1 + n_2 = n_1 + n_2\} \cup \{\mathbf{n} \in \sigma_4 \mid n_1 + n_2 = -n_2\} \\ &= \sigma_3 \end{aligned}$$

Take the face $\overline{l_{\sigma_2} l_{\sigma_3}}$ (1-dimensional face) and calculate

$$\begin{aligned}
(\overline{l_{\sigma_2} l_{\sigma_3}})^* &= \{\mathbf{n} \in N_{\mathbb{R}} \mid \langle \mathbf{m},\ \mathbf{n} \rangle = h(\mathbf{n}) \ \text{ for all } \ \mathbf{m} \in \overline{l_{\sigma_2} l_{\sigma_3}}\ \} \\
&= \{\mathbf{n} \in N_{\mathbb{R}} \mid n_1 + n_2 = h(n_1, n_2),\ n_2 = h(n_1, n_2), \\
&\qquad\qquad -n_1 + n_2 = h(n_1, n_2)\} \\
&= \{\mathbf{n} \in N_{\mathbb{R}} \mid n_1 = 0,\ n_2 = h(n_1, n_2)\} \\
&= \sigma_2 \cap \sigma_3.
\end{aligned}$$

The other direction, take for instance the cone σ_2 and calculate

$$\begin{aligned}
\sigma_2^* &= \{\mathbf{m} \in P(\Delta, h) \mid m_1 n_1 + m_2 n_2 = h(n_1, n_2) \ \text{ for all } n_1 \geqslant 0,\ n_2 \leqslant 0\} \\
&= \{\mathbf{m} \in P(\Delta, h) \mid m_1 n_1 + m_2 n_2 = -n_1 + n_2 \ \text{ for all } n_1 \geqslant 0,\ n_2 \leqslant 0\} \\
&= \{\mathbf{m} \in P(\Delta, h) \mid (m_1 + 1) n_1 + (m_2 - 1) n_2 = 0 \ \text{ for all } n_1 \geqslant 0,\ n_2 \leqslant 0\} \\
&= \{\mathbf{m} = [-1,\ 1]\} \\
&= l_{\sigma_2}
\end{aligned}$$

Another case, take $\sigma_1 \cap \sigma_2$ and calculate

$$\begin{aligned}
(\sigma_1 \cap \sigma_2)^* &= \{\mathbf{m} \in P(\Delta, h) \mid m_1 n_1 + m_2 n_2 = h(n_1, n_2),\ \text{for all } n_1 \geqslant 0,\ n_2 = 0\} \\
&= \{\mathbf{m} \in P(\Delta, h) \mid m_1 n_1 = -n_1,\ \text{for all } n_1 \geqslant 0,\ n_2 = 0\} \\
&= \{\mathbf{m} \in P(\Delta, h) \mid m_1 = -1\} \\
&= \overline{l_{\sigma_1} l_{\sigma_2}}.
\end{aligned}$$

10.7. From polytopes to fans with support function. Many properties which could fit into this subsection actually were included in Subsections 10.2 and 10.6. Here we note that a convex polytope P determines a strictly upper convex support function on the fan Δ_P in the following way:

$$h_P(\mathbf{n}) = \inf\{\langle \mathbf{m}, \mathbf{n} \rangle \mid \ \mathbf{m} \in P\}. \tag{30}$$

For each $\mathbf{n} \in N_{\mathbb{R}}$ the affine half space

$$H^+(\mathbf{n}; h_P(\mathbf{n})) = \{\mathbf{m} \in M_{\mathbb{R}} \mid \langle \mathbf{m}, \mathbf{n} \rangle \geqslant h_P(\mathbf{n})\}$$

contains the polytope P which has a nonempty intersection with the boundary

$$\partial H^+(\mathbf{n}; h_P(\mathbf{n})) = \{\mathbf{m} \in M_{\mathbb{R}} \mid \langle \mathbf{m}, \mathbf{n}\rangle = h_P(\mathbf{n})\}.$$

It is not difficult to prove (see [13], Theorem A.11) that

$$P = \bigcap_{\mathbf{n}\in N_{\mathbb{R}}} H^+(\mathbf{n}; h_P(\mathbf{n})) = \{\mathbf{m} \in M_{\mathbb{R}} \mid \langle \mathbf{m}, \mathbf{n}\rangle \geqslant h_P(\mathbf{n}),\ \forall \mathbf{n} \in N_{\mathbb{R}}\}.$$

Also we should note that $h_P(\mathbf{n})$ additively depends on translations of the polytope P, more precisely we have

$$\begin{aligned} h_{P+\mathbf{m}_0}(\mathbf{n}) &= \inf\{\langle \mu, \mathbf{n}\rangle \mid \ \mu \in P + \mathbf{m}_0\} \\ &= \inf\{\langle \mathbf{m} + \mathbf{m}_0, \mathbf{n}\rangle \mid \ \mathbf{m} \in P\} \\ &= \inf\{\langle \mathbf{m}, \mathbf{n}\rangle \mid \ \mathbf{m} \in P\} + \langle \mathbf{m}_0, \mathbf{n}\rangle \\ &= h_P(\mathbf{n}) + \langle \mathbf{m}_0,\, \mathbf{n}\rangle. \end{aligned} \tag{31}$$

Some considerations are needed to prove that there is a fan in $N_{\mathbb{R}}$ on which

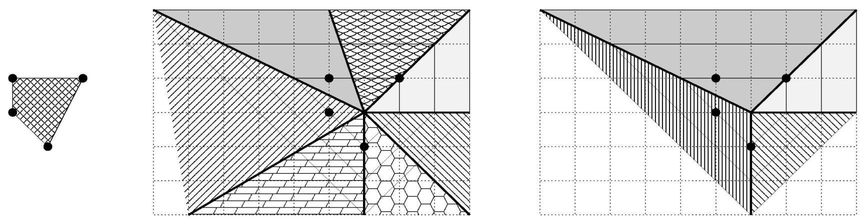


Fig. 31. From a polytope to fan and coarsest fan

the function (30) is strictly upper convex. We formulate this in the following proposition

Proposition 10.11 (compare [13], Theorem A.18 and Corollary A.19). *Let for a convex polytope $P \subset M_{\mathbb{R}}$ the function h be defined as in* (30).

1. *Then there exists a complete fan in $N_{\mathbb{R}}$ of convex polyhedral cones such that the restriction of h to each cone is a linear function.*

2. *Moreover, there exists the unique coarsest fan in $N_{\mathbb{R}}$ for which there is a one-to-one correspondence between the faces of the polytope and cones of the fan (as in* (27) *and* (28)*).*

We note that because of the translation property (31), the coarsest fan mentioned in the decomposition does not depend on translation of P, i.e., h_P and $h_{P+\mathbf{m}_0}$ determine the same coarsest fan in $N_{\mathbb{R}}$.

Example 10.12. Consider the polytope given in Fig. 32 with vertices $(-1,-1)$, $(0,-1)$ and $(1,2)$. We notice that the only integral point inside is $(0,0)$. The function h_P determined by the polytope is

$$h_P(\mathbf{n}) = \min\{\langle \mathbf{m}, \mathbf{n}\rangle \mid \mathbf{m} \in P\} = \min\{0,\ -n_1 - n_2,\ -n_2,\ n_1 + 2n_2\}.$$

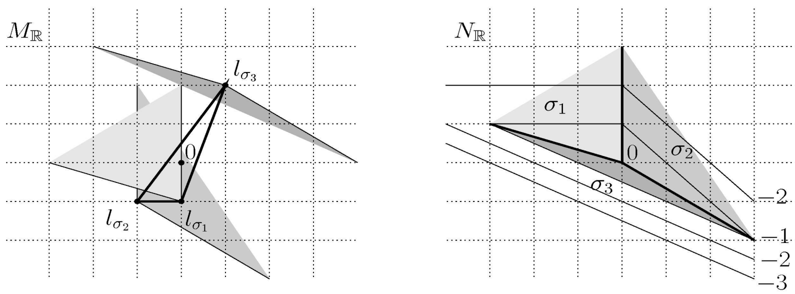


Fig. 32. From a polygon to a fan with support function

To evaluate the function explicitly we will consider three cases with respect when $\mathbf{n}$ lies, namely three dual cones σ_1, σ_2 and σ_3 as in the figure.

$$h_P(\mathbf{n}) = \begin{cases} \min\{0,\ -n_1 - n_2,\ -n_2,\ n_1 + 2n_2\} & \text{if} \quad [n_1, n_2] \in \sigma_1 \\ \min\{0,\ -n_1 - n_2,\ -n_2,\ n_1 + 2n_2\} & \text{if} \quad [n_1, n_2] \in \sigma_2 \\ \min\{0,\ -n_1 - n_2,\ -n_2,\ n_1 + 2n_2\} & \text{if} \quad [n_1, n_2] \in \sigma_3 \end{cases}$$

$$= \begin{cases} \min\{0,\ -n_1 - n_2,\ -n_2,\ n_1 + 2n_2\} & \text{if} \quad n_1 \leqslant 0 & \text{and} \quad n_1 + 3n_2 \geqslant 0 \\ \min\{0,\ -n_1 - n_2,\ -n_2,\ n_1 + 2n_2\} & \text{if} \quad n_1 \geqslant 0 & \text{and} \quad 2n_1 + 3n_2 \geqslant 0 \\ \min\{0,\ -n_1 - n_2,\ -n_2,\ n_1 + 2n_2\} & \text{if} \quad n_1 + 3n_2 \leqslant 0 & \text{and} \quad 2n_1 + 3n_2 \leqslant 0 \end{cases}$$

$$= \begin{cases} -n_2 & \text{if} \quad n_1 \leqslant 0 & \text{and} \quad 3n_1 + n_2 \geqslant 0 \\ -n_1 - n_2 & \text{if} \quad n_1 \geqslant 0 & \text{and} \quad 2n_1 + 3n_2 \geqslant 0 \\ n_1 + 2n_2 & \text{if} \quad n_1 + 3n_2 \leqslant 0 & \text{and} \quad 2n_1 + 3n_2 \leqslant 0 \end{cases}$$

$$= \begin{cases} \langle [0,-1],\ [n_1, n_2]\rangle & \text{if} \quad \mathbf{n} = [n_1,\ n_2] \in \sigma_1 & \rightsquigarrow \quad l_{\sigma_1} = [0,-1] \\ \langle [-1,-1],\ [n_1, n_2]\rangle & \text{if} \quad \mathbf{n} = [n_1,\ n_2] \in \sigma_2 & \rightsquigarrow \quad l_{\sigma_2} = [-1,-1] \\ \langle [1,2],\ [n_1, n_2]\rangle & \text{if} \quad \mathbf{n} = [n_1,\ n_2] \in \sigma_3 & \rightsquigarrow \quad l_{\sigma_3} = [1,2] \end{cases}$$

11. Equivariant line bundles over toric varieties. Equivariant line bundles and Cartier divisors (considered in the next section) are very useful tools in the theory of toric varieties. In particular they provide methods to characterize projective toric varieties, i.e., toric varieties which can be embedded into projective spaces.

11.1. Picard group and equivariant line bundles.
Line bundles and Picard group. Before we consider line bundles over toric varieties, we very shortly define the Picard group over any variety (not necessarily toric).

Let X be a complex variety covered by open sets $\{X_\alpha\}_{\alpha\in I}$. Assume that holomorphic maps are given

$$f_{\alpha\beta} : X_\alpha \cap X_\beta \longrightarrow \mathbb{C}^* \quad \text{that satisfy} \quad f_{\alpha\gamma} = f_{\alpha\beta} f_{\beta\gamma} \quad \text{for} \quad \alpha, \beta, \gamma \in I.$$

We define a line bundle over X by gluing $X_\alpha \times \mathbb{C}$ in the following way

$$(32) \qquad (X_\alpha \cap X_\beta) \times \mathbb{C} \ni (x, c_\beta) \longleftrightarrow (x, f_{\alpha\beta}\, c_\beta) = (x, c_\alpha) \in (X_\alpha \cap X_\beta) \times \mathbb{C}.$$

After such identification we get the line bundle $L \xrightarrow{\pi} X$.

It is a standard property that the set of line bundles over X forms a group, where the multiplication is the tensor product $L \otimes L'$ with the transition functions $f_{\alpha\beta} f'_{\alpha\beta}$ and inverse is by taking $f^{-1}_{\alpha\beta}$. The group of the isomorphism classes of line bundles on X is denoted by $\mathrm{Pic}(X)$.

Equivariant line bundles. From now on as X we take a toric variety $T(\Delta)$ of dimension r. Since we have a torus action on toric varieties, therefore it is natural to involve this action on line bundles over X. Such line bundles with an additional torus action are called *equivariant line bundles* and a precise definition is given below.

Let $\pi : L \longrightarrow X$ be a line bundle over a toric variety $X = T(\Delta)$, $\dim_{\mathbb{C}} X = r$. The fiber L_x is isomorphic to $\mathbb{C}$.

Definition 11.1. *The bundle $\pi : L \longrightarrow X$ is an* equivariant line bundle *if the complex torus T acts on L, linearly on each fiber, and its action is compatible with the projection π, i.e., $\pi(t\lambda) = t\pi(\lambda)$ for all $t \in T$ and $\lambda \in L$.*

$$\begin{array}{ccc} L & \xrightarrow{t} & L \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{t} & X \end{array} \qquad\qquad \begin{array}{ccc} \lambda & \xrightarrow{t} & t\lambda \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \pi(\lambda) & \xrightarrow{t} & t\pi(\lambda) = \pi(t\lambda) \end{array}$$

The set of equivalence classes of equivariant line bundles is denoted by $\mathrm{ELB}(X)$. It forms a commutative group with respect to the tensor product. The identity element in $\mathrm{ELB}(X)$ is the trivial bundle $X \times \mathbb{C}$ with the torus action $t(u, c) = (tu, c)$ for each $t \in T$, $(u, c) \in X \times \mathbb{C}$. Disregarding the torus action on an equivariant line bundle we obtain a homomorphism

$$\mathrm{ELB}(X) \longrightarrow \mathrm{Pic}(X).$$

11.2. Support function determines a line bundle. We will see that there is a natural homomorphism $\mathrm{SF}(\Delta) \longrightarrow \mathrm{ELB}(X)$, which associates an equivariant line bundle L_h to each Δ-linear support function h. This homomorphism is not necessarily onto, but for compact toric varieties actually it is.

Let Δ be a fan in $N_{\mathbb{R}}$ and $T(\Delta)$ the corresponding toric variety. For each $\sigma \in \Delta$ we have the affine toric variety U_σ which is open in $T(\Delta)$. To construct a bundle, it is enough to define transition functions $f_{\sigma\tau} : U_{\sigma\tau} \longrightarrow \mathbb{C}^*$, which satisfy the cocycle property $f_{\sigma\rho} f_{\rho\tau} = f_{\sigma\tau}$ and the identifications are as in (32). If a support function $h \in \mathrm{SF}(\Delta)$ is given, then we describe a natural procedure how to construct the transition functions $f_{\tau\sigma}$.

Let $h \in \mathrm{SF}(\Delta)$. This function determines (not uniquely) the set $\{l_\sigma \mid \sigma \in \Delta\}$ such that $h(\mathbf{n}) = \langle l_\sigma, \mathbf{n}\rangle$ for $\mathbf{n} \in \sigma_{\mathbb{Z}}$. For the intersection of two cones $\sigma \cap \tau$ we have

$$h(\mathbf{n}) = \langle l_\sigma, \mathbf{n}\rangle = \langle l_{\sigma\cap\tau}, \mathbf{n}\rangle = \langle l_\tau, \mathbf{n}\rangle \quad \text{for} \quad \mathbf{n} \in \sigma_{\mathbb{Z}} \cap \tau_{\mathbb{Z}}.$$

Thus $l_\sigma - l_\tau$ and $l_\tau - l_\sigma$ are contained in $M_{\mathbb{Z}} \cap (\sigma\cap\tau)^{\perp} \subset (\sigma\cap\tau)^{\check{}}$, so in particular they determine a mapping

$$\begin{aligned} U_{\sigma\cap\tau} &\longrightarrow \mathbb{C} \qquad \text{given by} \qquad U_{\sigma\cap\tau} \ni u \longmapsto u(l_\sigma - l_\tau), \\ U_{\sigma\cap\tau} &\longrightarrow \mathbb{C} \qquad \text{given by} \qquad U_{\sigma\cap\tau} \ni u \longmapsto u(l_\tau - l_\sigma). \end{aligned}$$

Both these mappings are holomorphic on $U_{\sigma\cap\tau}$ and because

$$u(l_\sigma - l_\tau)u(l_\tau - l_\sigma) = u(0) = 1$$

they are nonzero. The transition functions we choose are $f_{\tau\sigma}(u) = u(l_\sigma - l_\tau)$.

To show that this bundle L_h is equivariant, we just write explicitly the torus action:

$$T \ni t \longrightarrow t(u, c) = (tu,\, t(-l_\sigma)c) \quad \text{for} \quad (u, c) \in U_\sigma \times \mathbb{C}, \quad \sigma \in \Delta.$$

This action is compatible with the gluing maps $f_{\sigma\tau}$:

$$\begin{tikzcd} U_\sigma \times \mathbb{C} \arrow[r, "t"] \arrow[d, "f_{\tau\sigma}"'] & U_\sigma \times \mathbb{C} \arrow[d, "f_{\tau\sigma}"] \\ U_\tau \times \mathbb{C} \arrow[r, "t"] & U_\tau \times \mathbb{C} \end{tikzcd}$$

or

$$\begin{tikzcd} (u,c) \arrow[r, "t"] \arrow[d, "f_{\tau\sigma}"'] & (tu,\, t(-l_\sigma)c) \arrow[d, "f_{\tau\sigma}"] \\ (u,\, f_{\tau\sigma}(u)c) \arrow[r, "t"] & t(u,\, f_{\tau\sigma}(u)c) = (tu,\, f_{\tau\sigma}(tu)t(-l_\sigma)c) \end{tikzcd}$$

where the equality in the diagram holds because

$$t(u,\ f_{\tau\sigma}(u)\,c) \quad = \quad (tu,\ t(-l_\tau)u(l_\sigma - l_\tau)c)$$

and

$$\begin{aligned} (tu,\ f_{\tau\sigma}(tu)\,t(-l_\sigma)c) &= (tu,\ (tu)(l_\sigma - l_\tau)t(-l_\sigma)c) \\ &= (tu,\ t(l_\sigma - l_\tau)u(l_\sigma - l_\tau)t(-l_\sigma)c) \\ &= (tu,\ u(l_\sigma - l_\tau)t(-l_\tau)c). \end{aligned}$$

In a similar way as above, it is easy to prove that a different set $\{l'_\sigma \mid \sigma \in \Delta\}$ corresponding to the same h determines an isomorphic equivariant line bundle.

Example 11.2. The simplest support function is just a linear function on $N_\mathbb{R}$, independently on the fan Δ in $N_\mathbb{R}$. Such function is determined by a single element $\mathbf{m} \in M_\mathbb{Z}$. Since $l_\sigma = \mathbf{m}$ for all $\sigma \in \Delta$, therefore the transition functions are just 1, $f_{\tau\sigma}(u) = u(l_\sigma - l_\tau) = u(0) = 1$. The bundle is trivial, but still we have a nontrivial action given by

$$T \ni t \longrightarrow t(u,c) = (tu,\ t(-\mathbf{m})c) \quad \text{for} \quad (u,c) \in U_\sigma \times \mathbb{C}, \quad \sigma \in \Delta.$$

This equivariant line bundle is denoted by $L_\mathbf{m}$ or $\mathbf{1}_{-\mathbf{m}}$.

11.3. Relation between equivariant line bundles and the Picard group. In general, the homomorphism $\mathrm{SF}(\Delta) \longrightarrow \mathrm{ELB(X)}$, $X = T(\Delta)$, is not onto, however we get some equivalence if X is compact. First some comments.

Since $\mathbf{m} \in M_{\mathbb{Z}}$ determines a linear mapping on $N_{\mathbb{Z}}$, therefore $M_{\mathbb{Z}}$ can be considered as a subspace of $\mathrm{SF}(\Delta)$ and consequently we have $\mathrm{SF}(\Delta)/M_{\mathbb{Z}}$. Similarly, the set of bundles $\{\mathbf{1_m} \mid \mathbf{m} \in M_{\mathbb{Z}}\}$ can be considered as a subset of $\mathrm{ELB}(X)$ and also $\mathrm{ELB}(X)/\{\mathbf{1_m} \mid \mathbf{m} \in M_{\mathbb{Z}}\}$ makes sense. We have the diagram

$$\begin{tikzcd}
\mathrm{SF}(\Delta) \arrow[r] \arrow[d] & \mathrm{ELB}(X) \arrow[d] & \\
\mathrm{SF}(\Delta)/M_{\mathbb{Z}} \arrow[r] & \mathrm{ELB}(X)/M_{\mathbb{Z}} \arrow[r] & \mathrm{Pic}(X)
\end{tikzcd}$$

Lemma 11.3 ([13], Corollary 2.5). *For any compact toric variety $X = T(\Delta)$ we have a canonical isomorphisms*

$$\mathrm{SF}(\Delta)/M_{\mathbb{Z}} \xrightarrow{\sim} \mathrm{ELB}(X)/\{\mathbf{1_m} \mid \mathbf{m} \in M_{\mathbb{Z}}\} \xrightarrow{\sim} \mathrm{Pic}(X).$$

11.4. Sections of line bundles over toric varieties. If $L \xrightarrow{\pi} X$ is a line bundle over a variety X (both analytic), then by a section of the bundle we mean an analytic mapping $\varphi : X \longrightarrow L$ such that $\pi(\varphi(\xi)) = \pi(\xi)$. In general it is a difficult question about existence of nontrivial sections of the bundle.

Let $X = T(\Delta)$ be an r-dimensional toric variety and L_h the line bundle determined by $h \in \mathrm{SF}(\Delta)$. It appears that there is a simple procedure to construct sections for some elements $\mathbf{m} \in M_{\mathbb{Z}}$. We describe this procedure in the following lemma:

Lemma 11.4 ([13], Proposition 2.1(ii)). *Suppose that $h \in \mathrm{SF}(\Delta)$. If $\mathbf{m} \in M_{\mathbb{Z}}$ satisfies*

$$\langle \mathbf{m}, \mathbf{n} \rangle \geqslant h(\mathbf{n}) \quad \text{for all} \quad \mathbf{n} \in |\Delta|, \tag{33}$$

then there is a section $\varphi = \varphi_{\mathbf{m}} : X \longrightarrow L_h$ that is T-semi-invariant, i.e., satisfies $\varphi(tu) = t(u)\,(t\varphi(u))$, where the right hand side is the scalar multiple by $t(u) \in \mathbb{C}^*$ of the element $t\varphi(u)$ in the fiber over tu. The section $\varphi_{\mathbf{m}}$ is locally on U_σ given by the formula

$$\varphi_\sigma : U_\sigma \longrightarrow U_\sigma \times \mathbb{C}, \quad \varphi_\sigma = (\mathrm{id},\ \chi(\mathbf{m} - l_\sigma)), \quad \varphi_\sigma(u) = (u,\ u(\mathbf{m} - l_\sigma)). \tag{34}$$

Proof. We just give an explicit formula for the section φ. If $h \in \mathrm{SF}(\Delta)$ and determined by $\{l_\sigma\}_{\sigma\in\Delta}$, then

$$\begin{aligned}\langle \mathbf{m}, \mathbf{n}\rangle \geqslant h(\mathbf{n}) \quad \forall \mathbf{n} \in |\Delta| &\iff \langle \mathbf{m}, \mathbf{n}\rangle \geqslant \langle l_\sigma, \mathbf{n}\rangle \quad \forall \mathbf{n} \in \sigma, \quad \sigma \in \Delta \\ &\iff \langle \mathbf{m} - l_\sigma, \mathbf{n}\rangle \geqslant 0 \quad \forall \mathbf{n} \in \sigma, \quad \sigma \in \Delta \\ &\iff m - l_\sigma \in \check{\sigma}_{\mathbb{Z}}\,, \qquad \sigma \in \Delta \\ &\iff U_\sigma \ni u \mapsto u(\mathbf{m} - l_\sigma) \in \mathbb{C} \quad \text{is holomorphic.}\end{aligned}$$

Now we can define the section explicitly over each U_σ:

$$\varphi_\sigma : U_\sigma \longrightarrow U_\sigma \times \mathbb{C}, \quad \varphi_\sigma(u) := (u,\, u(\mathbf{m} - l_\sigma)).$$

To check that it defines a global section, we check compatibility with transition functions

$$f_{\tau\sigma}(u)\, u(\mathbf{m} - l_\sigma) = u(l_\sigma - l_\tau)\, u(\mathbf{m} - l_\sigma) = u(\mathbf{m} - l_\tau).$$

$$\begin{tikzcd} L_h \arrow[r, "\tilde t"] & L_h \\ X \arrow[u, "\varphi"] \arrow[r, "t"] & X \arrow[u, "\varphi"'] \end{tikzcd} \qquad \begin{tikzcd} L_{h,u} \arrow[r, "\tilde t"] & L_{h,tu} \\ u \arrow[u, "\varphi"] \arrow[r, "t"] & tu \arrow[u, "\varphi"'] \end{tikzcd} \qquad \begin{tikzcd} \varphi(u) \arrow[r, "\tilde t"] & \varphi(tu) = t(u)(t\varphi(u)) \\ u \arrow[u, "\varphi"] \arrow[r, "t"] & tu \arrow[u, "\varphi"'] \end{tikzcd}$$

To check that φ is T-semi-invariant is straightforward. $\square$

Example 11.5. If $h \in \mathrm{SF}(\Delta)$ is the linear mapping on $N_{\mathbb{R}}$, i.e.,

$$h(\mathbf{n}) = \langle \mathbf{m}_0, \mathbf{n}\rangle \quad \forall \mathbf{n} \in N_{\mathbb{Z}} \quad \text{and some} \quad \mathbf{m}_0 \in M_{\mathbb{Z}},$$

then the only $\mathbf{m}$ that satisfies $\langle \mathbf{m}, \mathbf{n}\rangle \geqslant \langle \mathbf{m}_0, \mathbf{n}\rangle$ for all $\mathbf{n}$ is $\mathbf{m} = \mathbf{m}_0$. The bundle $L_{\mathbf{m}}$ is holomorphically trivial and the section φ is just $X \ni u \longrightarrow (u,\, 1) \in X \times \mathbb{C}$.

Example 11.6. In this example we show that not for all $h \in \mathrm{SF}(\Delta)$ there is $\mathbf{m} \in M_{\mathbb{Z}}$ that satisfy (33). As in Fig. 33 that represents a fan for $\mathbb{CP}^2$, let the level lines be for values 1, 2, etc., then the corresponding vectors in $M_{\mathbb{Z}}$ are

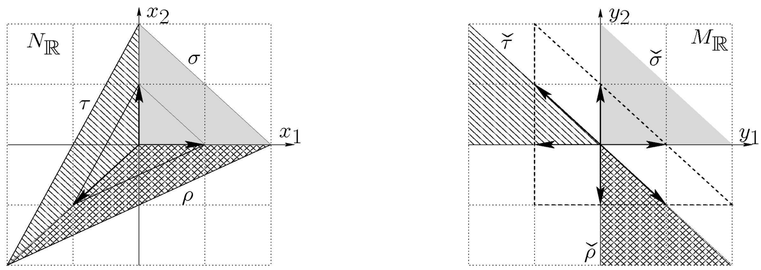


Fig. 33. Support function on $\mathbb{CP}^2$ and line bundles

$l_\sigma = [1,\, 1]$, $l_\tau = [-2,\, 1]$, $l_\rho = [1,\, -2]$. If such $\mathbf{m} = [a,\, b]$ exists, then it satisfies

$$\begin{cases} [a,\, b] - [1,\, 1] \in \check{\sigma}_{\mathbb{Z}} \\ [a,\, b] - [-2,\, 1] \in \check{\tau}_{\mathbb{Z}} \\ [a,\, b] - [1,\, -2] \in \check{\rho}_{\mathbb{Z}} \end{cases} \Longrightarrow \begin{cases} a - 1 \geqslant 0 \quad \text{and} \quad b - 1 \geqslant 0 \\ b - 1 \geqslant 0 \quad \text{and} \quad b - 1 \leqslant -a - 2 \\ a - 1 \geqslant 0 \quad \text{and} \quad b + 2 \leqslant -a + 1 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} a \geqslant 1 \qquad \text{and} \quad b \geqslant 1 \\ b + a \leqslant -1 \end{cases}$$

contradiction

Example 11.7. If we consider the above example with the only difference that the support function takes the values -1, -2, etc., instead of positive integers, then such $\mathbf{m}$'s exist. In this case, the corresponding vectors in $M_{\mathbb{Z}}$ are $l_\sigma = [-1,\, -1]$, $l_\tau = [2,\, -1]$, $l_\rho = [-1,\, 2]$. So we have

$$\begin{cases} [a,\, b] - [-1,\, -1] \in \check{\sigma}_{\mathbb{Z}} \\ [a,\, b] - [2,\, -1] \in \check{\tau}_{\mathbb{Z}} \\ [a,\, b] - [-1,\, 2] \in \check{\rho}_{\mathbb{Z}} \end{cases} \Longrightarrow \begin{cases} a + 1 \geqslant 0 \quad \text{and} \quad b + 1 \geqslant 0 \\ b + 1 \geqslant 0 \quad \text{and} \quad b + 1 \leqslant -a + 2 \\ a + 1 \geqslant 0 \quad \text{and} \quad b - 2 \leqslant -a - 1 \end{cases}$$

$$\Downarrow$$

$$\begin{matrix} [-1,\, 2],\ [-1,\, 1],\ [-1,\, 0], \\ [-1,\, -1],\ [0,\, 1],\ [0,\, 0], \\ [0,\, -1],\ [1,\, 0],\ [1,\, -1], \\ [2,\, -1] \end{matrix} \quad \xleftarrow{\text{Fig. 33, right}} \quad \begin{cases} a \geqslant -1 \qquad \text{and} \quad b \geqslant -1 \\ b + a \leqslant 1 \end{cases}$$

12. Cartier divisors on toric varieties.

12.1. Definition of various classes of divisors. We consider few classes of divisors on toric varieties.

Weil divisors. $\mathrm{Div}(X)$. A *divisor* or *Weil divisor* is a formal finite $\mathbb{Z}$-linear combination of closed irreducible subvarieties of codimension one of a toric variety $X = T(\Delta)$:

$$D = \sum_{j \in I} a_j V_j, \quad a_j \in \mathbb{Z}, \quad V_j \subset X, \quad \operatorname{codim} V_j = 1, \quad j \in I,$$

with the natural addition and multiplication by integers. This set of divisors together with these operations forms a commutative group.

Effective divisors. A divisor D is called *effective* and denoted by $D \geqslant 0$ if the coefficients $a_j \geqslant 0$ for $j \in I$.

***T*-invariant divisors.** T-$\mathrm{Div}(X)$. A divisor D is called T*-invariant* if the torus action preserves each V_j, i.e., $t(V_j) = V_j$ for $t \in T$. Since V_j is T-invariant, it should contain orbits of codimension one. Orbits were considered in Section 8 and we have seen that codimension one orbits $V(\rho)$ are in one-to-one correspondence with one-dimensional cones ρ in the fan Δ. Consequently, T-invariant divisors are of the form

$$D = \sum_{\rho \in \Delta(1)} a_\rho V(\rho), \quad a_\rho \in \mathbb{Z} \quad \Longrightarrow \quad T\text{-}\mathrm{Div}(X) = \bigoplus_{\rho \in \Delta(1)} \mathbb{Z}\, V(\rho). \tag{35}$$

Principal divisors. $\mathrm{PDiv}(X)$. A divisor is called a *principal divisor* if it is of the form $\mathrm{div}(f) = \sum_V o_V(f)\, V$ for a nonzero rational function f on X, where $o_V(f)$ is the order of zero of f along each closed irreducible subspace V of X of codimension one. If f has a pole along V, then $o_V(f)$ is the negative of the order of pole.

Cartier divisors. $\mathrm{CDiv}(X)$. Cartier divisors are defined as locally principle Weil divisors. So we have $\mathrm{PDiv}(X) \subset \mathrm{CDiv}(X)$.

***T*-invariant Cartier divisors.** T-$\mathrm{CDiv}(X)$. This class is defined as

$$T\text{-}\mathrm{CDiv}(X) := T\text{-}\mathrm{Div}(X) \cap \mathrm{CDiv}(X)$$

and obviously

$$T\text{-}\mathrm{Div}(X) \cap \mathrm{PDiv}(X) \subset T\text{-}\mathrm{CDiv}(X).$$

Summarizing:

$\mathrm{Div}(X)$	set of *Weil divisors* on X.
T-$\mathrm{Div}(X)$	subgroup of $\mathrm{Div}(X)$ consisting of T-*invariant divisors* on X.
$\mathrm{PDiv}(X)$	subgroup of $\mathrm{Div}(X)$ consisting of *principal divisors* on X.
$\mathrm{CDiv}(X)$	*Cartier divisors*, i.e., locally principal Weil divisors.
T-$\mathrm{CDiv}(X) :=$	T-$\mathrm{Div}(X) \cap \mathrm{CDiv}(X)$ the subgroup consisting of T-*invariant Cartier divisors.*

12.2. How a support function determines a Cartier divisor? We know a natural way to assign an equivariant line bundle to a support function $h \in \mathrm{SF}(\Delta)$. Now we can define explicitly T-invariant Cartier divisors determined by a support function $h \in \mathrm{SF}(\Delta)$:

$$\mathrm{SF}(\Delta) \ni h \longmapsto D_h \in T\text{-}\mathrm{CDiv}(X), \quad D_h := - \sum_{\rho \in \Delta(1)} h(\mathbf{n}_\rho) V(\rho), \tag{36}$$

where $\mathbf{n}_\rho$ is the primitive element of the 1-dimensional cone ρ, i.e., $\mathbf{n}_\rho$ generates $\rho_{\mathbb{Z}}$. It appears that the above homomorphism is injective (see [13], p. 69). Moreover, if X is nonsingular, then we have an isomorphism:

$$\mathrm{SF}(\Delta) \xrightarrow{\sim} T\text{-}\mathrm{CDiv}(X) = T\text{-}\mathrm{Div}(X) = \bigoplus_{\rho \in \Delta(1)} \mathbb{Z}\, V(\rho).$$

Actually the formula (36) needs some explanation. First start with a concrete example below.

12.3. Calculation of the principal divisor in dimension two.

Example 12.1. For each $\mathbf{m} \in M_{\mathbb{Z}}$ let $\chi(\mathbf{m})$ be a character defined on the torus $T = T_r$ by $\chi(\mathbf{m})(t) = t(\mathbf{m})$. For a cone σ in $\mathbb{N}_{\mathbb{R}} \simeq \mathbb{R}^2$ and any $\mathbf{m} \in M_{\mathbb{Z}} \simeq \mathbb{Z}^2$, we will show the formula

$$\mathrm{div}(\chi(\mathbf{m})) = \sum_{\rho \in \Delta(1)} \langle \mathbf{m}, \mathbf{n}_\rho \rangle V(\rho). \tag{37}$$

The cone σ is spanned over $\mathbb{R}$ by $\mathbf{n}_{\rho_1} = [4,\ 1]$ and $\mathbf{n}_{\rho_2} = [1,\ 3]$. The vector $\mathbf{n}^*_{\rho_1} = [3,\ -1]$ (from the dual space) is orthogonal to $\mathbf{n}_{\rho_2}$, i.e., $\langle \mathbf{n}^*_{\rho_1}, \mathbf{n}_{\rho_2} \rangle = 0$, and the vector $\mathbf{n}^*_{\rho_2} = [-1,\ 4]$ is orthogonal to $\mathbf{n}_{\rho_1}$. In the general case, which we will

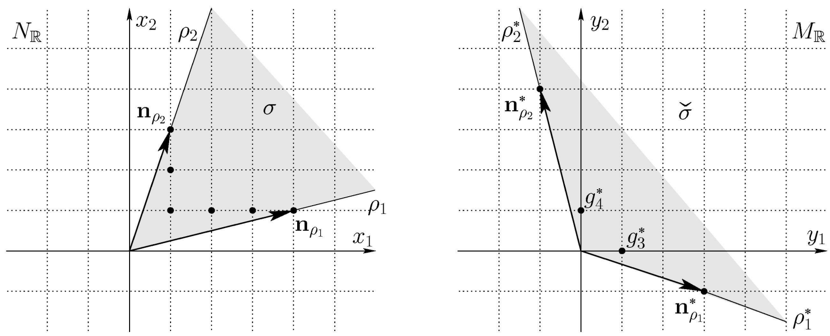


Fig. 34. Linear function determining a divisor

consider later on, if an r-dimensional cone is spanned by $\mathbf{n}_{\rho_1}, \ldots, \mathbf{n}_{\rho_r}$, the vector $\mathbf{n}_{\rho_\alpha}^*$ will be orthogonal to all vectors $\mathbf{n}_{\rho_\beta}$, $\alpha \neq \beta$.

Since $\mathbf{n}_{\rho_1}^*$ and $\mathbf{n}_{\rho_2}^*$ do not generate $\check{\sigma}_{\mathbb{Z}}$, we add additional generators $g_3^* = [1,\ 0]$ and $g_4^* = [0,\ 1]$. We have obvious relations

$$\left\{ \begin{aligned} 4\mathbf{n}_{\rho_1}^* + \mathbf{n}_{\rho_2}^* &= 11g_3^* \\ \mathbf{n}_{\rho_1}^* + 3\mathbf{n}_{\rho_2}^* &= 11g_4^* \end{aligned} \right. \qquad \Longrightarrow \qquad \left\{ \begin{aligned} z_1^4 z_2 &= z_3^{11} \\ z_1 z_2^3 &= z_4^{11} \end{aligned} \right. \tag{38}$$

Any $u \in U_\sigma$ can be represented by $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, where z satisfies (38).

Now take any element $\mathbf{m} \in M_{\mathbb{Z}}$ and the linear function $h(\mathbf{n}) = \langle \mathbf{m},\ \mathbf{n} \rangle$. The vector $\mathbf{m}$ is a linear combination of ρ_1^* and ρ_2^*

$$\mathbf{m} = \frac{1}{11}(4m_1 + m_2)\mathbf{n}_{\rho_1}^* + \frac{1}{11}(m_1 + 3m_2)\mathbf{n}_{\rho_2}^*.$$

Assuming that $z_1 \neq 0$ and $z_2 \neq 0$, the character function $\chi(\mathbf{m})$ is

$$\begin{aligned} \chi(\mathbf{m})(u) = u(\mathbf{m}) \quad &= \quad u\left(\frac{1}{11}(4m_1 + m_2)\mathbf{n}_{\rho_1}^* + \frac{1}{11}(m_1 + 3m_2)\mathbf{n}_{\rho_2}^*\right) \\ &= \quad z_1^{\frac{1}{11}(4m_1+m_2)} z_2^{\frac{1}{11}(m_1+3m_2)}. \end{aligned}$$

The function $\chi(\mathbf{m})(u)$ has a zero or a pole depending whether at least one z_1 or z_2 vanish. Suppose that $z_2 \neq 0$. Then to calculate the order of zero or pole of

$\chi(\mathbf{m})$ we have

$$z_1 z_2^3 = z_4^{11} \quad \Longrightarrow \quad \begin{aligned} \chi(\mathbf{m})(u) &= z_1^{\frac{1}{11}(4m_1+m_2)} z_2^{\frac{1}{11}(m_1+3m_2)} \\ &= \left(\frac{z_4^{11}}{z_2^3}\right)^{\frac{1}{11}(4m_1+m_2)} z_2^{\frac{1}{11}(m_1+3m_2)} \\ &= z_4^{4m_1+m_2} z_2^{-m_1} \\ &= z_4^{\langle \mathbf{m},\, \mathbf{n}_{\rho_1}\rangle} z_2^{-m_1}. \end{aligned}$$

The order of zero or pole of $\chi(\mathbf{m})$ is $\langle \mathbf{m}, \rho_1\rangle = 4m_1 + m_2$. But this zero set corresponds to the variety $V(\rho_1)$ therefore as a divisor it is $\langle \mathbf{m},\ \mathbf{n}_{\rho_1}\rangle V(\rho_1)$.

In the same way we calculate for ρ_2. Consequently we obtained

$$\operatorname{div}(\chi(\mathbf{m})) = \langle \mathbf{m}, \rho_1\rangle V(\rho_1) + \langle \mathbf{m}, \rho_2\rangle V(\rho_2).$$

12.4. Calculation of principal divisor in any dimension. The lemma below generalizes Example 12.1.

Lemma 12.2. *If $\sigma_{\mathbb{R}} \subset N_{\mathbb{R}} \simeq \mathbb{R}^r$ is strongly convex rational polyhedral cone, then for any $\mathbf{m} \in M_{\mathbb{Z}}$ we have*

$$\operatorname{div}(\chi(\mathbf{m})) = \sum_{\rho\in\sigma(1)} \langle \mathbf{m},\ \mathbf{n}_\rho\rangle\, V(\rho). \tag{39}$$

P r o o f. We prove the above lemma if the cone is r-dimensional and is spanned by 1-dimensional faces $\rho_1, \ldots, \rho_r$ with primitive elements $\mathbf{n}_{\rho_1}, \ldots, \mathbf{n}_{\rho_r} \in N_{\mathbb{Z}}$, respectively. The proof of the general case is similar, only more technical.

Let $\mathbf{n}^*_{\rho_1}, \ldots, \mathbf{n}^*_{\rho_r} \in M_{\mathbb{Z}}$ be vectors such that $\langle \mathbf{n}^*_{\rho_j}, \mathbf{n}_{\rho_k}\rangle = 0$ for $j \neq k$, $\langle \mathbf{n}^*_{\rho_j}, \mathbf{n}_{\rho_j}\rangle > 0$, and $\mathbf{n}^*_{\rho_j}$ generates over $\mathbb{Z}$ the corresponding 1-dimensional cone. We note that $\mathbf{n}^*_{\rho_j} \in \check{\sigma}_{\mathbb{Z}}$, $j = 1, \ldots, r$, and they can be a part of the generator set. Assume that the complete generator set of $\check{\sigma}_{\mathbb{Z}}$ is

$$\begin{aligned} &\mathbf{n}^*_{\rho_1}, \ldots, \mathbf{n}^*_{\rho_r},\ g^*_{r+1}, \ldots, g^*_k \\ &z_1 = u(\mathbf{n}^*_{\rho_1}), \ldots, z_r = u(\mathbf{n}^*_{\rho_r}), z_{r+1} = u(g^*_{r+1}), \ldots, z_k = u(g^*_k). \end{aligned} \tag{40}$$

We have

$$\mathbf{m} = \frac{\langle \mathbf{m},\ \mathbf{n}_{\rho_1}\rangle}{\langle \mathbf{n}^*_{\rho_1},\ \mathbf{n}_{\rho_1}\rangle}\mathbf{n}^*_{\rho_1} + \ldots + \frac{\langle \mathbf{m},\ \mathbf{n}_{\rho_r}\rangle}{\langle \mathbf{n}^*_{\rho_r},\ \mathbf{n}_{\rho_r}\rangle}\mathbf{n}^*_{\rho_r}.$$

Compute the function $U_\sigma \ni u \longmapsto u(\mathbf{m})$:

$$(41)\quad u(\mathbf{m}) = u\left(\frac{\langle \mathbf{m},\, \mathbf{n}_{\rho_1}\rangle}{\langle \mathbf{n}^*_{\rho_1},\, \mathbf{n}_{\rho_1}\rangle}\mathbf{n}^*_{\rho_1} + \ldots + \frac{\langle \mathbf{m},\, \mathbf{n}_{\rho_r}\rangle}{\langle \mathbf{n}^*_{\rho_r},\, \mathbf{n}_{\rho_r}\rangle}\mathbf{n}^*_{\rho_r}\right) = z_1^{\frac{\langle \mathbf{m},\, \mathbf{n}_{\rho_1}\rangle}{\langle \mathbf{n}^*_{\rho_1},\, \mathbf{n}_{\rho_1}\rangle}} \cdot \ldots \cdot z_r^{\frac{\langle \mathbf{m},\, \mathbf{n}_{\rho_r}\rangle}{\langle \mathbf{n}^*_{\rho_r},\, \mathbf{n}_{\rho_r}\rangle}}$$

If $\langle \mathbf{n}^*_{\rho_1},\, \mathbf{n}_{\rho_1}\rangle > 1$, then there is a generator, say g^*_{r+1} such that $\langle g^*_{r+1},\, \mathbf{n}_{\rho_1}\rangle = 1$. Otherwise the set (40) will not generate all $\check{\sigma}_{\mathbb{Z}}$. So we have

$$g^*_{r+1} = \frac{1}{\langle \mathbf{n}^*_{\rho_1},\, \mathbf{n}_{\rho_1}\rangle}\mathbf{n}^*_{\rho_1} + \frac{\langle g^*_{r+1},\, \mathbf{n}_{\rho_2}\rangle}{\langle \mathbf{n}^*_{\rho_2},\, \mathbf{n}_{\rho_2}\rangle}\mathbf{n}^*_{\rho_2} + \ldots + \frac{\langle g^*_{r+1},\, \mathbf{n}_{\rho_r}\rangle}{\langle \mathbf{n}^*_{\rho_r},\, \mathbf{n}_{\rho_r}\rangle}\mathbf{n}^*_{\rho_r}$$

and

$$z_{r+1} = z_1^{\frac{1}{\langle \mathbf{n}^*_{\rho_1},\, \mathbf{n}_{\rho_1}\rangle}} z_2^{\frac{\langle g^*_{r+1},\, \mathbf{n}_{\rho_2}\rangle}{\langle \mathbf{n}^*_{\rho_2},\, \mathbf{n}_{\rho_2}\rangle}} \cdot \ldots \cdot z_r^{\frac{\langle g^*_{r+1},\, \mathbf{n}_{\rho_r}\rangle}{\langle \mathbf{n}^*_{\rho_r},\, \mathbf{n}_{\rho_r}\rangle}}.$$

If $z_2 \neq 0, \ldots, z_r \neq 0$, then plugging the above formula into (41) we obtain

$$u(\mathbf{m}) = z_{r+1}^{\langle \mathbf{m}, \mathbf{n}_{\rho_1}\rangle} \cdot (\text{nonvanishing factor}).$$

Therefore the order of zero or infinity is $|\langle \mathbf{m},\, \mathbf{n}_{\rho_1}\rangle|$. The formula (37) follows immediately. □

Corollary 12.3. *If Δ is a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^r$, then for any $\mathbf{m} \in M_{\mathbb{Z}}$ we have*

$$(42)\qquad \operatorname{div}(\chi(\mathbf{m})) = \sum_{\rho \in \Delta(1)} \langle \mathbf{m},\, \mathbf{n}_\rho\rangle V(\rho).$$

Proof. For each cone $\sigma \in \Delta$ the equation (39) holds, which means that $\operatorname{div}(\chi(\mathbf{m}))|_{U_\sigma} = \sum_{\rho \in \sigma(1)} \langle \mathbf{m},\, \mathbf{n}_\rho\rangle V(\rho)$. Since on the intersection $U_\sigma \cap U_\tau$ the corresponding formulas are compatible, therefore (42) holds on the toric variety $X = T(\Delta)$. □

12.5. Bundle associated with Cartier divisor. From the general theory of divisors, it is well known how to associate a line bundle to a divisor. Let D be a Cartier divisor in X and let U be a sufficiently small neighborhood of a point in X. We can find a rational function $g : U \longrightarrow \mathbb{C}$ such that $\operatorname{div}(g)|_U = D|_U$. So we can find a locally finite covering $\{U_j\}_{j \in J}$ and rational functions $\{g_j\}_{j \in J}$ with the above property. We note that the quotient g_j/g_k is a non-vanishing holomorphic function on $U_j \cap U_k$ and consequently we can define a line bundle by gluing

$$(U_j \cap U_k) \times \mathbb{C} \ni (x, c_j) \longmapsto (x, \frac{g_k(x_k)}{g_j(x_j)} c_j) = (x, c_k) \in (U_k \cap U_j) \times \mathbb{C}.$$

A natural question appears, what is a relation between the bundle L_h which we defined previously and the divisor D_h. We have

Lemma 12.4. *The line bundle determined by the divisor D_h is the same as L_h.*

Proof. Actually it is enough to recall the definition of the bundle L_h and of the divisor D_h. The bundle L_h is defined by $\{U_\sigma, f_{\sigma\tau}\}$, where $f_{\sigma\tau}(u) = u(l_\sigma - l_\tau)$, $u \in U_\sigma \cap U_\tau$. The divisor D_h is defined by

$$D_h := - \sum_{\rho \in \Delta(1)} h(\mathbf{n}_\rho) V(\rho).$$

But

$$D_h|_{U_\sigma} = - \sum_{\rho \in \sigma(1)} \langle l_\sigma, \mathbf{n}_\rho \rangle V(\rho) = -\mathrm{div}(\chi(l_\sigma)).$$

On U_τ we have the same:

$$D_h|_{U_\tau} = - \sum_{\rho \in \tau(1)} \langle l_\tau, \mathbf{n}_\rho \rangle V(\rho) = -\mathrm{div}(\chi(l_\tau)).$$

Because these two divisors are the same, the quotient of $\chi(l_\sigma)$ and $\chi(l_\tau)$ is a nonzero holomorphic function and

$$\chi(l_\sigma)(u)/\chi(l_\tau)(u) = u(l_\sigma)/u(l_\tau) = u(l_\sigma - l_\tau),$$

which are the same as the transition functions for the line bundle L_h. Consequently, the divisor D_h determines the bundle L_h. □

Sections of the bundle. In the next section we will consider the space of analytic sections of the bundle L_h. If G is an open subset of a toric variety X, then we denote $\Gamma(G, L_h)$ the space of analytic sections of L_h over G. To reflect the property that complex analytic sections are taken, sometimes it is more convenient to consider the sheaf of germs of analytic sections $\mathscr{O}_X(D_h)$ and the space of sections $\Gamma(G, \mathscr{O}_X(D_h))$.

13. Projective toric varieties. Not all toric varieties can be embedded into complex projective spaces, however there is a nice characterization in terms of fans with support functions and corresponding polygons. In order to have an embedding, sufficiently many "independent" global holomorphic functions on X are needed. This, as we will see in a moment, is related to how "rich" is the space of sections of the sheaf $\mathscr{O}_X(D_h)$, i.e., $\Gamma(X, \mathscr{O}_X(D_h))$.

The minimal assumptions in this section are that Δ is a complete fan and $h \in \mathrm{SF}(\Delta)$ is upper convex.

13.1. How to define a mapping into the projective space? The following mapping will play an important role in embedding of a toric variety into a projective space:

If $P(\Delta, h)$ is the integral convex polytope in $M_{\mathbb{R}}$ determined by Δ and h (as in Section 10), and $\{\mathbf{m}_0, \dots, \mathbf{m}_k\}$ are the integral points of the polytope, then define

$$\psi_h : T(\Delta) \longrightarrow \mathbb{CP}^k, \quad \psi_h(u) := [u(\mathbf{m}_0) : u(\mathbf{m}_1) : \dots : u(\mathbf{m}_k)] \tag{43}$$

in homogeneous coordinates.

The mapping ψ_h needs some explanation. To understand it better, we recall Lemma 11.4 again.
If $\mathbf{m} \in M_{\mathbb{Z}}$ is such that

$$\langle \mathbf{m},\ \mathbf{n} \rangle \geqslant h(\mathbf{n}) \quad \text{for all} \quad \mathbf{n} \in |\Delta|, \tag{44}$$

then $\mathbf{m}$ determines a global section of the bundle L_h (or the sheaf $\mathscr{O}_X(D_h)$) given by the formula (34)

$$\begin{aligned} &\varphi_\sigma : U_\sigma \longrightarrow U_\sigma \times \mathbb{C}, && \varphi_\sigma = (\mathrm{id},\ \chi(\mathbf{m} - l_\sigma)), \\ &\varphi_\sigma(u) = (u,\ u(\mathbf{m} - l_\sigma)), && u(\mathbf{m} - l_\tau) = u(\mathbf{m} - l_\sigma) u(l_\sigma - l_\tau), \end{aligned} \qquad \sigma, \tau \in \Delta. \tag{45}$$

Recall the definition of the polytope

$$P(\Delta, h) = \{\mathbf{m} \in M_{\mathbb{R}} \mid \langle \mathbf{m},\ \mathbf{n} \rangle \geqslant h(\mathbf{n}),\ \ \forall\, \mathbf{n} \in N_{\mathbb{R}}\}.$$

We see that each $\mathbf{m} \in P(\Delta, h) \cap M_{\mathbb{Z}}$ satisfies (44), therefore determines a global section of the sheaf $\mathscr{O}_X(D_h)$. Now we list all integral points of $P(\Delta, h) \cap M_{\mathbb{Z}}$:

$$P(\Delta, h) \cap M_{\mathbb{Z}} = \{\mathbf{m}_0,\ \mathbf{m}_1, \dots, \mathbf{m}_k\}.$$

Since each $\mathbf{m}_\alpha$ determines a global section of $\mathscr{O}_X(D_h)$ as in (45) and taking only the second components of φ_σ, define the mapping

$$\Psi_{h\sigma} : U_\sigma \longrightarrow \mathbb{C}^{k+1}, \quad \Psi_{h\sigma}(u) = (u(\mathbf{m}_0 - l_\sigma), \ldots, u(\mathbf{m}_k - l_\sigma)), \quad u \in U_\sigma, \quad \sigma \in \Delta.$$

Using once again (45) for change between σ and τ, we get

$$\Psi_{h\tau}(u) = u(l_\tau - l_\sigma)\Psi_{h\sigma}(u), \qquad u \in U_\sigma \cap U_\tau,$$

which gives the global map, independent of σ, if we use homogeneous coordinates in $\mathbb{CP}^k$:

$$\begin{aligned} &\psi_h : T(\Delta) \longrightarrow \mathbb{CP}^k \\ &\psi_h(u) = [u(\mathbf{m}_0 - l_\sigma) : \ldots : u(\mathbf{m}_k - l_\sigma)], \quad \text{for} \quad u \in U_\sigma, \quad \sigma \in \Delta. \end{aligned} \tag{46}$$

If $u(l_\sigma) \neq 0$, the above mapping becomes

$$\psi_h(u) = [u(\mathbf{m}_0) : \ldots : u(\mathbf{m}_k)],$$

which is the same as (43). To see that not all components $u(\mathbf{m}_\alpha - l_\sigma)$ of (46) are zero, please note that if h is upper convex, then $\{l_\sigma\}$ are also vertices of $P(\Delta, h)$. By a change of indices, if necessary, we can assume that $\mathbf{m}_0 = l_\sigma$ for some σ, so we have $u(\mathbf{m}_0 - l_\sigma) = 1$. The mapping (46) becomes

$$\psi_h(u) = [1 : u(\mathbf{m}_0 - l_\sigma) : \ldots : u(\mathbf{m}_k - l_\sigma)], \quad \text{for} \quad u \in U_\sigma.$$

13.2. When the mapping is an embedding? To answer this question, we formulate a couple of theorems. Let Δ be a complete fan in $N_\mathbb{R} \simeq \mathbb{R}^r$, $h \in \mathrm{SF}(\Delta)$, and $X = T(\Delta)$ the toric variety.

Definition 13.1. *The sheaf $\mathscr{O}_X(D_h)$ is very ample if it is generated by global sections from $\Gamma(X, \mathscr{O}_X(D_h))$ and the mapping $\psi_h : X \longrightarrow \mathbb{CP}^k$, defined above, is a closed embedding. The sheaf $\mathscr{O}_X(D_h)$ is ample if $\mathscr{O}_X(\nu D_h)$ is very ample for large enough positive integers ν.*

There are two basic properties of the sheaf $\mathscr{O}_X(D_h)$, which we formulate as a theorem.

Theorem 13.2 (Theorem 2.13, [13])**.** *Let $X = T(\Delta)$ be an r-dimensional compact toric variety and let $h \in \mathrm{SF}(\Delta)$. Then, the sheaf $\mathscr{O}_X(D_h)$ is very ample if and only if one of the (equivalent) conditions is satisfied:*

(1) *h is strictly upper convex with respect to Δ and for each $\sigma \in \Delta(r)$, the subset $M_\mathbb{Z} \cap P(\Delta, h) - l_\sigma$ generates the semigroup $\check{\sigma}_\mathbb{Z}$.*

(2) *The integral convex polytope $P(\Delta, h)$ is r-dimensional and has exactly $\{l_\sigma \mid \sigma \in \Delta(r)\}$ as the set of its vertices. Moreover, $M_{\mathbb{Z}} \cap P(\Delta, h) - l_\sigma$ generates the semigroup $\check{\sigma}_{\mathbb{Z}}$.*

Example 13.3. Let consider the fan and the support function h as in Fig. 35. We have $h(0, 0) = 0$, on the first level polygon the value is -1, on the second level polygon the value is -2 and so on. The support function h is strictly upper convex. Also we have

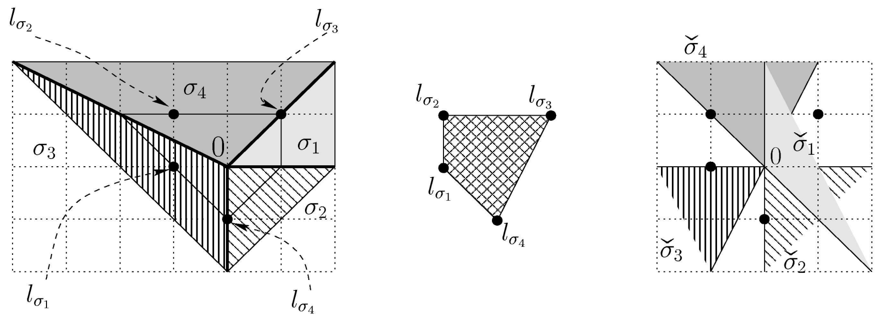


Fig. 35. Polygon and a support function

$$l_{\sigma_1} = [-1,\ 0], \quad l_{\sigma_2} = [-1,\ 1], \quad l_{\sigma_3} = [1,\ 1], \quad l_{\sigma_4} = [0,\ -1].$$

The polygon $l_{\sigma_1} l_{\sigma_2} l_{\sigma_3} l_{\sigma_4}$ contains six integral points

$$M_{\mathbb{Z}} \cap P(\Delta, h) = \{[0,\ 0],\ [-1,\ 0], [-1,\ 1],\ [0,\ 1],\ [1,\ 1],\ [0,\ -1]\}\,.$$

It is easy to check that for each α the set $M_{\mathbb{Z}} \cap P(\Delta, h) - l_{\sigma_\alpha}$ generates $\check{\sigma}_{\mathbb{Z}}$. The assumptions of the theorem are satisfied.

Corollary 13.4 (Corollary 2.14, [13])**.** *Let $X = T(\Delta)$ be an r-dimensional compact toric variety and let $h \in \mathrm{SF}(\Delta)$. Then, the sheaf $\mathscr{O}_X(D_h)$ is ample if and only if one of the (equivalent) conditions is satisfied:*

(1) *h is strictly upper convex with respect to Δ.*

(2) *The integral convex polytope $P(\Delta, h)$ is r-dimensional and has exactly $\{l_\sigma \mid \sigma \in \Delta(r)\}$ as the set of its vertices. Moreover, $l_\sigma \neq l_\tau$ holds for each pair $\sigma \neq \tau$ in $\Delta(r)$.*

Corollary 13.5 (Corollary 2.16, [13]). *A compact toric variety $T(\Delta)$ can be embedded equivariantly into a projective space as a closed subvariety if and only there exists $h \in SF(\Delta)$ which is strictly upper convex with respect to Δ.*

This nice corollary is one of the basic results in the theory of toric varieties. From the properties of the strictly upper convex functions we see that only toric varieties which are determined by integral convex polytopes are projective.

14. Homogeneous coordinates for toric varieties. In this section we will see that toric varieties can be defined in a similar way as projective spaces. We give a very rough sketch of this approach, mainly following [5]. For more detailed and precise description see [4] or [6].

It is well-known that

$$\mathbb{CP}^r \simeq (\mathbb{C}^{r+1} \setminus \{0\})/\mathbb{C}^*, \text{ where } (z_0, z_1, \ldots, z_r) \sim (\lambda z_0, \lambda z_1, \ldots, \lambda z_r), \ \lambda \in \mathbb{C}^*.$$

For toric varieties, we remove from $\mathbb{C}^k$ (for some k, which will be determined later) a set Z and the group action of G on $\mathbb{C}^k \backslash Z$ will be more complicated, but anyway we get

$$T(\Delta) = (\mathbb{C}^k \setminus Z)/G.$$

14.1. How to define *Z*? Take a fan Δ in $N_{\mathbb{R}} \simeq \mathbb{R}^r$. Let $\rho_1, \ldots, \rho_k$ be the 1-dimensional cones of Δ and let $\mathbf{n}_{\rho_j} \in N_{\mathbb{R}} \simeq \mathbb{Z}^r$ be the primitive element of ρ_j, i.e., the generator of $\rho_j \cap N_{\mathbb{Z}}$. Then we consider the space $\mathbb{C}^k$, assign the variable z_j to $\mathbf{n}_{\rho_j}$ and define Z as follows:

For each cone $\sigma \in \Delta$, we define the monomial

$$\mathbf{n}_{\rho_j} \rightsquigarrow z_j\,, \qquad z^\sigma = \prod_{\mathbf{n}_{\rho_j} \notin \sigma} z_j$$

which is the product of all variables not coming from the edges of σ. Then we put

$$Z = \mathbf{V}(z^\sigma \mid \sigma \in \Delta) = \{(z_1, \ldots, z_k) \mid z^\sigma = 0, \ \sigma \in \Delta\} \subset \mathbb{C}^k. \tag{47}$$

We note that Z is contained in the set where at least one coordinate is zero, that is, it is contained in $\{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_1 \cdot \ldots \cdot z_k = 0\}$. Looking at the definition of Z in (47), we see that if $\tau \prec \sigma$ then $\{z^\tau = 0\} \supset \{z^\sigma = 0\}$, therefore it is enough to take the maximal cones only (those cones not contained in any larger cone).

Another characterization of Z. There is another description of Z due to Batyrev which is useful in practice. We say that a set of edge generators $\{\mathbf{n}_{j_1}, \ldots, \mathbf{n}_{j_s}\}$ is *primitive* if they don't lie in any cone of Δ but every proper subset does. Then one can show that

$$Z = \bigcup_{\{\mathbf{n}_{j_1},\ldots,\mathbf{n}_{j_s}\} \text{ primitive}} \mathbf{V}(z_{j_1}, \ldots, z_{j_s}).$$

This shows that Z is the union of coordinate subspaces.

Example 14.1. Let $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^r$ and the fan Δ contain all faces of σ, including itself. Then there are no primitive sets of generators, therefore the set Z is empty.

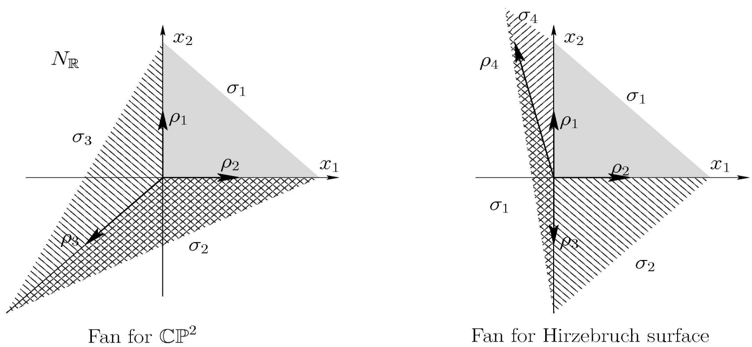


Fig. 36. Fans for $\mathbb{CP}^2$ and Hirzebruch surface

Example 14.2 (Projective space – set Z.)**.** For the two-dimensional projective space (see Fig. 36) we have

$$\begin{cases} \rho_1 \rightsquigarrow z_1 \\ \rho_2 \rightsquigarrow z_2 \\ \rho_3 \rightsquigarrow z_3 \end{cases} \Longrightarrow \begin{cases} z^{\sigma_1} = z_3 \\ z^{\sigma_2} = z_1 \\ z^{\sigma_3} = z_2 \end{cases} \Longrightarrow Z = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 = z_2 = z_3 = 0\}.$$

For the projective space $\mathbb{CP}^r$ it is easy to find the homogeneous coordinates. Let $\mathbf{n}_0 = -\mathbf{e_1} - \ldots - \mathbf{e_r}$, $\mathbf{n}_1 = \mathbf{e}_1, \ldots, \mathbf{n}_r = \mathbf{e}_r$, and the fan contains cones spanned by r vectors chosen from the above set of $r+1$ vectors. To calculate the set Z, we note that the only primitive set of vectors is $\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_r$, so we have

$$Z = \mathbf{V}(z_0, z_1, \ldots, z_r) = \{(0, \ldots, 0)\} \in \mathbb{C}^{r+1}.$$

Example 14.3 (Hirzebruch surfaces – set Z.)**.** For a Hirzebruch surface (see Fig. 36) we have

$$\begin{cases} \rho_1 \rightsquigarrow z_1 \\ \rho_2 \rightsquigarrow z_2 \\ \rho_3 \rightsquigarrow z_3 \\ \rho_4 \rightsquigarrow z_4 \end{cases} \Longrightarrow \begin{cases} z^{\sigma_1} = z_3 z_4 \\ z^{\sigma_2} = z_1 z_4 \\ z^{\sigma_3} = z_1 z_2 \\ z^{\sigma_4} = z_2 z_3 \end{cases} \Longrightarrow$$

$$\Longrightarrow Z = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z_3 z_4 = z_1 z_4 = z_1 z_2 = z_2 z_3 = 0\}.$$

14.2. How to define G? Let again Δ be a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^r$ and $\rho_1, \ldots, \rho_k$ be the 1-dimensional cones with primitive generators $\mathbf{n}_{\rho_1}, \ldots, \mathbf{n}_{\rho_k}$.

The group G can be defined in two equivalent ways

$$\begin{aligned} G &= \{(\mu_1, \ldots, \mu_k) \in (\mathbb{C}^*)^k \mid \prod_{j=1}^{k} \mu_j^{\langle \mathbf{m}, \mathbf{n}_{\rho_j} \rangle} = 1 \quad \text{for all} \quad \mathbf{m} \in \mathbb{Z}^r\} \\ &= \{(\mu_1, \ldots, \mu_k) \in (\mathbb{C}^*)^k \mid \prod_{j=1}^{k} \mu_j^{\langle \mathbf{e}_1, \mathbf{n}_{\rho_j} \rangle} = \ldots = \prod_{j=1}^{k} \mu_j^{\langle \mathbf{e}_r, \mathbf{n}_{\rho_j} \rangle} = 1\} \end{aligned}$$

The group G acts on $\mathbb{C}^k \setminus Z$ in the following way:

$$\mu = (\mu_1, \ldots, \mu_k) \in G, \ \ z = (z_1, \ldots, z_k) \in \mathbb{C}^k \setminus Z \implies \mu(z) = (\mu_1 z_1, \ldots, \mu_k z_k).$$

Therefore

$$(\mathbb{C}^k \setminus Z)/G \ : \qquad (z_1, \ldots, z_k) \sim (\mu_1 z_1, \ldots, \mu_k z_k).$$

It is possible to prove that the toric variety $T(\Delta)$ can be identified with $(\mathbb{C}^k \setminus Z)/G$. For details see [5] and [6].

Example 14.4 (Projective space – group G)**.** To define the group G, to each generator $\mathbf{n}_j$ we assign the variable μ_j, $j = 0, 1, \ldots, r$, and

$$\begin{aligned} G &= \{(\mu_0, \ldots, \mu_r) \in (\mathbb{C}^*)^{r+1} \mid \mu_0^{-1}\mu_1 = \mu_0^{-1}\mu_2 = \ldots = \mu_0^{-1}\mu_r = 1\} \\ &= \{(\mu, \mu, \ldots, \mu) \mid \mu \in \mathbb{C}\}. \end{aligned}$$

The group G acts on $\mathbb{C}^{r+1} \setminus \{0\}$ in the way

$$(\mu, \ldots, \mu)(z_0, z_1, \ldots, z_r) = (\mu z_0, \mu z_1, \ldots, \mu z_r),$$

that means that we identify points

$$(\mu z_0, \mu z_1, \dots, \mu z_r) \sim (z_0, z_1, \dots, z_r),$$

which gives precisely $\mathbb{CP}^r$.

Example 14.5 (Hirzebruch surfaces – group G)**.** In this example we find homogeneous coordinates for Hirzebruch surfaces.

We calculate the group G acting on $\mathbb{C}^4 \backslash Z$:

$$\begin{aligned} G &= \{(\mu_1, \mu_2, \mu_3, \mu_4) \in (\mathbb{C}^*)^4 \mid \mu_2 \mu_4^{-1} = \mu_1 \mu_3^{-1} \mu_4^a = 1\} \\ &= \{(\mu_1, \mu_2, \mu_3, \mu_4) \in (\mathbb{C}^*)^4 \mid \mu_2 = \mu_4,\ \mu_3 = \mu_1 \mu_4^a\} \\ &= \{(\mu_1, \mu_4, \mu_1 \mu_4^a, \mu_4) \in (\mathbb{C}^*)^4 \mid \mu_1,\ \mu_4 \in \mathbb{C}^*\} \end{aligned}$$

We identify two points from $\mathbb{C}^4 \backslash Z$ if and only if

$$(z_1, z_2, z_3, z_4) \sim (\mu z_1, \eta z_2, \mu \eta^a z_3, \eta z_4) \quad \text{for some} \quad \mu, \eta \in \mathbb{C}^*.$$

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