Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **33** (2007), 321–338

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

INVARIANT FUNCTIONS ON NEIL PARABOLA IN $\mathbb{C}^n$

Paweł Zapałowski

Communicated by P. Pflug

Abstract. We present the Carathéodory and the inner Carathéodory distances and the Carathéodory-Reiffen metric on generalized Neil parabolas in $\mathbb{C}^n$. It is a generalization of the results from [4] and [5].

1. Introduction and results. In the paper [3] the authors had asked for an effective formula for the Carathéodory distance on the Neil parabola in the bidisc. Such a formula was presented by G. Knese in [4], where he also computed the formula for the Carathéodory-Reiffen pseudometric. It should be pointed out that these are the first effective formulas for the Carathéodory distance and the Carathéodory-Reiffen pseudometric of a non-trivial complex space. In [5] N. Nikolov and P. Pflug generalized Knese's result. The authors presented formula for the inner Carathéodory distance in so called generalized

2000 *Mathematics Subject Classification:* Primary 32F45.

Key words: generalized Neil parabola, Carathéodory pseudodistance, Kobayashi pseudodistance, Carathéodory-Reiffen pseudometric, Kobayashi-Royden pseudometric.

This work is a part of the Research Grant No. 1 PO3A 005 28, which is supported by public means in the programme promoting science in Poland in the years 2005–2008.

Neil parabola (but still in bidisc) and, as a corollary, they obtained sufficient and necessary condition for the Carathéodory distance on the Neil parabola to be inner. Moreover, they presented also formula for the Carathéodory-Reiffen pseudometric on the two-dimensional generalized Neil parabola.

In this paper we present next possible generalization of the definition of Neil parabola, namely we embed the unit disc in $\mathbb{C}^n$. It turns out that in such a generalized Neil parabola all the results obtained in [5] are still valid. The aim of this paper is to translate the results from the two-dimensional case onto the n-dimensional one. Below we present all the necessary definitions.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. For $M = (m_1, \dots, m_n) \in \mathbb{N}^n$, where m_j's are relatively prime and such that $m_1 \leqslant \dots \leqslant m_n$ define

$$\mathbb{D} \ni \lambda \xrightarrow{p} (\lambda^{m_1}, \dots, \lambda^{m_n}) \in A := p(\mathbb{D}) \subset \mathbb{D}^n.$$

A is called the *n-dimensional generalized parabola*. Note that A is one-dimensional analytic subset of $\mathbb{D}^n$ with $\operatorname{reg} A = A_* := A \setminus \{0\}$. Recall that G. Knese worked with $M = (3, 2)$ while N. Nikolov and P. Pflug obtained their results for $M = (n, m)$, where n, m are relatively prime.

The mapping p is a global bijective holomorphic parametrization for A. Observe that there exist $r_1, \dots, r_n \in \mathbb{Z}$ such that $r_1 m_1 + \dots + r_n m_n = 1$.

Define $q : A \to \mathbb{C}$ with the formula

$$q(z_1, \dots, z_n) = \begin{cases} z_1^{r_1} \dots z_n^{r_n}, & z_1 \dots z_n \neq 0 \\ 0, & z_1 \dots z_n = 0 \end{cases}.$$

Observe that $q = p^{-1}$. Note that q is continuous on A and holomorphic on A_*. Thus the mapping $q|_{A_*} : A_* \to \mathbb{D}_* := D \setminus \{0\}$ is biholomorphic.

Let

$$\mathcal{O}_M(\mathbb{D}) := \{h \in \mathcal{O}(\mathbb{D}, \mathbb{D}) : h^{(s)}(0) = 0, s \in S\},$$

where $S := \{s \in \mathbb{N} : s \notin m_1\mathbb{Z}_+ + \dots + m_n\mathbb{Z}_+\}$. Note that if $m_1 = 1$ then $S = \varnothing$ and if $m_1 > 1$ then $\max_{s \in S} =: s^* < nrm_1 \dots m_n$, where $r := \max_{j=1,\dots,n} |r_j|$.

Observe that if $f \in \mathcal{O}(A, \mathbb{D})$, i.e. f is locally the restriction of a holomorphic function on an open neighborhood of A in $\mathbb{C}^n$, then $f \circ p \in \mathcal{O}_M(\mathbb{D})$. Moreover, the converse is true. Indeed, we have the following

Lemma 1 (cf. Section 5 in [4]). *If $h \in \mathcal{O}_M(\mathbb{D})$, then $h \circ q \in \mathcal{O}(A, \mathbb{D})$.*

All the proof will be presented in Section 2. We will also use the following identification.

Lemma 2. $\mathcal{O}(\mathbb{D}, A) = \{p \circ \psi : \psi \in \mathcal{O}(\mathbb{D}, \mathbb{D})\}$.

For $a \in A$ let $T_a A$ denote the tangent space of A at a. Recall that if $a = p(\lambda)$, $\lambda \in \mathbb{D}_*$, then $T_{p(\lambda)}A$ is spanned by the vector $p'(\lambda)$. If $a = 0$ then

$$T_0 A = \begin{cases} \lambda p'(0),\ \lambda \in \mathbb{C}, & \text{if} \quad m_1 = 1 \\ \mathbb{C}^n & \text{if} \quad m_1 > 1 \end{cases}.$$

We will study some invariant functions. So let us recall the objects we will deal with in this paper. For details we refer the Reader to [2] and [3]. For $z, w \in A$ and $X \in T_z A$ we define

$$\begin{aligned} c_A(z,w) &:= \sup\{p_{\mathbb{D}}(f(z), f(w)) : f \in \mathcal{O}(A, \mathbb{D})\}, \\ m_A(z,w) &:= \sup\{m_{\mathbb{D}}(f(z), f(w)) : f \in \mathcal{O}(A, \mathbb{D})\}, \\ \gamma_A(z;X) &:= \max\{|f'(z)X| : f \in \mathcal{O}(A, \mathbb{D})\}, \\ \tilde{k}_A(z,w) &:= \inf\{p_{\mathbb{D}}(\zeta, \xi) : \exists_{\varphi \in \mathcal{O}(\mathbb{D}, A)} : \varphi(\zeta) = z,\ \varphi(\xi) = w\}, \\ k_A &:= \text{the largest distance on } A \text{ below of } \tilde{k}_A, \\ \kappa_A(z;X) &:= \inf\{\alpha > 0 : \exists_{\varphi \in \mathcal{O}(\mathbb{D}, A)} : \varphi(0) = z,\ \alpha\varphi'(0) = X\}, \end{aligned}$$

where $p_{\mathbb{D}} := \tanh^{-1} m_{\mathbb{D}}$ denotes the *Poincaré distance* and $m_{\mathbb{D}}(a,b) := \left|\dfrac{a-b}{1-a\bar{b}}\right|$, $a, b \in \mathbb{D}$, is the *Möbius distance* on $\mathbb{D}$. We set $\tilde{k}_A(z,w) := \infty$ or $\kappa_A(z;X) := \infty$ if there are no respective discs φ. We call c_A the *Carathéodory distance*, m_A is the *Möbious distance*, γ_A is the *Carathéodory-Reiffen metric*, $\tilde{k}_A$ is the *Lempert function*, k_A is the *Kobayashi distance* and κ_A is the *Kobayashi-Royden metric* for A.

Recall that the associated inner Carathéodory distance c^i_A is given by

$$c^i_A(z,w) := \inf\{L_{c_A}(\alpha) : \alpha \text{ is a } \|\cdot\|\text{-rectifiable curve in } A \text{ connecting } z, w\}, \quad z, w \in A,$$

where L_{c_A} denotes the c_A-length. We say that the curve α is $\|\cdot\|$-*rectifiable* if its Euclidean length is finite. Obviously, $c_A \leqslant c^i_A$.

Theorem 3 (cf. Theorem 3 in [5]). *Let* $\lambda \in \mathbb{D}$. *Then*

$$\gamma_A(p(\lambda); p'(\lambda)) = \frac{m_1|\lambda|^{m_1-1}}{1-|\lambda|^{2m_1}}.$$

Theorem 4 (cf. Theorem 1 in [5]). *Let $\lambda, \mu \in \mathbb{D}$. Then*

$$c_A^i(p(\lambda), p(\mu)) = \begin{cases} p_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1}) & \text{if } \operatorname{Re}(\lambda\bar{\mu}) \geqslant \cos(\pi/m_1)|\lambda\mu| \\ p_{\mathbb{D}}(\lambda^{m_1}, 0) + p_{\mathbb{D}}(0, \mu^{m_1}) & \text{otherwise} \end{cases}.$$

Theorem 5 (cf. Theorem 4.1 in [4]). *Let $\lambda, \mu \in \mathbb{D}$.*
(a) *If $S = \varnothing$, i.e. $m_1 = 1$, then*

$$c_A(p(\lambda), p(\mu)) = p_{\mathbb{D}}(\lambda, \mu).$$

(b) *If $S = \{1\}$, i.e. $m_1 = 2, m_j = 3$ for some $1 < j \leqslant n$, then*

$$c_A(p(\lambda), p(\mu)) = \begin{cases} p_{\mathbb{D}}(\lambda^2, \mu^2) & \text{if} \quad |a| \geqslant 1 \\ p_{\mathbb{D}}\left(\lambda^2 \dfrac{a-\lambda}{1-\bar{a}\lambda}, \mu^2 \dfrac{a-\mu}{1-\bar{a}\mu}\right) & \text{if} \quad |a| < 1 \end{cases},$$

where $a = a_{\lambda,\mu} := \dfrac{1}{2}\left(\lambda + \dfrac{1}{\bar{\lambda}} + \mu + \dfrac{1}{\bar{\mu}}\right)$. In the case when $\lambda\mu = 0$ the formula should be read as in the case $|a| \geqslant 1$.

Due to the results above we have the following correspondence between the Carathéodory distance and its associated inner one.

Corollary 6 (cf. Corollary 2 in [5]). *Let $\lambda, \mu \in \mathbb{D}$.*
(a) *If $\operatorname{Re}(\lambda\bar{\mu}) \geqslant \cos(\pi/m_1)|\lambda\mu|$ then*

$$c_A^i(p(\lambda), p(\mu)) = c_A(p(\lambda), p(\mu)).$$

(b) *If $\operatorname{Re}(\lambda\bar{\mu}) < \cos(\pi/m_1)|\lambda\mu|$ then*

$$c_A^i(p(\lambda), p(\mu)) = c_A(p(\lambda), p(\mu)) \text{ iff } (\lambda\bar{\mu})^{m_1} < 0.$$

Thus, the following conditions are equivalent
- $c_A^i(p(\lambda), p(\mu)) = c_A(p(\lambda), p(\mu))$;
- $c_A^i(p(\lambda), p(\mu)) = p_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1})$;
- $\operatorname{Re}(\lambda\bar{\mu}) \geqslant \cos(\pi/m_1)|\lambda\mu|$ *or* $(\lambda\bar{\mu})^{m_1} < 0$.

In particular, c_A is inner iff $m_1 = 1$.

It turns out that (as in the case of domains in $\mathbb{C}^n$) γ_A is the infinitesimal form of c_A outside the origin.

Corollary 7 (cf. [5]). *Let $\lambda \in \mathbb{D}_*$ (if $m_1 = 1$ we may take $\lambda \in \mathbb{D}$). Then*

$$\lim_{\mu\to\lambda} \frac{c_A(p(\lambda), p(\mu))}{|\lambda - \mu|} = \gamma_A(p(\lambda); p'(\lambda)).$$

Now assume that $m_1 > 1$. Let $X \in T_0A = \mathbb{C}^n$. Observe that

$$\gamma_A(0; X) = \max\{|f'(0)X| : f \in \mathcal{O}(A, \mathbb{D}), f(0) = 0\}.$$

Then for such an f we have $(f \circ p)(\lambda) = \lambda^{m_1} h(\lambda),\ \lambda \in \mathbb{D}$, where $h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})$. Observe that

$$\frac{\partial f}{\partial z_j}(0) = \frac{h^{(m_j - m_1)}(0)}{(m_j - m_1)!}, \quad j = 1, \dots, n.$$

Thus, for $X = (X_1, \dots, X_n) \in \mathbb{C}^n$ we have

$$\begin{aligned}\gamma_A(0; X) &= \max\Big\{\Big|\sum_{j=1}^n \frac{h^{(m_j)}(0)}{m_j!} X_j\Big| : h \in \mathcal{O}_M(\mathbb{D}), h(0) = 0\Big\} \\ &= \max\Big\{\Big|\sum_{j=1}^n \frac{h^{(m_j - m_1)}(0)}{(m_j - m_1)!} X_j\Big| : \\ &\qquad\qquad h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}}), h^{(j)}(0) = 0, j + m_1 \in S\Big\}.\end{aligned} \tag{1}$$

In particular, $\gamma_A(0; X) = \|X\|$ if $n - 1$ coordinates of X is equal 0. Using the first equality above, we will prove the following infinitesimal result at the origin.

Proposition 8 (cf. Prop. 4 in [5]). *Let $X_{\lambda,\mu} := (\lambda^{m_1} - \mu^{m_1}, \dots, \lambda^{m_n} - \mu^{m_n})$. Then*

$$\lim_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{c_A(p(\lambda), p(\mu))}{\gamma_A(0; X_{\lambda,\mu})} = 1.$$

Corollary 9 (cf. Corollary 5 in [5]). *Let $m_1 > 1$. For any $j \in \{2, \dots, n\}$ there are points $\lambda, \mu \in \mathbb{D}$ such that*

$$c_A(p(\lambda), p(\mu)) > \max\{p_{\mathbb{D}}(\lambda^{m_1}, \mu^{m_1}), p_{\mathbb{D}}(\lambda^{m_j}, \mu^{m_j})\}. \tag{2}$$

In the proof of Proposition 8 we use the following

Lemma 10 (cf. [5]). *There exists a constant $c > 0$ such that for any $\lambda, \mu \in \mathbb{D}$*

$$c_A(p(\lambda), p(\mu)) \geqslant \max\{p_{\mathbb{D}}(\lambda^{m_j}, \mu^{m_j}) : j = 1, \dots, n\} \geqslant c\|X_{\lambda,\mu}\|, \tag{3}$$

$$\max\{|\lambda|^{k-m_n}, |\mu|^{k-m_n}\}\|X_{\lambda,\mu}\| \geqslant \frac{c}{k}|\lambda^k - \mu^k|, \quad m_n < k, \tag{4}$$

$$\gamma_A(0; X_{\lambda,\mu}) \geqslant c\|X_{\lambda,\mu}\|. \tag{5}$$

Moreover,

$$\lim_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \sum_{j=m_1+1}^{\infty} \frac{|\lambda^j - \mu^j|}{\|X_{\lambda,\mu}\|} = 0. \tag{6}$$

Proposition 11 (cf. Proposition 7 in [5]). *Let $M = (m_1, \dots, m_n)$ be such that $m_1 = \dots = m_j = 2$, $m_{j+1} = 2k+1$ for some $1 \leqslant j \leqslant n-1$ and $k \in \mathbb{N}$. Then*

$$m_A(p(\lambda), p(-\lambda)) = \frac{2|\lambda|^{2k+1}}{1 + |\lambda|^{4k+2}}, \quad \lambda \in \mathbb{D}.$$

Finally, we discuss the Kobayashi distance and Kobayashi-Royden metric on A. Due to Lemma 2, we have the following result.

Proposition 12 (cf. Proposition 8 in [5]). (a) *Let $\lambda, \mu \in \mathbb{D}$. Then*

$$k_A(p(\lambda), p(\mu)) = \tilde{k}_A(p(\lambda), p(\mu)) = p_{\mathbb{D}}(\lambda, \mu).$$

(b) *If $\lambda \in \mathbb{D}_*$ then*

$$\kappa_A(p(\lambda); p'(\lambda)) = \gamma_{\mathbb{D}}(\lambda; 1).$$

If $\lambda = 0$ and $X = (X_1, \dots, X_n) \in T_0A$, $X \neq 0$ then

$$\kappa_A(0; X) = \begin{cases} |X_1| & \text{if} \quad m_1 = 1 \\ \infty & \text{if} \quad m_1 > 1 \end{cases}.$$

We conclude this note by generalizing the example of the coordinate cross discussed in [5]. Let $e_j = (\underbrace{0,\dots,0}_{j-1},1,0,\dots,0) \in \mathbb{C}^n,\ j=1,\dots,n$. Put

$$V_1 := \bigcup_{j=1}^{n} \mathbb{D} e_j.$$

Proposition 13 (cf. Remark in [5]). (a) *Let $\lambda,\mu \in \mathbb{D}$. Then*

$$c_{V_1}(\lambda e_j, \mu e_k) = k_{V_1}(\lambda e_j, \mu e_k) = \begin{cases} p_{\mathbb{D}}(\lambda,\mu) & \text{if} \quad j=k \\ p_{\mathbb{D}}(\lambda,0) + p_{\mathbb{D}}(0,\mu) & \text{if} \quad j \neq k \end{cases}, \tag{7}$$

$$\tilde{k}_{V_1}(\lambda e_j, \mu e_k) = \begin{cases} p_{\mathbb{D}}(\lambda,\mu) & \text{if} \quad j=k \\ \infty & \text{if } j \neq k,\ \lambda\mu \neq 0 \end{cases}. \tag{8}$$

(b) *If $\lambda \in \mathbb{D}_*$ then*

$$\gamma_{V_1}(\lambda e_j; e_j) = \kappa_{V_1}(\lambda e_j; e_j) = \gamma_{\mathbb{D}}(\lambda; 1). \tag{9}$$

If $\lambda = 0$ and $X = (X_1,\dots,X_n) \in \mathbb{C}^n$ then

$$\gamma_{V_1}(0;X) = \sum_{j=1}^{n} |X_j|, \tag{10}$$

$$\kappa_{V_1}(0;X) = \begin{cases} |X_j| & \text{if } X = X_j e_j,\ j=1,\dots,n \\ \infty & \text{otherwise} \end{cases}. \tag{11}$$

2. Proofs.

P r o o f o f L e m m a 1. $h \circ q$ is holomorphic on A_* because it may be extended to a holomorphic function on the set

$$\Omega := \Big\{ (z_1,\dots,z_n) \in \mathbb{D}^n : \prod_{j\in M^+} |z_j|^{r_j} < \prod_{k \notin M^+} |z_k|^{-r_k} \Big\},$$

where $M^+ := \{ j \in \{1,\dots,n\} : r_j \in \mathbb{Z}_+\}$, and Ω is an open neighborhood of A_*.

To prove that $h \circ q$ is holomorphic at the origin observe that

$$h(\lambda) = \sum_{j \in \mathbb{Z}_+ \setminus S} a_j \lambda^j, \quad \lambda \in \mathbb{D}. \tag{12}$$

Moreover, the following identities hold

$$z_j^{m_k} = z_k^{m_j}, \quad j, k \in \{1, \ldots, n\}, \ (z_1, \ldots, z_n) \in A.$$

Hence for any $j = m_1 b_{j,1} + \cdots + m_n b_{j,n} \in \mathbb{Z}_+ \setminus S$ and $z = (z_1, \ldots, z_n) \in A$ we obtain

$$\begin{aligned} q^j(z) &= z_1^{r_1 m_1 b_{j,1}} \ldots z_1^{r_1 m_n b_{j,n}} \ldots z_n^{r_n m_1 b_{j,1}} \ldots z_n^{r_n m_n b_{j,n}} \\ &= z_1^{r_1 m_1 b_{j,1}} \ldots z_n^{r_1 m_1 b_{j,n}} \ldots z_1^{r_n m_n b_{j,1}} \ldots z_n^{r_n m_n b_{j,n}} \\ &= z_1^{(r_1 m_1 + \cdots + r_n m_n) b_{j,1}} \ldots z_n^{(r_1 m_1 + \cdots + r_n m_n) b_{j,n}} = z_1^{b_{j,1}} \ldots z_n^{b_{j,n}}. \end{aligned}$$

Using the equality above and (12) we get

$$(h \circ q)(z) = \sum_{j \in \mathbb{Z}_+ \setminus S} a_j z_1^{b_{j,1}} \ldots z_n^{b_{j,n}}, \quad z = (z_1, \ldots, z_n) \in A, \tag{13}$$

where $b_{j,k} \in \mathbb{Z}_+$ for all $j \in \mathbb{Z}_+ \setminus S$ and $k = 1, \ldots, n$.

The series (13) is convergent for $z = (\lambda^{m_1}, \ldots, \lambda^{m_n})$, $|\lambda| \leqslant R < 1$. Thus it converges for $z \in R^{m_1}\mathbb{D} \times \cdots \times R^{m_n}\mathbb{D}$ which gives us holomorphicity of the extension of $h \circ q$ in some neigborhood of the origin. $\square$

Proof of Lemma 2. Since $p \in \mathcal{O}(\mathbb{D}, A)$, we have that $p \circ \psi \in \mathcal{O}(\mathbb{D}, A)$. Now assume that $f \in \mathcal{O}(\mathbb{D}, A)$. Since $f = p \circ q \circ f$ it suffices to show that $q \circ f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$.

Fix $\lambda \in \mathbb{D}$. If $f(\lambda) \neq 0$ then $q \circ f$ is holomorphic in some neighborhood of λ. If $f(\lambda) = 0$, i.e. $f_1(\lambda) = \cdots = f_n(\lambda) = 0$, where $f = (f_1, \ldots, f_n)$, then $f_j(\zeta) = (\zeta - \lambda)^{s_j} \tilde{f}_j(\zeta)$ for some $s_j \in \mathbb{N}$ and $\tilde{f}_j \in \mathcal{O}(U_\lambda)$, $\tilde{f}_j(\zeta) \neq 0$, $\zeta \in U_\lambda$, $j = 1, \ldots, n$, where $U_\lambda \subset \mathbb{D}$ is some neighborhood of λ. Since

$$(\zeta - \lambda)^{s_j m_k} \tilde{f}_j^{m_k}(\zeta) = (\zeta - \lambda)^{s_k m_j} \tilde{f}_k^{m_j}(\zeta), \quad \zeta \in U_\lambda, \ j, k \in \{1, \ldots, n\}, \tag{14}$$

there exists $l \in \mathbb{N}$ such that $s_j = l m_j$, $j = 1, \ldots, n$. Indeed, from (14) it follows that

$$s_j m_k = s_k m_j, \quad j, k \in \{1, \ldots, n\}. \tag{15}$$

Fix $j \in \{1, \ldots, n\}$. Observe that $m_j = p_{j,1} \ldots p_{j,s(j)}$, where $p_{j,s}$'s are prime numbers. Since $m_1, \ldots, m_n$ are relatively prime, for any $1 \leqslant s \leqslant s(j)$ there exists $1 \leqslant k \leqslant n$ such that $p_{j,s} \nmid m_k$. Then (15) implies that $s_j = p_{j,1} \ldots p_{j,s(j)} l_j$ for some $l_j \in \mathbb{N}$. Using (15) again, we conclude that $l_j = l_k =: l$ for all $j, k \in \{1, \ldots, n\}$.

Hence

$$(q \circ f)(\zeta) = f_1^{r_1}(\zeta) \ldots f_n^{r_n}(\zeta) = (\zeta - \lambda)^l \tilde{f}_1^{r_1}(\zeta) \ldots \tilde{f}_n^{r_n}(\zeta), \quad \zeta \in U_\lambda.$$

Thus $q \circ f \in \mathcal{O}(U_\lambda)$ and the proof is complete. □

Proof of Theorem 3. Recall that

$$\gamma_A(p(\lambda); p'(\lambda)) = \max\left\{\frac{|h'(\lambda)|}{1 - |h(\lambda)|^2} : h \in \mathcal{O}_M(\mathbb{D})\right\}.$$

Observe that if $\phi \in \mathrm{Aut}(\mathbb{D})$ and $h \in \mathcal{O}_M(\mathbb{D})$ then $\phi \circ h \in \mathcal{O}_M(\mathbb{D})$ and

$$\frac{|h'(\lambda)|}{1 - |h(\lambda)|^2} = \frac{|(\phi \circ h)'(\lambda)|}{1 - |(\phi \circ h)(\lambda)|^2}.$$

Thus

$$\begin{aligned}
&\gamma_A(p(\lambda); p'(\lambda)) \\
&\quad = \max\left\{\frac{|h'(\lambda)|}{1 - |h(\lambda)|^2} : h \in \mathcal{O}_M(\mathbb{D}), h(0) = 0\right\} \\
&\quad = \max\left\{\frac{|(\lambda^{m_1}\tilde{h}(\lambda))'|}{1 - |\lambda^{m_1}\tilde{h}(\lambda)|^2} : \tilde{h} \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}}), \tilde{h}^{(j)}(0) = 0, j + m_1 \in S\right\} \\
&\quad = |\lambda|^{m_1 - 1} \max\left\{\frac{|m_1 h(\lambda) + \lambda h'(\lambda)|}{1 - |\lambda^{m_1} h(\lambda)|^2} :\right. \\
&\qquad\qquad \left. h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}}), h^{(j)}(0) = 0, j + m_1 \in S\right\} = \frac{m_1 |\lambda|^{m_1 - 1}}{1 - |\lambda|^{2m_1}}.
\end{aligned}$$

The last equality may be proved exactly as in the proof of Theorem 3 in [5] with m_1 instead of m. □

Proof of Theorem 4. The proof follows the proof of Theorem 1 in [5] with m_1 instead of m. □

Proof of Theorem 5. Ad (a). It is a consequence of Theorem 4, since $m_1 = 1$.

A d (b). Since $S = \{1\}$, the proof of Theorem 4.1 from [4] may be repeated. $\square$

P r o o f o f C o r o l l a r y 6. The proof follows the proof of Corollary 2 in [5] with m_1 instead of m. $\square$

Remark 14 (cf. Remark (a) in [5]). In [5] for $m \in \mathbb{N}$ the following distance was introduced

$$p_{\mathbb{D}}^{(m)}(\lambda,\mu) := \max\{p_{\mathbb{D}}(\lambda^m h(\lambda), \mu^m h(\mu)) : h \in \mathcal{O}(\mathbb{D}, \bar{\mathbb{D}})\}.$$

Note that

$$\begin{aligned}\lim_{\substack{\varepsilon\to 0\\ \varepsilon\neq 0}} \frac{p_{\mathbb{D}}^{(m_1)}(\lambda,\lambda+\varepsilon)}{|\varepsilon|} &= |\lambda|^{m_1-1}\max\left\{\frac{|m_1 h(\lambda)+\lambda h'(\lambda)|}{1-|\lambda^{m_1}h(\lambda)|^2} : h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})\right\}\\ &= \gamma_A(p(\lambda);p'(\lambda))\end{aligned}$$

by the proof of Theorem 3. So it follows that the associated inner distance $\int p_{\mathbb{D}}^{(m_1)}$ of $p_{\mathbb{D}}^{(m_1)}$ equals to $c_A^i(p(\cdot),p(\cdot))$. Then

$$\begin{aligned}c_A^i(p(\lambda),p(\mu)) &\geqslant p_{\mathbb{D}}^{(m_1)}(\lambda,\mu)\\ &\geqslant c_A(p(\lambda),p(\mu)) \geqslant p_{\mathbb{D}}(\lambda^{m_1},\mu^{m_1}).\end{aligned}$$

Moreover, the proof of Corollary 6 shows that the following conditions are equivalent

- $c_A^i(p(\lambda),p(\mu)) = p_{\mathbb{D}}^{(m_1)}(\lambda,\mu)$;
- $c_A^i(p(\lambda),p(\mu)) = c_A(p(\lambda),p(\mu))$;
- $c_A^i(p(\lambda),p(\mu)) = p_{\mathbb{D}}(\lambda^{m_1},\mu^{m_1})$;
- $\mathrm{Re}(\lambda\bar{\mu}) \geqslant \cos(\pi/m_1)|\lambda\mu|$ or $(\lambda\bar{\mu})^{m_1} < 0$.

P r o o f o f C o r o l l a r y 7. Since

$$c_A^i(p(\lambda),p(\mu)) \geqslant c_A(p(\lambda),p(\mu)) \geqslant p_{\mathbb{D}}(\lambda^{m_1},\mu^{m_1}), \quad \lambda\in\mathbb{D},$$

for $\lambda \in \mathbb{D}_*$ (if $m_1 = 1$ we may take $\lambda \in \mathbb{D}$) we have

$$\begin{aligned}\lim_{\mu\to\lambda}\frac{c_A(p(\lambda),p(\mu))}{|\lambda-\mu|} = \lim_{\mu\to\lambda}\frac{p_{\mathbb{D}}(\lambda^{m_1},\mu^{m_1})}{|\lambda-\mu|} &= \lim_{\mu\to\lambda}\frac{m_{\mathbb{D}}(\lambda^{m_1},\mu^{m_1})}{|\lambda-\mu|}\\ &= \frac{m_1|\lambda|^{m_1-1}}{1-|\lambda|^{2m_1}} = \gamma_A(p(\lambda);p'(\lambda)).\end{aligned}$$

$\square$

Proof of Lemma 10. Fix $\lambda, \mu \in \mathbb{D}$. Without loss of generality we may assume that $\lambda \neq \mu$ and $|\mu| \leqslant |\lambda|$. Moreover, it suffices to obtain each inequality with different constant, since minimum of these constants will do the job.

Ad (3). The first inequality in (3) we obtain with help of the projection from A onto its m_j-th coordinate, while the second one is a consequence of the equivalence of norms in $\mathbb{C}^n$.

Ad (4). Let $\sqrt[m_j]{1} = \{\varepsilon_{m_j,0}, \dots, \varepsilon_{m_j,m_j-1}\}$ and let $R_{m_j,s} := \varepsilon_{m_j,s}[0,1]$, $s = 0, \dots, m_j - 1$. Observe that there is a constant $\delta = \delta(M) > 0$ such that

$$\Lambda_{m_j,s,\delta} \cap \Lambda_{m_l,t,\delta} = \varnothing \quad \text{if} \quad R_{m_j,s} \neq R_{m_l,t},$$

where $\Lambda_{m_j,s,\delta} := \{re^{i\varphi} : r \in R_{m_j,s}, \varphi \in (-\delta, \delta)\},\ s = 0, \dots, m_j - 1,\ j = 1, \dots, n.$

Observe that $\mu/\lambda \in \overline{\mathbb{D}}$. Since m_j's are relatively prime, one of the following two cases holds:

1° There exists $j \in \{1, \dots, n\}$ such that $\mu/\lambda \notin \bigcup_{s=0}^{m_j-1} \Lambda_{m_j,s,\delta}$;

2° $\mu/\lambda \in \Lambda_{m_n,0,\delta}$.

Ad 1°. Then there is a constant $c = c(\delta) > 0$ such that $|1-(\mu/\lambda)^{m_j}| \geqslant 2c$. Therefore

$$\begin{aligned}|\lambda|^{k-m_n} \|X_{\lambda,\mu}\| &\geqslant |\lambda|^{k-m_j} |\lambda^{m_j} - \mu^{m_j}| \\ &= |\lambda|^k |1 - (\mu/\lambda)^{m_j}| \geqslant 2c|\lambda|^k \geqslant \frac{c}{k}|\lambda^k - \mu^k|.\end{aligned}$$

Ad 2°. To obtain (4) in this case it suffices to prove that there exists $c > 0$ such that

$$\frac{c}{k}\left|\frac{1-(\mu/\lambda)^k}{1-(\mu/\lambda)^{m_n}}\right| \leqslant 1, \quad k > m_n.$$

Since $\lim\limits_{\mu/\lambda \to 1}\left|\frac{1-(\mu/\lambda)^k}{1-(\mu/\lambda)^{m_n}}\right| = \frac{k}{m_n}$, there is a constant $r > 0$ such that

$$\left|\frac{1-(\mu/\lambda)^k}{1-(\mu/\lambda)^{m_n}}\right| \leqslant \frac{2k}{m_n}, \quad |1-\mu/\lambda| < r,\ k > m_n.$$

Hence in case $|1 - \mu/\lambda| < r$, a constant $c_1 := \frac{m_n}{2}$ will do the job.

On the other hand, if $|1-\mu/\lambda| \geqslant r$ then there is a constant $c_2 = c_2(r) > 0$ such that $|1 - (\mu/\lambda)^{m_n}| \geqslant 2c_2$. Therefore

$$\frac{c_2}{k}\left|\frac{1-(\mu/\lambda)^k}{1-(\mu/\lambda)^{m_n}}\right| \leqslant \frac{2c_2}{2c_2 k} \leqslant 1.$$

Finally we take $c:=\min\{c_1,c_2\}$.

A d (5). Let $l\in\{1,\dots,n\}$ be such that $|\lambda^{m_l}-\mu^{m_l}|=\max\{|\lambda^{m_j}-\mu^{m_j}|: j=1,\dots,n\}$. Let $h(\zeta)=\zeta^{m_l},\ \zeta\in\mathbb{D}$. Observe that $h\in\mathcal{O}_M(\mathbb{D})$ and $h(0)=0$. Thus

$$\gamma_A(0;X_{\lambda,\mu})\geqslant\Big|\sum_{j=1}^{n}\frac{h^{(m_j)}(0)}{m_j!}(\lambda^{m_j}-\mu^{m_j})\Big|=|\lambda^{m_l}-\mu^{m_l}|\geqslant c\|X_{\lambda,\mu}\|,$$

where $c>0$ is a constant from the inequality (3).

A d (6). First assume that $m_1+1\leqslant j\leqslant m_n$. Then

$$\frac{|\lambda^j-\mu^j|}{\|X_{\lambda,\mu}\|}\leqslant\Big|\frac{\lambda^j-\mu^j}{\lambda^{m_1}-\mu^{m_1}}\Big|=\Big|\frac{\lambda^{\alpha_j}-\mu^{\alpha_j}}{\lambda-\mu}\Big|\leqslant\alpha_j(|\lambda|+|\mu|),$$

where $\alpha_j>1$. Therefore

$$\lim_{\substack{\lambda,\mu\to0\\ \lambda\neq\mu}}\sum_{j=m_1+1}^{m_n}\frac{|\lambda^j-\mu^j|}{\|X_{\lambda,\mu}\|}=0. \tag{16}$$

Observe that, using (4), we have

$$\sum_{j=m_n+1}^{\infty}\frac{|\lambda^j-\mu^j|}{\|X_{\lambda,\mu}\|}\leqslant\frac{1}{c}\sum_{j=m_n+1}^{\infty}j|\lambda|^{j-m_n}=\frac{1}{c}\sum_{j=1}^{\infty}(m_n+j)|\lambda|^j$$

$$\leqslant\frac{m_n+1}{c}\sum_{j=1}^{\infty}j|\lambda|^j=\frac{(m_n+1)|\lambda|}{c(1-|\lambda|)^2}.$$

Hence, letting $\lambda,\mu\to0,\ \lambda\neq\mu$, and using (16) we obtain (6). $\square$

P r o o f o f P r o p o s i t i o n 8. Let $h^+_{\lambda,\mu}\in\mathcal{O}_M(\mathbb{D})$ be an extremal function for $c_A(p(\lambda),p(\mu))$. Then

$$h^+_{\lambda,\mu}(\zeta)=\sum_{j\in\mathbb{Z}_+\setminus S}a_{\lambda,\mu,j}\zeta^j.$$

Since $|a_{\lambda,\mu,j}|\leqslant1$, it follows that

$$|h^+_{\lambda,\mu}(\lambda)-h^+_{\lambda,\mu}(\mu)|$$
$$\leqslant H^+(\lambda,\mu):=\Big|\sum_{j=1}^{n}a_{\lambda,\mu,m_j}(\lambda^{m_j}-\mu^{m_j})\Big|+\sum_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|.$$

Thus, using (3), (6), and (1)

$$1 \leqslant \liminf_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{H^+(\lambda,\mu)}{|h^+_{\lambda,\mu}(\lambda)-h^+_{\lambda,\mu}(\mu)|} = \liminf_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{H^+(\lambda,\mu)}{c_A(p(\lambda),p(\mu))}$$

$$\leqslant \liminf_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \Bigg(\frac{\Big|\sum\limits_{j=1}^{n} a_{\lambda,\mu,m_j}(\lambda^{m_j}-\mu^{m_j})\Big|}{c_A(p(\lambda),p(\mu))} + \frac{\sum\limits_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|}{c\|X_{\lambda,\mu}\|}\Bigg)$$

$$= \liminf_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{\Big|\sum\limits_{j=1}^{n} a_{\lambda,\mu,m_j}(\lambda^{m_j}-\mu^{m_j})\Big|}{c_A(p(\lambda),p(\mu))} \leqslant \liminf_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{\gamma_A(0;X_{\lambda,\mu})}{c_A(p(\lambda),p(\mu))}.$$

Let now $h^-_{\lambda,\mu}\in\mathcal{O}_M(\mathbb{D})$ be an extremal function for $\gamma_A(0;X_{\lambda,\mu})$. Then

$$h^-_{\lambda,\mu}(\zeta) = \sum_{j\in\mathbb{Z}_+\setminus S} a_{\lambda,\mu,j}\zeta^j.$$

Since $|a_{\lambda,\mu,j}|\leqslant 1$, it follows that

$$|h^-_{\lambda,\mu}(\lambda)-h^-_{\lambda,\mu}(\mu)| \geqslant \Big|\sum_{j=1}^{n} a_{\lambda,\mu,m_j}(\lambda^{m_j}-\mu^{m_j})\Big| - \sum_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|.$$

Then, using (5) and (6), we have

$$\lim_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{\sum\limits_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|}{\gamma_A(0;X_{\lambda,\mu})} \leqslant \lim_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \sum_{j=m_1+1}^{\infty}\frac{|\lambda^j-\mu^j|}{c\|X_{\lambda,\mu}\|} = 0,$$

and, consequently,

$$\lim_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{\sum\limits_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|}{|h^-_{\lambda,\mu}(\lambda)-h^-_{\lambda,\mu}(\mu)|} \leqslant \lim_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{\sum\limits_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|}{\gamma_A(0;X_{\lambda,\mu}) - \sum\limits_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|} = 0.$$

Thus, using (3), (6), and the last equality,

$$\begin{aligned}
1 &\geqslant \limsup_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{\gamma_A(0;X_{\lambda,\mu}) - \sum\limits_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|}{|h^-_{\lambda,\mu}(\lambda)-h^-_{\lambda,\mu}(\mu)|}\\
&\geqslant \limsup_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \left(\frac{\gamma_A(0;X_{\lambda,\mu})}{c_A(p(\lambda),p(\mu))} - \frac{\sum\limits_{j=m_1+1}^{\infty}|\lambda^j-\mu^j|}{|h^-_{\lambda,\mu}(\lambda)-h^-_{\lambda,\mu}(\mu)|}\right)\\
&= \limsup_{\substack{\lambda,\mu\to 0\\ \lambda\neq\mu}} \frac{\gamma_A(0;X_{\lambda,\mu})}{c_A(p(\lambda),p(\mu))}.
\end{aligned}$$

□

P r o o f o f C o r o l l a r y 9. Observe that for any neighborhood U of 0 one may find points $\lambda,\mu\in U$ such that $\lambda^{m_1}-\mu^{m_1}=\lambda^{m_j}-\mu^{m_j}\neq 0$. Then, by Proposition 8, it suffices to show that

$$\gamma_A(0;X_0)>1,\quad X_0:=(X_1,\dots,X_n),\ X_1=X_j=1. \tag{17}$$

Indeed, having (17) and using the equality (cf. Corollary 1.13 (d) in [2])

$$\lim_{\substack{\lambda',\lambda''\to 0\\ \lambda'\neq\lambda''}} \frac{p_{\mathbb{D}}(\lambda',\lambda'')}{|\lambda'-\lambda''|}=1$$

we obtain the required result.

By the second equality in (1) and the fact that $\max\limits_{s\in S} s = s^* < \infty$,

$$\gamma_A(0;X_0)\geqslant \max\{|a+b| : (a,b)\in T_{m_j-m_1}\},$$

where $T_{m_j-m_1}:=\{(a,b)\in\mathbb{C}^2 : \exists_{h\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})} : h(\zeta)=a+b\zeta^{m_j-m_1}+o(\zeta^{s^*-m_1})\}$.

Let $k\in\mathbb{N}$ be such that $k(m_j-m_1)\geqslant s^*-m_1$. We shall show that there is a function $f\in\mathcal{O}(\mathbb{D},\bar{\mathbb{D}})$ of the form $f(\zeta)=a+b\zeta+o(\zeta^k)$, where $a,b>0$ and $a+b>1$, which will imply (17).

From now on the rest of the proof of Corollary 5 in [5] may be repeated. For convenience of the Reader we recall that proof.

Note that by Shur's theorem (cf. [1]) such a function f exists if and only if

$$(1-a^2-b^2)\sum_{j=1}^{k} X_j^2 \geqslant 2ab\sum_{j=2}^{k} X_{j-1}X_j,\quad (X_1,\dots,X_k)\in\mathbb{R}^k. \tag{18}$$

Since $\cos\dfrac{\pi}{k+1}$ is the maximal eigenvalue of the quadratic form defined by $\sum\limits_{j=2}^{k} X_{j-1}X_j$, it follows that

$$\cos\frac{\pi}{k+1}\sum_{j=1}^{k}X_j^2 \geqslant \sum_{j=2}^{k}X_{j-1}X_j, \quad (X_1,\dots,X_k)\in\mathbb{R}^k.$$

Then all pairs $(a,b)\in\mathbb{R}^2$ for which $2ab\cos\dfrac{\pi}{k+1}\leqslant 1-a^2-b^2$ satisfy (18); in particular, we may choose $a,b>0$ such that $2ab\cos\dfrac{\pi}{k+1}\leqslant 1-a^2-b^2<2ab$, i.e. $a+b>1$. $\square$

Proof of Proposition 11. Observe that in this case $S=\{2j-1: j=1,2,\dots,k\}$ and the proof of Proposition 7 from [5] may be repeated. $\square$

Proof of Proposition 12. Ad (a). $\tilde{k}_A(p(\lambda),p(\mu))\leqslant p_{\mathbb{D}}(\lambda,\mu)$, since $p\in\mathcal{O}(\mathbb{D},A)$. From Lemma 2 we already know that for any $\varphi\in\mathcal{O}(\mathbb{D},A)$ with $\varphi(\tilde{\lambda})=p(\lambda)$ and $\varphi(\tilde{\mu})=p(\mu)$ there exists some $\psi\in\mathcal{O}(\mathbb{D},\mathbb{D})$ such that $\psi(\tilde{\lambda})=\lambda$ and $\psi(\tilde{\mu})=\mu$. Hence $p_{\mathbb{D}}(\lambda,\mu)\leqslant p_{\mathbb{D}}(\tilde{\lambda},\tilde{\mu})$. Taking infimum over all appropriate $\varphi\in\mathcal{O}(\mathbb{D},A)$ we obtain $p_{\mathbb{D}}(\lambda,\mu)\leqslant\tilde{k}_A(p(\lambda),p(\mu))$. Hence, $p_{\mathbb{D}}(\lambda,\mu)=\tilde{k}_A(p(\lambda),p(\mu))$. In particular, $\tilde{k}_A$ is a distance and, consequently, $\tilde{k}_A=k_A$.

Ad (b). Again, using Lemma 2, we obtain

$$\begin{aligned}
&\kappa_A(p(\lambda);p'(\lambda))\\
&\qquad=\inf\{\alpha>0:\exists_{\varphi\in\mathcal{O}(\mathbb{D},A)}:\varphi(0)=p(\lambda),\ \alpha\varphi'(0)=p'(\lambda)\}\\
&\qquad\geqslant\inf\{\alpha>0:\exists_{\psi\in\mathcal{O}(\mathbb{D},\mathbb{D})}:\psi(0)=\lambda,\ \alpha\psi'(0)=1\}\\
&\qquad=\kappa_{\mathbb{D}}(\lambda;1)=\gamma_{\mathbb{D}}(\lambda;1).
\end{aligned}$$

On the other hand, for $\varphi:=p\circ\psi$, where $\psi\in\operatorname{Aut}(\mathbb{D})$ is such that $\psi(0)=\lambda$, we have that $\varphi\in\mathcal{O}(\mathbb{D},A)$, $\varphi(0)=p(\lambda)$, and $\gamma_{\mathbb{D}}(\lambda;1)\varphi'(0)=p'(\lambda)$. Therefore $\kappa_A(p(\lambda);p'(\lambda))\leqslant\gamma_{\mathbb{D}}(\lambda;1)$.

It remains to prove formula for $\lambda = 0$. Observe that

$$\begin{aligned}\kappa_A(0;X) &= \inf\{\alpha > 0 : \exists_{\varphi\in\mathcal{O}(\mathbb{D},A)} : \varphi(0) = 0,\ \alpha\varphi'(0) = X\}\\ &\geqslant \inf\{\alpha > 0 : \exists_{\psi\in\mathcal{O}(\mathbb{D},\mathbb{D})} : \psi(0) = 0,\ \alpha p'(0)\psi'(0) = X\}\\ &= \begin{cases}|X_1| & \text{if } m_1 = 1\\ \infty & \text{if } m_1 > 1\end{cases}.\end{aligned}$$

It suffices to prove the opposite inequality in case $m_1 = 1$. Fix $X \in (T_0A)_*$. Then there exists $k \in \mathbb{N}$ such that $X_1 = \dots = X_k \neq 0$ and $X_{k+1} = \dots = X_n = 0$. We define $\varphi(\lambda) := p(X_1|X_1|^{-1}\lambda)$, $\lambda \in \mathbb{D}$. Observe that $\varphi \in \mathcal{O}(\mathbb{D}, A)$, $\varphi(0) = 0$, and $|X_1|\varphi'(0) = X$. Hence $\kappa_A(0;X) \leqslant |X_1|$ which ends the proof. $\square$

P r o o f o f P r o p o s i t i o n 13. A d (7). Let $\varphi_j(z) := z_j$, $z = (z_1,\dots,z_n) \in V_1$, and $\psi_j(\zeta) := \zeta e_j$, $\zeta \in \mathbb{D}$, for $j = 1,\dots,n$. Since $\varphi_j \in \mathcal{O}(V_1,\mathbb{D})$ and $\psi_j \in \mathcal{O}(\mathbb{D}, V_1)$, then

$$p_{\mathbb{D}}(\lambda,\mu) \leqslant c_{V_1}(\lambda e_j,\mu e_j) \leqslant \tilde{k}_{V_1}(\lambda e_j,\mu e_j) \leqslant p_{\mathbb{D}}(\lambda,\mu). \tag{19}$$

Now assume that $j \neq k$. Since $\varphi := \sum_{j=1}^n \varphi_j \in \mathcal{O}(V_1,\mathbb{D})$, then

$$p_{\mathbb{D}}(\lambda,0) + p_{\mathbb{D}}(0,\mu) = p_{\mathbb{D}}(|\lambda|,-|\mu|) \leqslant c_{V_1}(|\lambda|e_j,-|\mu|e_k) = c_{V_1}(\lambda e_j,\mu e_k).$$

Moreover, using (19),

$$k_{V_1}(\lambda e_j,\mu e_k) \leqslant \tilde{k}_{V_1}(\lambda e_j,0) + \tilde{k}_{V_1}(0,\mu e_k) = p_{\mathbb{D}}(\lambda,0) + p_{\mathbb{D}}(0,\mu).$$

A d (8). It remains to consider the case $j \neq k$, $\lambda\mu \neq 0$. Suppose there is a disc $\psi \in \mathcal{O}(\mathbb{D}, V_1)$ such that $\psi(\zeta) = \lambda e_j$ and $\psi(\xi) = \mu e_k$ for some $\zeta,\xi \in \mathbb{D}$. However, these equalities imply, together with the identity principle, that $\psi \equiv 0$; a contradiction, since $\lambda\mu \neq 0$.

A d (9). Using again the functions φ_j and ψ_j, $j = 1,\dots,n$, defined in the part of the proof of (7), we obtain

$$\gamma_{\mathbb{D}}(\lambda;1) \leqslant \gamma_{V_1}(\lambda e_j;e_j) \leqslant \kappa_{V_1}(\lambda e_j;e_j) \leqslant \gamma_{\mathbb{D}}(\lambda;1).$$

A d (10). For $X = (X_1,\dots,X_n) \in \mathbb{C}^n$ let $\varphi_X(z) := \sum_{j=1}^n z_j e^{-i\operatorname{Arg} X_j}$, where $z = (z_1,\dots,z_n) \in V_1$. Since $\varphi \in \mathcal{O}(V_1,\mathbb{D})$, then

$$\sum_{j=1}^n |X_j| = \gamma_{\mathbb{D}}(\varphi_X(0);\varphi_X'(0)X) \leqslant \gamma_{V_1}(0;X).$$

Recall now that

$$\mathcal{O}(V_1,\mathbb{D}) = \Big\{ \sum_{j=1}^{n} f_j - (n-1)f_1(0) :$$

$$f_j \in \mathcal{O}(\mathbb{D}e_j,\mathbb{D})\ , f_j(0) = f_k(0),\ j,k = 1,\dots,n\Big\}.$$

Therefore

$$\gamma_{V_1}(0;X) = \sup\{\gamma_{\mathbb{D}}(F(0);F'(0)X) : F \in \mathcal{O}(V_1,\mathbb{D})\}$$

$$\leqslant \sum_{j=1}^{n} \sup\{\gamma_{\mathbb{D}}(f_j(0);f_j'(0)X_j) : f_j \in \mathcal{O}(\mathbb{D},\mathbb{D})\} = \sum_{j=1}^{n} |X_j|.$$

Ad (11). Assume that $X = X_je_j$. Define $\psi_{j,X}(\zeta) = \zeta e_j e^{i \operatorname{Arg} X_j}, \zeta \in \mathbb{D}$. Observe that $\psi_{j,X} \in \mathcal{O}(\mathbb{D},V_1)$, $\psi_{j,X}(0) = 0$ and $|X_j|\psi_{j,X}'(0) = X$. Hence $\kappa_{V_1}(0;X) \leqslant |X_j|$.

To prove the opposite inequality observe that for any $\psi \in \mathcal{O}(\mathbb{D},V_1)$ there exist j and $f \in \mathcal{O}(\mathbb{D},\mathbb{D})$ such that $\psi = fe_j$. Hence

$$\kappa_{V_1}(0;X) = \inf\{\alpha > 0 : \exists_{\psi\in\mathcal{O}(\mathbb{D},V_1)} : \psi(0) = 0,\ \alpha\psi'(0) = X\}$$

$$\geqslant \inf\{\alpha > 0 : \exists_{f\in\mathcal{O}(\mathbb{D},\mathbb{D})} : f(0) = 0,\ \alpha f'(0) = X_j\} = |X_j|.$$

Now assume that X is not of the form X_je_j for some $j = 1,\dots,n$. Then there are $X_j \neq 0 \neq X_k$ for some $j \neq k$. Suppose there is a disc $\psi \in \mathcal{O}(\mathbb{D},V_1)$ such that $\alpha\psi'(0) = X$ for some $\alpha > 0$. This, however, implies that $\psi_j \neq \text{const}$ and $\psi_k \neq \text{const}$; a contradiction, since $j \neq k$. $\square$

Acknowledgments. Author would like to thank N. Nikolov for helpful discussion, especially on the proof of Lemma 10.

REFERENCES

[1] J. B . Garnett. Bounded Analytic Functions. *Pure and Appl. Math.* **96**, Academic Press, New York, 1981.

[2] M. Jarnicki, P. Pflug. Invariant Distances and Metrics in Complex Analysis. de Gruyter Exp. Math. 9, de Gruyter, Berlin, 1993.

[3] M. Jarnicki, P. Pflug. Invariant distances and metrics in complex analysis—revisited. *Dissertationes Math.* **430** (2005), 1–192.

[4] G. Knese. Function theory on the Neile parabola. *Michigan Math. J.* **55** (2007), 139–154.

[5] N. Nikolov, P. Pflug. Invariant metrics and distances on generalized Neil parabolas. Preprint, 2006, arXiv:math.CV/0603574.

Instytut Matematyki
Uniwersytet Jagielloński
Reymonta 4
30-059 Kraków, Poland
e-mail: `Pawel.Zapalowski@im.uj.edu.pl`

Received November 3, 2006
Revised December 19, 2006