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ON NILPOTENT SUBSEMIGROUPS IN SOME MATRIX SEMIGROUPS

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Abstract. We describe maximal nilpotent subsemigroups of a given nilpotency class in the semigroup Ω_n of all $n \times n$ real matrices with non-negative coefficients and the semigroup $\mathbf{D}_n$ of all doubly stochastic real matrices.

1. Introduction. Let Ω_n denote the semigroup of all $n \times n$ real matrices with non-negative coefficients with respect to the usual matrix multiplication. This semigroup has two natural subsemigroups, the subsemigroup $\mathbf{P}_n$ of all *stochastic* matrices (that is all matrices from Ω_n such that the sum of the elements of each column is equal to 1), and the subsemigroup $\mathbf{D}_n$ of all *doubly stochastic* matrices (that is all matrices from $\mathbf{P}_n$ such that additionally the sum of the elements of each row is equal to 1). These semigroups are important and popular objects of study in many branches of modern mathematics. Surprisingly enough, the algebraic properties of these semigroups are studied rather superficially. One of the possible explanations for this fact is complexity of the algebraic structure of these semigroups.

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Among the classical results on the algebraic structure of these semigroups one should mention the descriptions of the maximal subgroups in the semigroup $\mathbf{P}_n$, see [20], and the descriptions of the maximal subgroups in the semigroup $\mathbf{D}_n$, see [1]. In particular, it was shown that every maximal subgroup in $\mathbf{P}_n$ is isomorphic to the symmetric group S_k for some $k \leq n$, and that every maximal subgroup in $\mathbf{D}_n$ is isomorphic to the direct product of symmetric groups. Later on there appeared alternative and easier proofs of these results, discovered by different mathematicians, see for example [21, 26, 27, 8, 24]. In [2] it was shown that each maximal subgroup of Ω_n is isomorphic to some full monomial group of degree $k \leq n$ over the group of positive real numbers (see also [8, 16, 24, 12]).

Apart from the study of maximal subgroups, a lot of attention was paid to the description of regular elements in these semigroups, see [26, 27, 24, 16, 13, 14, 17], and Green's relations. Green's relations were studied for regular elements in [19] and for arbitrary elements in [28, 29, 9, 25]. In the semigroup $\mathbf{D}_n$ Green's relations for regular elements were studied in [13, 26], and for arbitrary elements in [14]. The paper [24], apart from new results, contains several new proofs of already known facts about the regular elements, Green's relations and maximal subgroups of the semigroups Ω_n, $\mathbf{P}_n$ and $\mathbf{D}_n$.

Some algebraic properties of the set $E(\mathbf{D}_n)$ of idempotents of the semigroup $\mathbf{D}_n$ were studied in [22], and indecomposable elements of Ω_n and $\mathbf{D}_n$ were studied in [18, 15]. In [20, 11] the minimal ideals of $\mathbf{P}_n$ were studied. Maximal subsemigroups of Ω_n, $\mathbf{P}_n$ and $\mathbf{D}_n$ were studied in [8]. And, finally, in [5, 6] the present authors studied a homomorphism from the semigroup $\mathbf{D}_n$ to the semigroup B_n of all binary relations on an n-element set, the image of which has several interesting extremal properties.

In the present paper we describe maximal nilpotent subsemigroups of a given nilpotency class in the semigroups Ω_n (see Section 3) and $\mathbf{D}_n$ (see Section 4). As a preparatory work, we develop some general reduction technique for the study of maximal nilpotent subsemigroups via epimorphisms in Section 2.

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2. Epimorphisms of nilpotent semigroups. Let S be a semigroup with the zero element 0. A subsemigroup, $T \subset S$, is called *nilpotent* provided that there exists $k \in \mathbb{N}$ such that $T^k = 0$. The minimal k with this property is called the *nilpotency class* of T.

For a positive integer, m, we denote by $\mathrm{Nil}_m(S)$ the set of all nilpotent subsemigroups of S of nilpotency class at most m, and by $\mathrm{Nil}_m^{\max}(S)$ – the set of all maximal elements in $\mathrm{Nil}_m(S)$ with respect to inclusion. Abusing the language, the elements of $\mathrm{Nil}_m^{\max}(S)$ will be called *maximal nilpotent subsemigroups of nilpotency class m* of S. Set also

$$\mathrm{Nil}(S) = \cup_{m\in\mathbb{N}}\mathrm{Nil}_m(S)$$

and let $\mathrm{Nil}^{\max}(S)$ be the set of all maximal elements in $\mathrm{Nil}(S)$ with respect to inclusion. The elements of $\mathrm{Nil}^{\max}(S)$ are *maximal nilpotent subsemigroups* of S.

Lemma 2.1. *Let $\varphi : S \to T$ be a surjective homomorphism of semigroups with zero. Assume that T is a nilpotent semigroup of class n and $\varphi^{-1}(0)$ is a nilpotent semigroup of class m. Then S is a nilpotent semigroup of class at most mn.*

Proof. For arbitrary $a_1, \dots, a_{nm} \in S$ we have

$$\varphi(a_1 \cdots a_n) = \varphi(a_{n+1} \cdots a_{2n}) = \cdots = \varphi(a_{n(m-1)+1} \cdots a_{nm}) = 0$$

and hence $a_{ni+1} \cdots a_{n(i+1)} \in \varphi^{-1}(0)$ for every $i = 0, 1, \dots, m-1$. Therefore

$$(a_1 \cdots a_n) \cdot (a_{n+1} \cdots a_{2n}) \cdots (a_{n(m-1)+1} \cdots a_{nm}) = 0. \qquad \square$$

Theorem 2.2. *Let $\varphi : S \to T$ be a surjective homomorphism of semigroups with zero and assume $\mathrm{Nil}^{\max}(S) \neq \varnothing$. Then:*

(a) φ^{-1} induces a bijection between $\mathrm{Nil}^{\max}(T)$ and $\mathrm{Nil}^{\max}(S)$ if and only if $\varphi^{-1}(0)$ is a nilpotent subsemigroup of S.

(b) If $\varphi^{-1}(0) = 0$, then for any $U \in \mathrm{Nil}(S)$ the semigroups U and $\varphi(U)$ have the same nilpotency class; and for any $V \in \mathrm{Nil}(T)$ the semigroups V and $\varphi^{-1}(V)$ have the same nilpotency class. Moreover, for every $k \in \mathbb{N}$ the maps φ and φ^{-1} induce mutually inverse bijections between $\mathrm{Nil}_k^{\max}(S)$ and $\mathrm{Nil}_k^{\max}(T)$.

Proof. We start with (a). The necessity is obvious. To prove the sufficiency we observe that from Lemma 2.1 it follows that if $\varphi^{-1}(0)$ is nilpotent, then φ^{-1} induces a map from $\mathrm{Nil}(T)$ to $\mathrm{Nil}(S)$, which is obviously injective.

Let $A \in \mathrm{Nil}^{\max}(T)$. If $\varphi^{-1}(A) \notin \mathrm{Nil}^{\max}(S)$, then there exists $B \in \mathrm{Nil}(S)$ such that $\varphi^{-1}(A) \subsetneq B$. This implies $A \subsetneq \varphi(B)$. On the other hand, $\varphi(B)$ is obviously nilpotent since so is B. This contradicts the maximality of A. Hence $\varphi^{-1}(A) \in \mathrm{Nil}^{\max}(S)$.

Let now $B \in \mathrm{Nil}^{\max}(S)$. Then for every nilpotent subsemigroup A of T such that $\varphi(B) \subset A$ we have $B \subset \varphi^{-1}(A)$. Since $\varphi^{-1}(A)$ is nilpotent, from the maximality of B we get $B = \varphi^{-1}(A)$ and $\varphi(B) = A$. This means that $\varphi(B) \in \mathrm{Nil}^{\max}(T)$ and the map

$$\begin{array}{ccc} \mathrm{Nil}^{\max}(T) & \longrightarrow & \mathrm{Nil}^{\max}(S) \\ A & \mapsto & \varphi^{-1}(A) \end{array}$$

is surjective. This proves (a).

Since the homomorphic image of a nilpotent semigroup is a nilpotent semigroup, whose nilpotency class is not greater than the nilpotency class of the original semigroup, the first part of (b) follows from Lemma 2.1. Analogously to the proof of (a) one shows that φ induces the bijective map

$$\begin{array}{ccc} \mathrm{Nil}_k^{\max}(T) & \longrightarrow & \mathrm{Nil}_k^{\max}(S) \\ A & \mapsto & \varphi^{-1}(A), \end{array}$$

and the statement (b) follows. This completes the proof. $\square$

3. Maximal nilpotent subsemigroups in Ω_n. Consider the semigroup M_n^0 of all real $n \times n$ matrices, each row and each column of which contain at most one non-zero component. The semigroup M_n^0 is naturally identified with the Rees matrix semigroup $\mathrm{M}^0(\mathbb{R}^*; I, \Lambda; E)$, where $I = \Lambda = \{1, 2, \dots, n\}$ and the sandwich-matrix E is the identity matrix of order n. Since the identity matrix does not contain any zero rows or columns, the semigroup M_n^0 is a regular semigroup. Recall that Ω_n denotes the semigroup of all $n \times n$ matrices with non-negative real coefficients. Set $\tilde{\mathrm{M}}_n^0 = \mathrm{M}_n^0 \cap \Omega_n$. Then $\tilde{\mathrm{M}}_n^0$ can be identified with the Rees matrix semigroup $\mathrm{M}^0(\mathbb{R}^+; I, \Lambda; E)$, where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers.

Theorem 3.1. *Let S denote one of the semigroups M_n^0, $\tilde{\mathrm{M}}_n^0$, or Ω_n. Then*

(a) The semigroup S contains $n!$ maximal nilpotent subsemigroups, each of nilpotency class n. These subsemigroups are in a natural bijection with linear orders on the set $\{1, 2, \dots, n\}$.

(b) The maximal nilpotent subsemigroup of S, which corresponds to the linear order $i_1 \prec i_2 \prec \dots \prec i_n$, has the form

$$T = \{(a_{k,l}) \in S : a_{k,l} \neq 0 \Rightarrow k \prec l\}.$$

(c) *Let the maximal nilpotent subsemigroups T_1 and T_2 of S correspond to the linear orders $i_1 \prec_1 i_2 \prec_1 \cdots \prec_1 i_n$ and $j_1 \prec_2 j_2 \prec_2 \cdots \prec_2 j_n$ respectively. Then $T_2 = M^{-1}T_1M$, where $M = (m_{i,j})$ is the monomial matrix, which corresponds to the permutation*

$$\pi = \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$$

(that is M is a $(0,1)$-matrix such that $m_{i,j} = 1$ if and only if $j = \pi(i)$). In particular, all maximal nilpotent subsemigroups of S are isomorphic.

(d) *For every $k \in \mathbb{N}$, $k < n$, the semigroup S contains $\sum\limits_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n$ maximal nilpotent subsemigroups of nilpotency class k. These maximal nilpotent subsemigroups of nilpotency class k are in a natural bijection with decompositions of the set $\{1, 2, \dots, n\}$ into an ordered union of k pairwise disjoint and non-empty blocks.*

(e) *Let $N_1 \cup \cdots \cup N_k = \{1, \dots, n\}$ be a decomposition into an ordered union of pairwise disjoint non-empty blocks. Then the maximal nilpotent subsemigroup of nilpotency class k of S, which corresponds to this partition, is*

$$T = \{(a_{i,j}) \in S : a_{i,j} \neq 0 \text{ and } i \in N_p, j \in N_q \Rightarrow p < q\}.$$

P r o o f. Consider the map $\psi : (a_{i,j}) \mapsto (\hat{a}_{i,j})$ from the set $M_n(\mathbb{R})$ of all real $n \times n$ matrices to the set of all $(0,1)$-matrices of size $n \times n$, defined via

$$\hat{a}_{i,j} = \begin{cases} 1, & a_{i,j} \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

As $\mathrm{M}_n^0 = \mathrm{M}^0(\mathbb{R}^+; I, \Lambda; E)$, the restriction $\psi|_{\mathrm{M}_n^0}$ is a surjective homomorphism from M_n^0 to the multiplicative semigroup R_n of those $(0,1)$-matrices of size $n \times n$, each row and column of which contains at most one non-zero element. The semigroup R_n is usually called the *Rook monoid*, and it is canonically isomorphic to the symmetric inverse semigroup $\mathcal{IS}_n$ of all partial injections on the set $\{1, 2, \dots, n\}$, see for example [23]. Obviously, $\psi^{-1}(0) = 0$. Hence for the semigroup M_n^0 the statements (a)–(c) follow from Theorem 2.2 and the description of all maximal nilpotent subsemigroups in $\mathcal{IS}_n$, given in [3]. The statements (d)–(e) follow for M_n^0 from Theorem 2.2 and the description of all maximal nilpotent subsemigroups of nilpotency class k in $\mathcal{IS}_n$, given in [4].

For the semigroup $\tilde{\mathrm{M}}_n^0$ the proof is identical. Hence it remains to consider the case $S = \Omega_n$. If $M \in \Omega_n$, the matrix $\psi(M)$ can be considered as the matrix of some binary relation on $\{1, 2, \dots, n\}$. Then the restriction $\psi|_{\Omega_n}$ defines a surjective homomorphism from Ω_n on the semigroup B_n of all binary relations on $\{1, 2, \dots, n\}$. Moreover, $\psi^{-1}(0) = 0$. Therefore the statements (a)-(e) for the semigroup Ω_n follow from Theorem 2.2 and the description of all maximal nilpotent subsemigroups (of a given nilpotency class) in B_n, given in [7, Theorem 5.15 and Theorem 6.1]. □

Corollary 3.2. *Each nilpotent matrix from Ω_n contains at most $n(n-1)/2$ non-zero elements.*

Proof. This follows from Theorem 3.1(b). □

From Theorem 3.1 we obtain that the semigroup T of all upper triangular $n \times n$ matrices with non-negative real coefficients and zero diagonal is a maximal nilpotent subsemigroup of Ω_n, and all other maximal nilpotent subsemigroups are obtained from T via conjugation with monomial matrices. On the other hand, for every non-degenerate matrix $A \in \Omega_n$ the semigroup $A^{-1}TA$ is nilpotent. However, if A is not monomial, then for every monomial matrix B we have $A^{-1}TA \neq B^{-1}TB$ (this follows immediately from the obvious fact that $C^{-1}TC = T$ is possible only for a diagonal monomial matrix C). Hence $A^{-1}TA$ is not a maximal nilpotent subsemigroup of Ω_n, which means that $A^{-1}TA \not\subset \Omega_n$. By our choice of A we have that the matrix $A^{-1}MA$ can contain negative coefficients only if there are some negative coefficients in the matrix A^{-1}. This proves the following:

Corollary 3.3. *The group of invertible elements of Ω_n coincides with the complete monomial group of degree n over positive reals.*

We note that Corollary 3.3 can be also derived from the description of all maximal subgroups in Ω_n, see [2, Theorem 1], [16, Corollary 1], [24, Corollary 3.3].

4. Maximal nilpotent subsemigroups in $\mathbf{D}_n$. The structure of maximal nilpotent semigroups in the semigroup $\mathbf{D}_n$ of doubly stochastic matrices is more complicated. In [5] it is shown that the restriction of the map ψ from the proof of Theorem 3.1 to $\mathbf{D}_n$ is a surjective homomorphism on the factor power $\mathcal{FP}^+(\mathrm{S}_n)$ of the symmetric group S_n. However, the zero element of the semigroup $\mathcal{FP}^+(\mathrm{S}_n)$ (the latter being considered as a subsemigroup of B_n) is the full relation, and hence $\psi^{-1}(0)$ coincides with the semigroup of all doubly stochastic matrices, all coefficients of which are positive. However, this subsemigroup of $\mathbf{D}_n$ is not nilpotent. Indeed, the zero element of $\mathbf{D}_n$ is the matrix $\mathbb{O}_n$, all coefficients of which are equal to $1/n$. The matrix $A_t = t\mathbb{O}_n + (1-t)E$, where $0 < t < 1$ is doubly stochastic and has positive coefficients. However, A_t is not nilpotent since

$$A_t^k = (1-t)^k E + (1-(1-t)^k)\mathbb{O}_n \neq \mathbb{O}_n$$

for all $k \in \mathbb{N}$.

For a positive integer, k, we denote by $M_k(\mathbb{R})$ the semigroup of all real matrices of size $k \times k$. Set

$$\mathbf{Q}_n = \left\{ (a_{i,j}) \in M_n(\mathbb{R}) : \sum_{k=1}^{n} a_{k,i} = \sum_{k=1}^{n} a_{i,k} = 1 \text{ for all } i \right\}.$$

It is easy to check that $\mathbf{Q}_n$ is a subsemigroup of $M_n(\mathbb{R})$, moreover, that $\mathbb{O}_n$ is the zero element of $\mathbf{Q}_n$.

Lemma 4.1. *$A \in \mathbf{Q}_n$ if and only if the vector $\mathbf{v} = (1, 1, \ldots, 1)$ is an eigenvector for A with eigenvalue 1, and the subspace*

$$\mathbf{V} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$$

is invariant with respect to A.

Proof. The "only if" part is checked by a direct calculation. Let us prove the "if" part. The fact that $\mathbf{v} = (1, 1, \ldots, 1)$ is an eigenvector for A with eigenvalue 1 means that the sum of all elements in each row of A equals 1. Consider, for $i \neq j$, the vector $u_{i,j}$ defined as follows: the i-th and j-th coordinates of $u_{i,j}$ equal 1 and -1 respectively, and all other coordinates are zero. Since $u_{i,j} \in \mathbf{V}$ we have $Au_{i,j} \in \mathbf{V}$, which means that in the matrix A the sums of all elements in the i-th and in the j-th columns coincide. From the first part we have the the sum of all entries of A equals n. The lemma follows. $\square$

Proposition 4.2. $\mathbf{Q}_n \cong M_{n-1}(\mathbb{R})$.

Proof. Let $\mathbf{v}$ and $\mathbf{V}$ be as in Lemma 4.1. Let further $\mathbf{v}_2, \ldots, \mathbf{v}_n$ be an arbitrary basis of the vector space $\mathbf{V}$, and F be the transition matrix from the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$ to the basis $\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then, from Lemma 4.1 it follows that for every $A \in \mathbf{Q}_n$ we have

$$F^{-1}AF = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array} \right),$$

where $B \in M_{n-1}(\mathbb{R})$. It is straightforward that the map $A \mapsto B$ from $\mathbf{Q}_n$ to $M_{n-1}(\mathbb{R})$ is an isomorphism. $\square$

In [10, Section 7] it is shown that for an arbitrary field, $\mathbb{F}$, there exists a bijection between the maximal nilpotent subsemigroups of nilpotency class k in the multiplicative semigroup $M_n(\mathbb{F})$, and flags of length k in $\mathbb{F}^n$. Moreover, the maximal nilpotent subsemigroup T of nilpotency class k, which corresponds to the flag $0 = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_k = \mathbb{F}^n$, has the following form:

$$T = \{A \in M_n(\mathbb{F}) : AV_i \subset V_{i-1} \text{ for all } i\}.$$

In particular, it follow that the nilpotency class of each maximal nilpotent subsemigroup of $M_n(\mathbb{F})$ equals n, and that all such subsemigroups are isomorphic and correspond bijectively to complete flags in $\mathbb{F}^n$. Choose some basis $\mathbf{f}_1, \dots, \mathbf{f}_n$ in $\mathbb{F}^n$ such that $V_i = \langle \mathbf{f}_1, \dots, \mathbf{f}_i \rangle$ for each i. Let F be the transition matrix from the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $\mathbb{F}^n$ to the basis $\mathbf{f}_1, \dots, \mathbf{f}_n$. Then the maximal nilpotent subsemigroup T, which corresponds to the flag $0 = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_n = \mathbb{F}^n$, has the form $F\mathrm{T}_nF^{-1}$, where T_n is the semigroup of all upper triangular matrices from $M_n(\mathbb{F})$, having zero diagonal. This and Proposition 4.2 immediately implies:

Theorem 4.3. *Let* $\mathbf{v}$ *and* $\mathbf{V}$ *be as in Lemma* 4.1. *Then:*

(a) *There exists a bijection between maximal nilpotent subsemigroups of nilpotency class* k *in the semigroup* $\mathbf{Q}_n$ *and flags of length* k *in the space* $\mathbf{V}$.

(b) *If* $0 = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_k = \mathbf{V}$ *is a flag in* $\mathbf{V}$, *then the maximal nilpotent subsemigroup of nilpotency class* k, *which corresponds to this flag, has the form*

$$T = \{A \in \mathbf{Q}_n : AV_i \subset V_{i-1} \text{ for all } i\}.$$

(c) *The nilpotency class of each maximal nilpotent subsemigroup of* $\mathbf{Q}_n$ *equals* $n-1$, *all such semigroups are isomorphic and they correspond bijectively to complete flags in* $\mathbf{V}$.

(d) *Let* $0 = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_n = \mathbf{V}$ *be a complete flag in* $\mathbf{V}$ *and* $\mathbf{f}_2, \dots, \mathbf{f}_n$ *be a basis in* $\mathbf{V}$ *such that* $V_i = \langle \mathbf{f}_2, \dots, \mathbf{f}_{i+1} \rangle$ *for each* i. *Then the maximal nilpotent subsemigroup* T, *which corresponds to this flag, has the form*

$$T = F \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & T_{n-1} \end{array} \right) F^{-1},$$

where F *is the transition matrix from the standard basis* $\mathbf{e}_1, \dots, \mathbf{e}_n$ *of* $\mathbb{R}^n$ *to the basis* $\mathbf{v}, \mathbf{f}_2, \dots, \mathbf{f}_n$, *and* T_{n-1} *is the semigroup of all upper triangular matrices from* $M_{n-1}(\mathbb{R})$ *with zero diagonal.*

Question 4.4. *Does the analogue of the semigroup* $\mathbf{Q}_n$ *for fields of positive characteristic* p *have any interesting properties in the case* $p|n$*?*

Lemma 4.5. *Let the semigroup* S_2 *be such that every nilpotent subsemigroup of* S_2 *is contained in some maximal nilpotent subsemigroup of* S_2. *Let further* S_1 *be a subsemigroup of* S_2. *Then every maximal nilpotent subsemigroup* T_1 *of* S_1 *of nilpotency class* k *has the form* $T_1 = S_1 \cap T_2$, *where* T_2 *is some maximal nilpotent subsemigroup of* S_2 *of nilpotency class* k.

P r o o f. We have $T_1 = S_1 \cap T_2$, where T_2 is some maximal nilpotent subsemigroup of S_2 of nilpotency class k, containing T_1 (which exists because of our assumptions). $\square$

Lemma 4.6. *Let T be a maximal nilpotent subsemigroup of nilpotency class k in $\mathbf{Q}_n$, $A \in \mathbf{Q}_n$, and $\alpha \neq 0$. Then $A \in T$ if and only if $\alpha A + (1-\alpha)\mathbb{O}_n \in T$.*

P r o o f. Let T' be a maximal nilpotent subsemigroup of nilpotency class k in $M_{n-1}(\mathbb{R})$. From the result of [10, Section 7] mentioned above it follows that for any $B \in M_{n-1}(\mathbb{R})$ and $\alpha \neq 0$ we have that $B \in T'$ if and only if $\alpha B \in T'$. Since

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \alpha B \end{array}\right) = \alpha \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right) + (1-\alpha) \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \alpha 0 \end{array}\right),$$

the necessary statement follows from Theorem 4.3 and the isomorphism $\mathbf{Q}_n \cong M_{n-1}(\mathbb{R})$ from the proof of Proposition 4.2. $\square$

Theorem 4.7. *Let $k \in \mathbb{N}$. Then the map $\tau : T \mapsto T \cap \mathbf{D}_n$ defines a bijection between the maximal nilpotent subsemigroups of nilpotency class k in $\mathbf{Q}_n$ and the maximal nilpotent subsemigroups of nilpotency class k in $\mathbf{D}_n$. In particular, every nilpotent subsemigroup of $\mathbf{D}_n$ is contained in some maximal nilpotent subsemigroup, and every maximal nilpotent subsemigroup of $\mathbf{D}_n$ has nilpotency class $n-1$.*

P r o o f. Taking Lemma 4.5 into account, we have just to prove that τ is injective and preserves the nilpotency class. Let T_1 and T_2 be two different maximal nilpotent subsemigroups of nilpotency class k from $\mathbf{Q}_n$ and let $A = (a_{i,j}) \in T_2 \setminus T_1$. Then $\alpha A + (1-\alpha)\mathbb{O}_n \in T_2 \setminus T_1$ for any $\alpha \neq 0$ by Lemma 4.6. However, if $\alpha \in \big(0, \min\{(2n|a_{i,j}|)^{-1} : a_{i,j} \neq 0\}\big)$, then we have $\alpha A + (1-\alpha)\mathbb{O}_n \in \mathbf{D}_n$. Hence $(T_2 \cap \mathbf{D}_n) \setminus (T_1 \cap \mathbf{D}_n) \neq \varnothing$. The injectivity of τ follows.

Let now T be a maximal nilpotent subsemigroup of nilpotency class k in $\mathbf{Q}_n$, and $A_1, \dots, A_m \in T$ be such that $A_1 \cdots A_m \neq \mathbb{O}_n$. Then for any non-zero $\alpha_1, \dots, \alpha_m$ there exists $\beta \in \mathbb{R}$ such that

$$(\alpha_1 A_1 + (1-\alpha_1)\mathbb{O}_n) \cdots (\alpha_m A_m + (1-\alpha_m)\mathbb{O}_n) = \alpha_1 \cdots \alpha_m A_1 \cdots A_m + \beta \mathbb{O}_n \neq \mathbb{O}_n.$$

Analogously to the previous arguments, all α_i can be chosen such that all corresponding $\alpha_i A_i + (1-\alpha_i)\mathbb{O}_n$ belong to $\mathbf{D}_n$. Hence the nilpotency classes of T and $T \cap \mathbf{D}_n$ coincide. This completes the proof. $\square$

Corollary 4.8. *There is a bijection between the maximal nilpotent subsemigroups of a given nilpotency class, k, in $\mathbf{D}_n$ and the flags of length k in the $(n-1)$-dimensional real vector space $\mathbf{V}$. In particular, the maximal nilpotent* *subsemigroups in* $\mathbf{D}_n$ *correspond to complete flags* $0 = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_{n-1} = \mathbf{V}$.

P r o o f. This follows from Theorem 4.3 and Theorem 4.7. □

The elements of the semigroup T_{n-1} of all upper triangular matrices with the zero diagonal can be naturally identified with the elements of the vector space $\mathbb{R}^{n(n-1)/2}$. The condition that some matrix from the nilpotent subsemigroup $F\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & T_{n-1}\end{array}\right)F^{-1}$ belongs to $\mathbf{D}_n$ is equivalent to the condition of non-negativity of the elements of the matrix. If the transition matrix F is fixed, our condition reduces to a system of linear inequalities for the coefficients of matrices in T_{n-1}. This means that every maximal nilpotent subsemigroup T of $\mathbf{D}_n$ corresponds to some convex polyhedron $\mathrm{P}(T)$ from $\mathbb{R}^{n(n-1)/2}$.

Proposition 4.9. *The polyhedron* $\mathrm{P}(T)$ *is bounded (compact) for every maximal nilpotent subsemigroup* T *of* $\mathbf{D}_n$.

P r o o f. It is obvious that the point $O = (0, \ldots, 0)$, which corresponds to the zero element $\mathbb{O}_n$ of the semigroup T is an inner point of $\mathrm{P}(T)$. Hence it is enough to show that the intersection of every straight line from $\mathbb{R}^{n(n-1)/2}$, which contains O, with $\mathrm{P}(T)$ is a bounded segment.

Let $A \in T \setminus \{\mathbb{O}_n\}$ and $M \in \mathrm{P}(T)$ be the corresponding point of $\mathrm{P}(T)$. From the proof of Lemma 4.6 it follows that the elements of the straight line $OM = \{\alpha M : \alpha \in \mathbb{R}\}$ correspond to the elements of the subset $\tilde{M} = \{\alpha A + (1-\alpha)\mathbb{O}_n : \alpha \in \mathbb{R}\}$ of $\mathbf{Q}_n$. Consider the intersection $\tilde{M} \cap \mathbf{D}_n$. Since $A \neq \mathbb{O}_n$, there exists coefficients a' and a'' of A such that $a' > 1/n$ and $a'' < 1/n$. From the inequalities

$$\alpha a' + (1-\alpha)1/n \geq 0 \quad \text{and} \quad \alpha a'' + (1-\alpha)1/n \geq 0$$

we obtain $(1-na')^{-1} \leq \alpha \leq (1-na'')^{-1}$. Hence $\tilde{M} \cap \mathbf{D}_n$ is a bounded segment and therefore $OM \cap \mathrm{P}(T)$ is a bounded segment as well. □

We remark that the geometric structure of $\mathrm{P}(T)$ heavily depends on the choice of T:

Example 4.10. Let $n = 4$. We identify the matrix $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ from T_3 with the point $(a, b, c) \in \mathbb{R}^3$. Then for the transition matrix

$$F' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

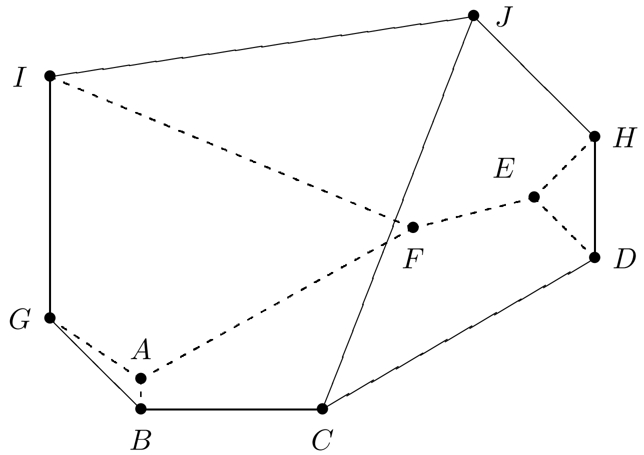

Fig. 1. Polyhedron $ABCDEFGHIJ$

the corresponding maximal nilpotent subsemigroup $T_{F'}$ of $\mathbf{D}_4$ is given by the following system of linear inequalities:

$$\begin{cases} 1+a+b+c \geq 0, \\ 1-3a+b+c \geq 0, \\ 1+a-3b-3c \geq 0, \\ 1-a+3b \geq 0, \\ 1+3a-b \geq 0, \\ 1-a-b \geq 0, \\ 1+3c \geq 0, \\ 1-c \geq 0. \end{cases}$$

The set of solutions to this system is the convex polyhedron with vertexes $A = (-5/12, -1/4, -1/3)$, $B = (-1/4, -5/12, -1/3)$, $C = (1/8, -7/24, -1/3)$, $D = (5/12, 7/12, -1/3)$, $E = (1/4, 3/4, -1/3)$, $F = (-1/8, 5/8, -1/3)$, $G = (-1/2, -1/2, 0)$, $H = (1/2, 1/2, 0)$, $I = (-1/2, -1/2, 2/3)$, and $J = (1/2, -1/6, 2/3)$. This polyhedron has seven faces, one of which is a hexagon, two are pentagons, two are quadrangles and the remaining two are triangles (see Figure 1).

At the same time for the transition matrix $F'' = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & -3 & -1 \end{pmatrix}$ the corresponding maximal nilpotent subsemigroup $T_{F''}$ of $\mathbf{D}_4$ is given by the following system of linear inequalities:

$$\begin{cases} 1-a+4b+4c \geq 0, \\ 1+3a-4b-4c \geq 0, \\ 1+a-4b-12c \geq 0, \\ 1-3a+4b+12c \geq 0, \\ 1+4c \geq 0, \\ 1-4c \geq 0, \\ 1-a \geq 0, \\ 1+a \geq 0. \end{cases}$$

The set of solutions to this system is the tetrahedron with the following vertexes: $X = (1, 5/4, -1/4)$, $Y = (1, -1/4, 1/4)$, $Z = (-1, -1/4, -1/4)$, and $W = (-1, -3/4, 1/4)$.

Question 4.11. *Is there any connection between the algebraic properties of a maximal nilpotent subsemigroup, $T \subset \mathbf{D}_n$, and the geometric properties of the polyhedron* $\mathrm{P}(T)$*?*

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