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ON THOM POLYNOMIALS FOR $A_4(-)$ VIA SCHUR FUNCTIONS

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Abstract. We study the structure of the Thom polynomials for $A_4(-)$ singularities. We analyze the Schur function expansions of these polynomials. We show that partitions indexing the Schur function expansions of Thom polynomials for $A_4(-)$ singularities have at most four parts. We simplify the system of equations that determines these polynomials and give a recursive description of Thom polynomials for $A_4(-)$ singularities. We also give Thom polynomials for $A_4(3)$ and $A_4(4)$ singularities.

1. Introduction. Thom polynomials, express invariants of singularities of a general map $f : X \to Y$ between complex analytic manifolds in terms of invariants of X and Y. Knowing the Thom polynomial of a singularity η one can compute the cohomology classes represented by η-points of f. The existence of these polynomials are guaranteed by an early theorem of Thom (cf. [22]).

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Different methods, such as desingularization, have been developed to compute these polynomials. A survey of these methods can be found in [7]. Although these methods gave formulas of Thom polynomials for $A_1(-)$, $A_2(-)$ and Σ^i-singularities they became very difficult for more complicated singularities. One can see some of the difficulties of these methods in [6], where Gaffney computed the Thom polynomial for $A_4(1)$ singularity.

Recently a new method, the "method of restriction equations" (developed mainly by Rimanyi) converted the problem into an algebraic one. When r is fixed and small using this method to compute the Thom polynomial for a singularity $\eta(r)$ is easier than previous methods (Compare [6] with [19], see also [21]). However, if we want to find the Thom polynomials for a series of singularities, containing r as a parameter, then we have to solve *simultaneously* a countable family of systems of linear equations (these formulas were asked in [20] and [2]).

In [15], [16] and [17] Pragacz (in collaboration with Lascoux) combined this method with the techniques of *Schur functions* and obtained many new results including more transparent proofs of formulas of Thom, Porteous and Ronga; formulas for the Thom polynomials for the singularities $I_{2,2}(-)$ and $A_3(-)$ (for all r, as desired) and a strategy for computing Thom polynomials for $A_i(-)$ singularities.

In [18], Pragacz and Weber proved that the coefficients of Schur function expansions of the Thom polynomials of stable singularities are nonnegative[1].

In this paper we study the structure of the Thom polynomials for $A_4(-)$ singularities using this strategy and other methods of [15], [16] and [17]. We prove that partitions indexing the Schur function expansions of Thom polynomials for $A_4(-)$ singularities have at most four parts. We simplify the system of equations that determines these polynomials. We give a recursive description of Thom polynomials for $A_4(-)$ singularities. We also give Thom polynomials for $A_4(3)$ and $A_4(4)$ singularities (note that the Thom polynomial for $A_4(2)$ singularity was computed in [20]).

In the next two sections we will collect the necessary information on Schur functions and Thom polynomials.

2. Schur functions. In this section we aim to give a quick introduction to Schur functions and introduce their notation. We use the approach and notation of Lascoux's book [8] (also of [9] and [15]) and refer to this book or to [10] for a more detailed study.

[1]This result was conjectured in [2] and [15]

An *alphabet* is a multi-set[2] of elements from a commutative ring.

Definition 1. *For alphabets $\mathbb{A}$ and $\mathbb{B}$, the ith complete function $S_i(\mathbb{A}-\mathbb{B})$ is defined as the coefficient of z^i in the generating series*

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i = \frac{\prod_{b\in\mathbb{B}}(1-bz)}{\prod_{a\in\mathbb{A}}(1-az)}. \tag{1}$$

Note that if $\mathbb{B} = \{0\}$ then $S_i(\mathbb{A}-\mathbb{B})$ gives the complete homogeneous symmetric function of degree i in $\mathbb{A}$. Similarly when $\mathbb{A} = \{0\}$ we see that $(-1)^i S_i(\mathbb{A}-\mathbb{B})$ is the ith elementary function in $\mathbb{B}$. Disjoint union of two alphabets $\mathbb{A}$ and $\mathbb{A}'$ is denoted by $\mathbb{A}+\mathbb{A}'$ so that we can write

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i \cdot \sum S_j(\mathbb{A}'-\mathbb{B}')z^j = \sum S_i\big((\mathbb{A}+\mathbb{A}')-(\mathbb{B}+\mathbb{B}')\big)z^i\,.$$

By setting $\mathbb{A}'=\mathbb{B}'=\mathbb{C}$ and simplifying the factors $\prod_{c\in\mathbb{C}}(1-cz)$ on the RHS we obtain

$$\sum S_i((\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}))z^i = \sum S_i(\mathbb{A}-\mathbb{B})z^i\,.$$

This enables us to write

$$(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}) = \mathbb{A}-\mathbb{B}. \tag{2}$$

The finite alphabet $\mathbb{A}_m = (a_1, a_2, \ldots, a_m)$, being the disjoint union of its subsets of cardinality one, will be written as $a_1+\cdots+a_m$. Similarly we write

$$a_1+\cdots+a_m-b_1-\cdots-b_n$$

to denote $\mathbb{A}_m-\mathbb{B}_n$, the difference of (finite) alphabets $\mathbb{A}_m$ and $\mathbb{B}_n$. We also define the product of alphabets $\mathbb{A}\cdot\mathbb{B}$ and multiplication by a constant α in the usual way. However there is one point that we should clarify: For a constant α we have

$$S_i(\alpha) = \binom{\alpha+i-1}{i},$$

whereas

$$S_i(u) = u^i$$

[2]We allow the elements to be repeated.

when u is a monomial. For example $S_i(2) = \binom{i+1}{2}$ and $S_i(a) = a^i$ for a letter a. So to emphasize the difference and to avoid extra variables we write $S_i\left(\boxed{2}\right)$ to denote the result of $S_i(a)$ specialized at $a = 2$. Similarly

$$S_2(\mathbb{X}_2) = x_1^2 + x_1x_2 + x_2^2 \neq x_1^2 + 2x_1x_2 + x_2^2 = S_2\left(\boxed{x_1 + x_2}\right).$$

That is, we write $\boxed{r}$ and take it as a single variable when we want to specialize a letter to an expression r (e.g. to a number or to a sum).

By a *partition* $I = (i_1, \ldots, i_l)$ we mean a weakly increasing sequence $0 \leq i_1 \leq i_2 \leq \ldots \leq i_l$ of natural numbers. We write $\ell(I)$ for the number of non-zero parts of I, and $|I|$ for the sum $i_1 + \cdots + i_l$. Often, partitions are represented by their *Young diagrams*; left aligned $\ell(I)$ rows of boxes, where jth row consists of i_j boxes.

We shall use the the simplified notation $i_1i_2\cdots i_l$ or $i_1, i_2, \ldots, i_l$ for a partition $(i_1, i_2, \ldots, i_l)$ (the latter one if $i_l \geq 10$).

Definition 2. *Given a partition $I = (i_1, i_2, \ldots, i_l)$, and alphabets $\mathbb{A}$ and $\mathbb{B}$, the Schur function*[3] $S_I(\mathbb{A} - \mathbb{B})$ *is*

$$S_I(\mathbb{A} - \mathbb{B}) := \Big|S_{i_q+q-p}(\mathbb{A} - \mathbb{B})\Big|_{1\leq p,q\leq \ell(I)}. \tag{3}$$

For example, if $I = (1, 3, 3, 4, 5)$ then

$$S_I(\mathbb{A} - \mathbb{B}) = \begin{vmatrix} S_1(\mathbb{A} - \mathbb{B}) & S_4(\mathbb{A} - \mathbb{B}) & S_5(\mathbb{A} - \mathbb{B}) & S_7(\mathbb{A} - \mathbb{B}) & S_9(\mathbb{A} - \mathbb{B}) \\ 1 & S_3(\mathbb{A} - \mathbb{B}) & S_4(\mathbb{A} - \mathbb{B}) & S_6(\mathbb{A} - \mathbb{B}) & S_8(\mathbb{A} - \mathbb{B}) \\ 0 & S_2(\mathbb{A} - \mathbb{B}) & S_3(\mathbb{A} - \mathbb{B}) & S_5(\mathbb{A} - \mathbb{B}) & S_7(\mathbb{A} - \mathbb{B}) \\ 0 & S_1(\mathbb{A} - \mathbb{B}) & S_2(\mathbb{A} - \mathbb{B}) & S_4(\mathbb{A} - \mathbb{B}) & S_6(\mathbb{A} - \mathbb{B}) \\ 0 & 1 & S_1(\mathbb{A} - \mathbb{B}) & S_3(\mathbb{A} - \mathbb{B}) & S_5(\mathbb{A} - \mathbb{B}) \end{vmatrix}.$$

Note that by Equation (2), we have a *cancellation property* for Schur functions:

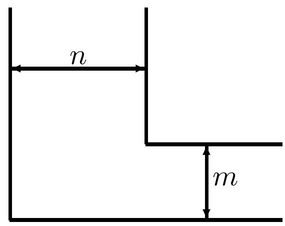
$$S_I((\mathbb{A} + \mathbb{C}) - (\mathbb{B} + \mathbb{C})) = S_I(\mathbb{A} - \mathbb{B}). \tag{4}$$

Definition 3. *Given two alphabets $\mathbb{A}, \mathbb{B}$, we define their resultant:*

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a - b). \tag{5}$$

[3]These functions are also called *supersymmetric Schur functions* or *Schur functions in difference of alphabets.*

Fix two positive integers m and n. We shall say that a partition I *is not contained in* the (m, n)-hook if the Young diagram of I is not contained in the following "tickened" hook:



That is, I *is not contained in* the (m, n)-hook if $\ell(I) > m$ and $i_{\ell(I)-m} > n$. Now consider the alphabets A_m and B_n. We have the following *vanishing property*: If a partition I is not contained in the (m, n)-hook then

$$S_I(\mathbb{A}_m - \mathbb{B}_n) = 0. \tag{6}$$

If on the other hand a partition is contained in the (m, n)-hook and contains the rectangular partition (n^m) then it is of the form

$$(J, I + (n^m)) := (j_1, \dots, j_h, i_1 + n, \dots, i_m + n)$$

for some partitions $I = (i_1, \dots, i_m)$ and $J = (j_1, \dots, j_h)$. Moreover, we have the following *factorization property*[4]:

$$S_{J,I+(n^m)}(\mathbb{A}_m - \mathbb{B}_n) = S_I(\mathbb{A}_m)\ R(\mathbb{A}_m, \mathbb{B}_n)\ S_J(-\mathbb{B}_n)\,. \tag{7}$$

Let $\mathbb{A}$ be an alphabet of cardinality m. Consider the function

$$F(\mathbb{A}, \bullet) := \sum_I S_I(\mathbb{A}) S_{n-i_m,\dots,n-i_1,n+|I|}(\bullet), \tag{8}$$

where the sum is over partitions $I = (i_1, i_2, \dots, i_m)$ such that $i_m \le n$. This function was introduced in [15], and it will be fundamental in the study of Thom polynomials for A_i singularities. The following properties of F are collected from [15] and [17].

[4]See [8] or [3] for a proof; the Sergeev-Pragacz formula of [13] (see also [14]) gives a symmetrization generalizing this factorization.

Lemma 4. *For a variable x and an alphabet $\mathbb{B}$ of cardinality n,*

$$F(\mathbb{A}, x-\mathbb{B}) = R(x+\mathbb{A}x, \mathbb{B})\,. \tag{9}$$

Setting $\mathbb{A} = \boxed{2} + \boxed{3} + \cdots + \boxed{i}$, $m = i-1$, and $n = r$, we obtain special cases $F_r^{(i)}$ of the function F:

$$F_r^{(i)}(\bullet) := \sum_J S_J\left(\boxed{2} + \boxed{3} + \cdots + \boxed{i}\right) S_{r-j_{i-1},\ldots,r-j_1,r+|J|}(\bullet), \tag{10}$$

where the sum is over partitions $J \subset (r^{i-1})$, and for $i = 1$ we understand $F_r^{(1)}(\bullet) = S_r(\bullet)$. In particular,

$$F_r^{(4)}(\bullet) = \sum_{j_1\le j_2\le j_3\le r} S_{j_1,j_2,j_3}\left(\boxed{2} + \boxed{3} + \boxed{4}\right) S_{r-j_3,r-j_2,r-j_1,r+j_1+j_2+j_3}(\bullet). \tag{11}$$

For example,

$$F_1^{(4)} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4. \tag{12}$$

Using Lemma 4 we obtain the most important algebraic property of $F_r^{(i)}$:

Proposition 5. *We have*

$$F_r^{(i)}(x-\mathbb{B}_r) = R\left(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}, \mathbb{B}_r\right)\,. \tag{13}$$

We close this section with the following corollary.

Corollary 6. *Fix an integer $i \ge 1$.*
(i) For $p \le i$, we have

$$F_r^{(i)}\left(x - \mathbb{B}_{r-1} - \boxed{px}\right) = 0. \tag{14}$$

(ii) Moreover, we have

$$\begin{aligned} &F_r^{(i)}\left(x - \mathbb{B}_{r-1} - \boxed{(i+1)x}\right) \\ &\qquad = R\left(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}, \mathbb{B}_{r-1} + \boxed{(i+1)x}\right)\,. \end{aligned} \tag{15}$$

3. Thom polynomials. In this section we outline our approach of computing Thom polynomials. We shall use a combination of the "method of restriction equations" (developed mainly by Rimanyi, cf. [20]) and the techniques of using Schur functions in this method (developed in [15], [16] and [17] by Pragacz). First we recall the necessary information about singularities, Thom polynomials and the "method of restriction equations".

Let $k \geq 0$ be a fixed integer and $\bullet \in \mathbf{N}$. Two stable germs $\kappa_1, \kappa_2 : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ are said to be right-left equivalent if there exist germs of biholomorphisms ϕ of $(\mathbf{C}^\bullet, 0)$ and ψ of $(\mathbf{C}^{\bullet+k}, 0)$ such that $\psi \circ \kappa_1 \circ \phi^{-1} = \kappa_2$. A suspension of a germ is its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$. Consider the equivalence relation (on stable germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$) generated by right-left equivalence and suspension. A *singularity* η is an equivalence class of this relation.

According to Mather's classification (cf. [4] or [1]), singularities are in one-to-one correspondence with finite dimensional (local) $\mathbf{C}$-algebras. We shall use the following notation:

– A_i (of Thom-Boardman type Σ^{1_i}) will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$;

– $I_{2,2}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy, x^2+y^2)$;

– $III_{a,b}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy, x^a, y^b)$, $b \geq a \geq 2$ (here $k \geq 1$).

Let $f : X \to Y$ be a general map between complex analytic manifolds and η be a singularity. Consider the closure of the set of η-points of f:

$$V^\eta(f) := \overline{\{x \in X : \text{the singularity of } f \text{ at } x \text{ is } \eta\}}.$$

By an early result of Thom there exists a universal polynomial $\mathcal{T}^\eta$, called the *Thom polynomial of* η, such that $\mathcal{T}^\eta(c_1, c_2, \ldots)$ gives the Poincaré dual of $V^\eta(f)$, after the substitution of the Chern classes c_i of the virtual bundle $f^*TY - TX$. That is, knowing the Thom polynomial of a singularity, we are able to express invariants of singularities of the map $f : X \to Y$ in terms of invariants of X and invariants of Y.

Let $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$ be a prototype of a stable singularity $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$. Although the *right-left symmetry group*

$$\operatorname{Aut}\kappa = \{(\phi, \psi) \in \operatorname{Diff}(\mathbf{C}^n, 0) \times \operatorname{Diff}(\mathbf{C}^{n+k}, 0) : \psi \circ \kappa \circ \phi^{-1} = \kappa\} \tag{16}$$

is much too large to be a finite dimensional Lie group, it is possible to define its maximal compact subgroup (up to conjugacy) in a sensible way (cf. [20]). Let G_η denote the *maximal compact subgroup* of $\operatorname{Aut}\kappa$. Note that if necessary we can replace G_η with one of its conjugates so that images of its projections to the factors $\operatorname{Diff}(\mathbf{C}^n, 0)$ and $\operatorname{Diff}(\mathbf{C}^{n+k}, 0)$ are linear. Let $\lambda_1(\eta)$ and $\lambda_2(\eta)$ denote the representations via the (linear) projections on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+k}$. Using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ we obtain vector bundles E'_η and E_η associated with the universal principal G_η-bundle $EG_\eta \to BG_\eta$. The *total Chern class of the singularity* η is defined in $H^\bullet(BG_\eta; \mathbf{Z})$ by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)}. \tag{17}$$

The *Euler class* of η is defined in $H^{2\operatorname{codim}(\eta)}(BG_\eta; \mathbf{Z})$ by

$$e(\eta) := e(E'_\eta), \tag{18}$$

where the codimension of a singularity η means the codimension of $V^\eta(f)$ in X.

Now we are ready to state the theorem of Rimanyi which explains the name "method of restriction equations".

Theorem 7. *Suppose, for a singularity η, that the Euler classes of all singularities of smaller codimension than* $\operatorname{codim}(\eta)$*, are not zero-divisors*[5]*. Then we have*
(i) if $\xi \neq \eta$ and $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$*, then $\mathcal{T}^\eta(c(\xi)) = 0$;*
(ii) $\mathcal{T}^\eta(c(\eta)) = e(\eta)$.
This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

Sometimes it is possible to use a subgroup of G_η instead of G_η itself. We can define these characteristic classes for a subgroup by using restrictions of the above representations and then use Theorem 7 with these characteristic classes (we take the homomorphic images of the equations). The equations will be still valid for any subgroup of G_η. But if the subgroup is small they may not contain necessary information to determine the Thom polynomial $\mathcal{T}^\eta$. For this reason, we should chose a subgroup as close to G_η as possible. For example, in case of $\eta = I_{2,2}$ we have $G_\eta = H \times U(k)$ where H is the extension of $U(1) \times U(1)$ by $\mathbf{Z}/2\mathbf{Z}$. To simplify computations we use the subgroup $U(1) \times U(1) \times U(k)$ instead G_η itself.

[5] This condition holds true for the singularities $A_4(-)$.

To determine the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ we follow Rimanyi (cf. [20], Theorem 4.1). Consider the algebraic automorphism group $\mathrm{Aut}(Q_\eta)$ of the local algebra Q_η. The group G_η is a subgroup of the maximal compact subgroup of $\mathrm{Aut}(Q_\eta)$ times the unitary group $U(k-d)$, where d is the deffect[6] of Q_η. The germ κ is the miniversal unfolding of another germ $\beta : (\mathbf{C}^m, 0) \to (\mathbf{C}^{m+k}, 0)$ with $d\beta = 0$. With β well chosen, G_η acts as right-left symmetry group on β with representations μ_1 and μ_2. Let μ_V be the representation of G_η on the unfolding space $V = \mathbf{C}^{n-m}$ given by

$$(\phi, \psi)\ \alpha = \psi \circ \alpha \circ \phi^{-1}, \tag{19}$$

where $\alpha \in V$ and $(\phi, \psi) \in G_\eta$. Then we have

$$\lambda_1 = \mu_1 \oplus \mu_V \quad \text{and} \quad \lambda_2 = \mu_2 \oplus \mu_V. \tag{20}$$

For example, for the singularity of type A_i: $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, we have $G_{A_i} = U(1) \times U(k)$. Let ρ_j denote the standard representation of the unitary group $U(j)$. Then

$$\mu_1 = \rho_1, \quad \mu_2 = \rho_1^{i+1} \oplus \rho_k, \quad \mu_V = \oplus_{j=2}^{i}\ \rho_1^j \oplus \oplus_{j=1}^{i}(\rho_k \otimes \rho_1^{-1}). \tag{21}$$

If we denote the Chern roots of the universal bundles on $BU(1)$ and $BU(k)$ by x and $y_1, \ldots, y_k$ then

$$c(A_i) = \frac{1+(i+1)x}{1+x} \prod_{j=1}^{k} (1+y_j), \tag{22}$$

and

$$e(A_i) = i!\ x^i \prod_{j=1}^{k} (ix - y_j) \cdots (2x - y_j)(x - y_j). \tag{23}$$

We list the characteristic classes of other singularities that we will use in the computation of Thom polynomials for A_4 singularities (cf. Theorem 7, see [15] or [20]).

$$c(I_{2,2}) = \frac{(1+2x_1)(1+2x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k} (1+y_j)\,, \tag{24}$$

[6] The deffect is the difference between the minimal number of relations and the number of generators.

$$c(III_{2,2}) = \frac{(1+2x_1)(1+2x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k-1} (1+y_j)\,, \tag{25}$$

$$c(III_{2,3}) = \frac{(1+2x_1)(1+3x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k-1} (1+y_j)\,. \tag{26}$$

Note that the last two Chern classes are defined for $k \geq 1$.

4. Thom polynomials for $A_4(3)$, $A_4(4)$ and towards $A_4(-)$.

We now focus on the structure of Thom polynomials for A_4 singularities. The results of this section are inspired by those from [15] and [17] but we use different methods. As in [15], we will use a "shifted" parameter r instead of k:

$$r := k+1\,. \tag{27}$$

We shall write $\eta(r)$ for the singularity $\eta : (\mathbf{C}^{\bullet}, 0) \to (\mathbf{C}^{\bullet+r-1}, 0)$, and $\mathcal{T}_r^{\eta}$ to denote the Thom polynomial for $\eta(r)$.

We have

$$c_i(f^*TY - TX) = S_i(TX^* - f^*(TY^*)),$$

where S_i means the Segre class. Working with Schur functions we shall follow of the notation on the RHS and that from Section 2.

Let us write the conditions imposed on $\mathcal{T}_r^{A_4}$ by the singularities with codimension at most $4r = \operatorname{codim} A_4(r)$. From $A_0(r)$, $A_1(r)$, $A_2(r)$ and $A_3(r)$ we have

$$\begin{aligned} P(-\mathbb{B}_{r-1}) &= P\left(x - \mathbb{B}_{r-1} - \boxed{2x}\right) = P\left(x - \mathbb{B}_{r-1} - \boxed{3x}\right) \\ &= P\left(x - \mathbb{B}_{r-1} - \boxed{4x}\right) = 0. \end{aligned} \tag{28}$$

Then $III_{2,2}(r)$, $I_{2,2}(r)$ and $III_{2,3}(r)$ impose that

$$P\left(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \boxed{x_1+x_2} - \mathbb{B}_{r-2}\right) = 0, \tag{29}$$

$$P\left(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \mathbb{B}_{r-1}\right) = 0, \tag{30}$$

and

$$P\left(\mathbb{X}_2 - \boxed{2x_1} - \boxed{3x_2} - \boxed{x_1 + x_2} - \mathbb{B}_{r-2}\right) = 0. \tag{31}$$

Additionally, $A_4(r)$ imposes

$$P\left(x - \mathbb{B}_{r-1} - \boxed{5x}\right) = R\left(x + \boxed{2x} + \boxed{3x} + \boxed{4x}, \mathbb{B}_{r-1} + \boxed{5x}\right). \tag{32}$$

Consider the functions of the form

$$F_r^{(4)} + \sum_{(r+1,r+1)\subset I} \alpha_I S_I$$

where the α_I are arbitrary integers. By Corollary 6 and the vanishing property all these functions satisfy the conditions imposed by $A_i(r)$ for $i = 0, 1, 2, 3$ and $i = 4$. Also notice that Equation (29) can be obtained from Equation (30) by substituting $b_{r-1} = x_1 + x_2$. Thus, to determine $\mathcal{T}_r^{A_4}$ we need only to find the coefficients α_I such that the equations

$$\left(F_r^{(4)} + \sum_{(r+1,r+1)\subset I} \alpha_I S_I\right)\left(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \mathbb{B}_{r-1}\right) = 0 \tag{33}$$

and

$$\left(F_r^{(4)} + \sum_{(r+1,r+1)\subset I} \alpha_I S_I\right)\left(\mathbb{X}_2 - \boxed{2x_1} - \boxed{3x_2} - \boxed{x_1 + x_2} - \mathbb{B}_{r-2}\right) = 0 \tag{34}$$

are satisfied.

Set $\mathbb{E} = \boxed{2x_1} + \boxed{2x_2}$ and $\mathbb{F} = \boxed{2x_1} + \boxed{3x_2} + \boxed{x_1 + x_2}$.

Lemma 8. *(i) $R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1})$ divides $F_r^{(4)}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})$ and*

(ii) $R(\mathbb{X}_2, \mathbb{F} + \mathbb{B}_{r-2})$ divides $F_r^{(4)}(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2})$.

Proof. (*i*) We use Proposition 5. Substituting $x_2 = 0$ we get

$$\begin{aligned} F_r^{(4)}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) &= F_r^{(4)}\left(x_1 - \boxed{2x_1} - \mathbb{B}_{r-1}\right) \\ &= R\left(x_1 + \boxed{2x_1} + \boxed{3x_1} + \boxed{4x_1},\ \mathbb{B}_{r-1} + \boxed{2x_1}\right) = 0. \end{aligned}$$

Substituting $x_2 = 2x_1$ we obtain

$$F_r^{(4)}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = F_r^{(4)}\left(x_1 - \boxed{4x_1} - \mathbb{B}_{r-1}\right)$$
$$= R\left(x_1 + \boxed{2x_1} + \boxed{3x_1} + \boxed{4x_1},\ \mathbb{B}_{r-1} + \boxed{4x_1}\right) = 0.$$

Substituting $b_{r-1} = x_2$ we have

$$F_r^{(4)}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = F_r^{(4)}\left(x_1 - \boxed{2x_1} - \boxed{2x_2} - \mathbb{B}_{r-2}\right)$$
$$= R\left(x_1 + \boxed{2x_1} + \boxed{3x_1} + \boxed{4x_1},\ \mathbb{B}_{r-2} + \boxed{2x_1} + \boxed{2x_2}\right) = 0.$$

Therefore by symmetry $R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1})$ divides $F_r^{(4)}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})$.

(ii) As in the proof of part (i) we use Proposition 5 with the substitutions $x_1 = 0$, $x_2 = 0$, $x_1 = 3x_2$, $x_2 = 2x_1$, $b_{r-2} = x_1$ and $b_{r-2} = x_2$. □

Lemma 9. (i) *If $S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})$ has degree d in b_{r-1} (as a polynomial over $\mathbf{Z}[x_1, x_2, b_1, \ldots, b_{r-2}]$) then I has at least d parts.*

(ii) *If a partition I appears as an index in the Schur function expansion of $\mathcal{T}_r^{A_4}$ then I has at most 4 parts.*

Proof. (i) Assume that $S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})$ has degree d in b_{r-1} and I has l parts. Let $I = (i_1, i_2, \ldots, i_l)$. Then

$$S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = \left|S_{i_p+p-q}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})\right|_{1\le p,q\le l}$$
$$= \left|S_{i_p+p-q}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2}) - b_{r-1}S_{i_p+p-q-1}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})\right|_{1\le p,q\le l}.$$

According to this determinant the degree in b_{r-1} is at most l. Hence $d \le l$.

(ii) It is proved in [17], the $1-$*part* of $\mathcal{T}_r^{A_4}$ is given by $F_r^{(4)}$. That is we can assume that $\mathcal{T}_r^{A_4} = F_r^{(4)} + \sum_{(r+1,r+1)\subset I} \alpha_I S_I$. Since the partitions indexing $F_r^{(4)}$ have at most 4 parts, it is enough to show that if I contains the partition $(r+1, r+1)$ and $\ell(I) \ge 5$ then $\alpha_I = 0$. For this we will show that for such a partition I, if $\alpha_J = 0$ for every partition J such that $\ell(J) > \ell(I)$ then $\alpha_I = 0$ too. Observe that $|I| = 4r$ and $(r+1, r+1) \subset I$ give us $\ell(I) \le 2r$. If $\ell(I) = 2r$ then I is necessarily $(1, \ldots, 1, r+1, r+1)$ (1 appears $2r-2$ times). The coefficient of b_{r-1}^{2r-2} in Equation (33) is zero. By part (i) this coefficient comes from $\alpha_I S_I$. Therefore if $\ell(I) = 2r$ then $\alpha_I = 0$. Next fix d between 5 and $2r-1$ and assume that $\ell(J) > d$ implies that $\alpha_J = 0$. Note that the set of coefficients of b_{r-1}^d's collected from all $S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})$s for I runs through the partitions containing $(r+1, r+1)$ and $\ell(I) = d$ is linearly independent over $\mathbf{Z}$. So we will look at the coefficient of b_{r-1}^d in the LHS of Equation (33). By our assumption and part (i) this coefficient comes from the sum

$$\sum_{\ell(I)=d,\quad (r+1,r+1)\subset I} \alpha_I S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}). \tag{35}$$

That is the coefficient of b_{r-1}^d in Equation (33) is a linear combination of **Z**-linearly independent functions. On the RHS of Equation (33) this coefficient is zero. Therefore $\alpha_I = 0$ for all I with $\ell(I) \geq 5$. $\square$

Denote by H_r the part of $\mathcal{T}_r^{A_4}$ corresponding to the sum of the Schur functions over the partitions containing the partition $(r+1, r+1)$. Using Lemma 9 we can obtain a recursive description of H_r. Let τ denote the linear endomorphism on the **Z**-module of Schur functions corresponding to partitions of length ≤ 4 that sends a Schur function

$$S_{i_1,i_2,i_3,i_4} \qquad \text{to} \qquad S_{i_1+1,i_2+1,i_3+1,i_4+1}.$$

Let H_r^o denote the sum of those terms in the Schur function expansion of H_r which corresponds to partitions of length ≤ 3.

Proposition 10. *Keeping the above notation, for $r \geq 2$, we have the following recursive equation:*

$$H_r = H_r^o + \tau(H_{r-1})\,.$$

Proof. Write

$$H_r = \sum_I \alpha_I S_I = \sum_J \alpha_J S_J + \sum_K \alpha_K S_K\,, \tag{36}$$

where J have at most 3 parts and $K = (k_1, k_2, k_3, k_4)$ have 4 parts (we assume that $\alpha_I \neq 0$). We set

$$Q = \sum_K \alpha_K S_{k_1-1,k_2-1,k_3-1,k_4-1}\,, \tag{37}$$

and our aim is to show that $Q = H_{r-1}$. For this it is enough to show that $\mathcal{T}_{r-1}^{A_4} = F_{r-1}^{(4)} + Q$. This is equivalent to saying the function $F_{r-1}^{(4)} + Q$ satisfies equations (30) and (31) (for $r-1$). We can also write $F_r^{(4)}$ as

$$F_r^{(4)} = \sum_I \alpha_I S_I = \sum_M \alpha_M S_M + \sum_N \alpha_N S_N, \tag{38}$$

where M have at most 3 parts and $N = (n_1, n_2, n_3, n_4)$ have 4 parts (we assume that $\alpha_I \neq 0$). Note that

$$\sum_N \alpha_N S_{n_1-1,n_2-1,n_3-1,n_4-1} = F^{(4)}_{r-1}. \tag{39}$$

Consider Equation (30) for r. We have

$$\mathcal{T}_r^{A_4}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = 0. \tag{40}$$

Since $\mathcal{T}_r^{A_4} = F_r^{(4)} + H_r$ we get

$$F_r^{(4)}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = -H_r(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}), \tag{41}$$

Using the previous lemma in the expansion of this equation we see that the coefficient of b_{r-1}^4 on the LHS is $F^{(4)}_{r-1}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$. On the RHS it is $-Q(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-2})$. Therefore we get Equation (30) for $r-1$. The case of Equation (31) is similar. □

Note that the function $F_r^{(4)}$ is given by the formula (11). Therefore using Proposition 10 we also obtain a description of $\mathcal{T}_r^{A_4}$:

$$\mathcal{T}_r^{A_4} = F_r^{(4)} + \tau(H_{r-1}) + H_r^o, \qquad \text{for } r \geq 2. \tag{42}$$

Then from Equation (33) we get

$$\mathcal{T}_r^{A_4}(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = \left(F_r^{(4)} + \tau(H_{r-1}) + H_r^o\right)(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = 0.$$

Therefore

$$H_r^o(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = -\left(F_r^{(4)} + \tau(H_{r-1})\right)(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}). \tag{43}$$

Similarly from Equation (34) we obtain

$$H_r^o(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}) = -\left(F_r^{(4)} + \tau(H_{r-1})\right)(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}). \tag{44}$$

Here $H_r^o = \sum \alpha_I S_I$ with $I = (a, r+1+p, r+1+q)$ and $a+p+q = 2r-2$. At this point one can expand the equations (43) and (44) to obtain a system of linear equations which determines α_I. However it is still possible to further simplify these equations: First using the factorization property on the LHS of Equation (43) we get

$$\begin{aligned}&\sum \alpha_I S_I(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})\\&= \sum \alpha_I S_a(-\mathbb{E} - \mathbb{B}_{r-1}) \cdot R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1}) \cdot S_{p,q}(\mathbb{X}_2)\\&= R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1}) \cdot \sum \alpha_I S_a(-\mathbb{E} - \mathbb{B}_{r-1}) \cdot S_{p,q}(\mathbb{X}_2).\end{aligned}$$

On the LHS of Equation (44) this property gives

$$\begin{aligned}&\sum \alpha_I S_I(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2})\\&= \sum \alpha_I S_a(-\mathbb{F} - \mathbb{B}_{r-2}) \cdot R(\mathbb{X}_2, \mathbb{F} + \mathbb{B}_{r-2}) \cdot S_{p,q}(\mathbb{X}_2)\\&= R(\mathbb{X}_2, \mathbb{F} + \mathbb{B}_{r-2}) \cdot \sum \alpha_I S_a(-\mathbb{F} - \mathbb{B}_{r-2}) \cdot S_{p,q}(\mathbb{X}_2).\end{aligned}$$

Next using once more the factorization property, this time for $\tau(H_{r-1})$, and Lemma 8 we define $U_r(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})$ and $V_r(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2})$ as the quotients

$$U_r(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = -\frac{\left(F_r^{(4)} + \tau(H_{r-1})\right)(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1})}{R(\mathbb{X}_2, \mathbb{E} + \mathbb{B}_{r-1})}$$

and

$$V_r(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}) = -\frac{\left(F_r^{(4)} + \tau(H_{r-1})\right)(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2})}{R(\mathbb{X}_2, \mathbb{F} + \mathbb{B}_{r-2})}.$$

Thus we obtain

$$\sum \alpha_I S_a(-\mathbb{E} - \mathbb{B}_{r-1}) \cdot S_{p,q}(\mathbb{X}_2) = U_r(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) \tag{45}$$

and

$$\sum \alpha_I S_a(-\mathbb{F} - \mathbb{B}_{r-2}) \cdot S_{p,q}(\mathbb{X}_2) = V_r(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}). \tag{46}$$

Now we can expand these equations and compare the coefficients of monomials on both sides to obtain a system of linear equations whose solution gives the coefficients α_I. Since H_2 is already computed we will start with $r = 3$. By Proposition 10 we have

$$H_3 = H_3^o + 21S_{1344} + 76S_{1155} + 104S_{1245} + 240S_{1146} + 10S_{2244}.$$

To determine H_3^o we need to study partitions I containing the partition $(4,4)$ and such that $\ell(I) \leq 3$. There are nine of them:

$$\begin{array}{lll} I_1 = 147, & I_4 = 255, & I_7 = 48, \\ I_2 = 156, & I_5 = 345, & I_8 = 57, \\ I_3 = 246, & I_6 = 444, & I_9 = 66. \end{array}$$

So we can write

$$H_3^o = \sum_{i=1}^{9} \alpha_i S_{I_i} \tag{47}$$

for some coefficients $\alpha_i \in \mathbf{Z}$. We will first use Equation (46). We will compare the coefficients of monomials on both sides of this equation. For this we need to expand the corresponding product $S_a(-\mathbb{F} - \mathbb{B}_1) \cdot S_{p,q}(\mathbb{X}_2)$ for $I = I_1, I_2, \ldots, I_9$ and $V_3(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2})$. For a fixed partition it is not difficult to do this by direct computation. For example, the corresponding product for I_1 is

$$\begin{aligned} S_1(-\mathbb{F} - \mathbb{B}_1) \cdot S_3(\mathbb{X}_2) &= -(2x_1 + 3x_2 + (x_1 + x_2) + b_1)(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) \\ &= -(3x_1 + 4x_2 + b_1)(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3). \end{aligned}$$

So the coefficient of $x_1^3 b_1$ in this product is -1. For I_3 the coefficient of this monomial is 3 since

$$\begin{aligned} S_2(-\mathbb{F} - \mathbb{B}_1) &= 3x_1 b_1 + 4x_2 b_1 + 2x_1^2 + 11 x_1 x_2 + 3x_2^2\,, \\ S_2(\mathbb{X}_2) &= x_1^2 + x_1 x_2 + x_2^2\,. \end{aligned}$$

However when we have too many partitions for which we need to repeat similar computations it is better to use a computer. So we use ACE 3.0 (cf. [23]) to expand these products and $V_3(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2})$. (In this paper this is the only use of computers.) We see that the coefficient of $x_1^3 b_1$ is zero in the corresponding product for partitions different from I_1 and I_3. The coefficient in $V_3(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2})$ is -568. Therefore we obtain the equation

$$2\alpha_3 - \alpha_1 = -568.$$

Comparing also the coefficients of other monomials in equations (45) and (46) we get the following system of equations:

$$\begin{array}{rrrrrl}
-\alpha_1 & +2\alpha_3 & & & & = -568 \\
\alpha_1 & -4\alpha_3 & +3\alpha_5 & & & = -284 \\
\alpha_1 & -3\alpha_3 & +2\alpha_5 & & & = 222 \\
\alpha_1+\alpha_2 & -7\alpha_3-3\alpha_4 & +13\alpha_5-6\alpha_6 & & & = -562 \\
\alpha_1+\alpha_2 & -7\alpha_3-4\alpha_4 & +14\alpha_5-6\alpha_6 & & & = -602 \\
3\alpha_1 & +2\alpha_3 & & -\alpha_7 & & = 664 \\
-4\alpha_1-2\alpha_2 & +4\alpha_3 & & +\alpha_7+\alpha_8 & & = -110 \\
-4\alpha_1-4\alpha_2 & +4\alpha_3+4\alpha_4 & & +\alpha_7+\alpha_8+\alpha_9 & & = -1392 \\
7\alpha_1+7\alpha_2 & -16\alpha_3-11\alpha_4 & +12\alpha_5 & -\alpha_7-\alpha_8-\alpha_9 & & = 2032
\end{array}$$

Solving this system we see that

$$\begin{array}{lll}
\alpha_1 = 1900, & \alpha_4 = 200, & \alpha_7 = 3704, \\
\alpha_2 = 804, & \alpha_5 = 160, & \alpha_8 = 1736, \\
\alpha_3 = 666, & \alpha_6 = 14, & \alpha_9 = 520.
\end{array}$$

Therefore we have computed

$$H_3^o = 1900S_{147} + 804S_{156} + 666S_{246} + 200S_{255} + 160S_{345} + 14S_{444} + 3704S_{48} + 1736S_{57} + 520S_{66}.$$

Repeating the same procedure for $r = 4$ we get

$$H_4^o = 116S_{556} + 280S_{466} + 889S_{457} + 1476S_{277} + 1490S_{367} + 3376S_{88} + 3520S_{358} + 5120S_{268} + 5504S_{178} + 10840S_{259} + 11520S_{79} + 13520S_{169} + 25280S_{6,10} + 27536S_{1,5,10} + 50624S_{5,11}.$$

These formulas enable us to write the Thom polynomials for the singularities $A_4(3)$ and $A_4(4)$.

Theorem 11. *We have*

$$\begin{aligned}
\mathcal{T}_3^{A_4} = {} & S_{3333} + 9S_{2334} + 24S_{2226} + 26S_{2235} + 55S_{1335} + 210S_{1236} \\
& + 216S_{1227} + 285S_{336} + 460S_{1137} + 576S_{1119} + 624S_{1128} \\
& + 1214S_{237} + 1320S_{228} + 3516S_{138} + 5040S_{129} + 5184S_{1,1,10} \\
& + 6920S_{39} + 11040S_{2,10} + 13824S_{0,12} + 14976S_{1,11} \\
& + 10S_{2244} + 21S_{1344} + 76S_{1155} + 104S_{1245} + 240S_{1146} \\
& + 14S_{444} + 160S_{345} + 200S_{255} + 520S_{66} + 666S_{246} \\
& + 804S_{156} + 1736S_{57} + 1900S_{147} + 3704S_{48},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{T}_4^{A_4} = {} & S_{4444} + 9S_{3445} + 24S_{3337} + 26S_{3346} + 55S_{2446} + 210S_{2347} \\
& + 216S_{2338} + 285S_{1447} + 460S_{2248} + 576S_{2,2,2,10} + 624S_{2239} \\
& + 1214S_{1348} + 1320S_{1339} + 1351S_{448} + 3516S_{1249} \\
& + 5040S_{1,2,3,10} + 5184S_{1,2,2,11} + 6090S_{349} + 6840S_{3,3,10} \\
& + 6920S_{1,1,4,10} + 11040S_{1,1,3,11} + 13824S_{1,1,1,13} + 14976S_{1,1,2,12} \\
& + 19684S_{2,4,10} + 29136S_{2,3,11} + 31680S_{2,2,12} + 51240S_{1,4,11} \\
& + 84384S_{1,3,12} + 95536S_{4,12} + 120960S_{1,2,13} + 124416S_{1,1,14} \\
& + 166080S_{3,13} + 264960S_{2,14} + 331776S_{0,16} + 359424S_{1,15} \\
& + 10S_{3355} + 21S_{2455} + 76S_{2266} + 104S_{1356} + 240S_{2257} \\
& + 14S_{1555} + 160S_{1456} + 200S_{1366} + 520S_{1177} + 666S_{1357} \\
& + 804S_{1267} + 1736S_{1168} + 1900S_{1258} + 3704S_{159} \\
& + 116S_{556} + 280S_{466} + 889S_{457} + 1476S_{277} + 1490S_{367} + 3376S_{88} \\
& + 3520S_{358} + 5120S_{268} + 5504S_{178} + 10840S_{259} + 11520S_{79} \\
& + 13520S_{169} + 25280S_{6,10} + 27536S_{1,5,10} + 50624S_{5,11}.
\end{aligned}
$$

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