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COMPLEX ANALOGUES OF THE ROLLE'S THEOREM

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Abstract. Classical Rolle's theorem and its analogues for complex algebraic polynomials are discussed. A complex Rolle's theorem is conjectured.

1. Introduction. The classical theorem of Rolle states that if $p(x)$ is a real polynomial, a, b are two different real numbers, $a < b$, and $p(a) = p(b)$, then there exists $\xi \in (a, b)$, such that $p'(\xi) = 0$. As linear transformations of the complex plane do not change the geometric relations between the zeros and the critical points of a polynomial, we may consider only the points $a = -1, b = 1$. There are many statements that are considered refinements of the classical Rolle theorem. Every such a refinement has the following structure:
Let $\mathcal{K}_n$ be the class of real polynomials $p(x)$ of degree n, $n \geq 2$, with $p(-1) = p(1)$ and $\alpha_n > 0$. Then every $p \in \mathcal{K}_n$ has at least one critical point in the interval $(-1 + \alpha_n, 1 - \alpha_n)$.
There are several refinements of Rolle's theorem in [1, pp. 203-208]. One of them is the classical Lagguerre-Cesàro Theorem 6.5.1.

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Theorem 1 (Lagguerre-Cesàro). *If $p(x)$ is a polynomial of degree $n \geq 2$ with only real zeros and $a = -1, b = 1$ are two consecutive zeros of $p(x)$, then at least one zero of $p'(x)$ is in the segment $[-1+2/n, 1-2/n]$. The segment $[-1+2/n, 1-2/n]$ is the smallest segment with this property.*

It is natural to consider the case when $\mathcal{K}_\infty = \bigcup_{n=1}^{\infty} \mathcal{K}_n$ is the set of all real polynomials with $p(-1) = p(1)$. This case was solved by Lubomir Tschakaloff [2], a leading Bulgarian mathematician from the first half of the last century.

Theorem 2 (L. Tschakaloff). *Let α_m be the biggest zero of the Legendre polynomial of degree m, see (20). If $p(x)$ is a real polynomial of degree $n \leq 2m$ and $p(-1) = p(1)$, then at least one zero of $p'(x)$ is in the open interval $(-\alpha_m, \alpha_m)$ for $n > 3$ and in the closed interval $[\alpha_2, \alpha_2]$ for $n = 3$. If $n = 2$, the single zero of p is $\alpha_1 = 0$. Moreover, for every $0 \leq \beta_m < \alpha_m$, there exists a polynomial of degree $n \leq 2m$ without zeros in the closet interval $[-\beta_m, \beta_m]$.*

As this result of Tschakaloff is missing in the basic reference book [1], it will be presented at the and of this paper.

1.1. Complex Rolle's theorem. An analogue of Rolle's theorem for complex polynomials must have the following structure:
Let Ω be a subset of the complex plane $\mathcal{C}$. If $p(z)$ is a complex polynomial with $p(-1) = p(1)$, then there exists $\zeta \in \Omega$, such that $p'(\zeta) = 0$.
Call such a domain Ω, a **Rolle's domain**. The smallest Rolle's domain is denoted by R. As the distances between the zeros and the critical points of a polynomial, and the relation $p(-1) = p(1)$ do not change by the transformations $z \Rightarrow -z$ and $z \Rightarrow \overline{z}$, we consider only domains Ω, which are symmetric with respect to both the real and the imaginary axis. We do not know much about the smallest Rolle domain R. It follows from Theorem 4 below that every Rolle domain obeys

$$\Omega = \mathcal{C} \setminus \{x : \ x \in (-\infty, -1) \cup (1, \infty)\} \supset R.$$

In this paper we conjecture that

$$R = \left\{ z : \ |Im(z)| > \frac{1}{\pi} \right\} \cup \{z : \ |z| < 1\}$$

and prove the inclusion

$$R \supset \left\{ z : \ |Im(z)| > \frac{1}{\pi} \right\} \cup \{z : \ |z| < 1\}.$$

1.2. Refinements of complex Rolle's theorem. A refinement of the complex Rolle's theorem has the following structure: *For every natural $n \geq 2$,* *let K_n be the class of complex polynomials of degree n with $p(-1) = p(1)$ and Ω_n be a subset of the complex plane. If $p \in K_n$, then there exists $\zeta \in \Omega_n$, such that $p'(\zeta) = 0$.* In the literature a theorem is usually called an "analogue of Rolle's theorem for complex polynomials", when in fact it is a refinement of the Rolle theorem. The reason may be that nontrivial complex Rolle's theorem does not exist. The book of Q. I. Rahman and G. Schmeisser [1] contains several refinements of the complex Rolle theorem. The most famous one is the Grace-Heawood theorem [1, p. 126].

Theorem 3 (Grace-Heawood). *If p is a polynomial of degree $n \geq 2$ and $p(-1) = p(1)$, then there exists*

$$\zeta \in D\left(0; \cot\frac{\pi}{n}\right) = \left\{z:\ |z| \leq \cot\frac{\pi}{n}\right\},$$

such that $p'(\zeta) = 0$.

Another refinement of the complex Rolle theorem is the following:

Theorem 4([1, Theorem 4.3.4, p. 128]). *If p is a polynomial of degree $n \geq 2$ and $p(-1) = p(1)$, then there exists*

$$\zeta \in D\left(-i\cot\frac{\pi}{n-1}; \sin^{-1}\frac{\pi}{n-1}\right) \cup D\left(i\cot\frac{\pi}{n-1}; \sin^{-1}\frac{\pi}{n-1}\right).$$

such that $p'(\zeta) = 0$.

Definition 1. *For every natural number $n > 2$, let R_n be the smallest domain, such that, for every polynomial $p(z)$ of degree n with $p(-1) = p(1)$, there exists $\zeta \in R_n$, for which $p'(\zeta) = 0$.*

It is easy to verify that

$$R_n \subset R_{n+1} \tag{1}$$

and

$$R = \cup_{n=2}^{\infty} R_n. \tag{2}$$

The problem to determine R_n for every natural n was formulated by L. Tschakaloff [3].

2. The domains R_n.

In this section we define disks, which belong to R_n.

Definiton 2. *Call a polynomial $p(z)$ of degree n with $p(-1) = p(1)$* **extremal** *for R_n if $p(z)$ has no critical points inside R_n.*

Let

$$p'(z) = (z-z_1)(z-z_2)\cdots(z-z_{n-1}) = \sum_{k=0}^{n-1}(-1)^{n-k-1}S_{n-1,n-k-1}(z_1, z_2, \ldots, z_{n-1})z^k,$$

where $S_{n-1,k}(z_1, z_2, \ldots, z_{n-1}),\ k = 0, 1, \ldots, n-1$, are the elementary symmetric functions of degree k of the numbers $z_1, z_2, \ldots, z_{n-1}$ and $S_{n-1,0}(z_1, z_2, \ldots, z_{n-1}) = 1$. The condition $p(-1) = p(1)$ is equivalent to the equation

$$\sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1} S_{n-1,n-2k-1}(z_1, z_2, \ldots, z_{n-1}) = 0. \tag{3}$$

The fact that the expression on the left-hand side of (3) is linear in respect to each critical point of $p(z)$ yields:

Statement 1. *A necessary and sufficient condition for the polynomial $p(z)$ to be extremal for R_n, is that all critical points of $p(z)$ are on the boundary of R_n.*

It follows from Theorem 3 that the point $z = i\nu_n,\ \nu = \cot(\pi/n)$ is on the boundary of R_n and the polynomial

$$g_n(z) = \int_1^z (u - \nu_n i)^{n-1}\, du \tag{4}$$

is extremal for R_n. Extremal is also the polynomial

$$g_n^*(z) = \int_1^z (u + \nu_n i)^{n-1}\, du,$$

and the segment with the end points $\nu_n i$ and $-\nu_n i$ is the diameter of R_n over the imaginary axis. Setting

$$z_1 = -\overline{z_2} = a + bi, \quad z_3 = z_4 = \cdots = z_{n-1} = \nu_n i,$$

in (3), we obtain

$$\sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{2k+1} \left[\binom{n-3}{2k-2}\nu_n^2 + 2\binom{n-3}{2k-1} b\nu_n + \binom{n-3}{2k}(a^2+b^2) \right] \nu_n^{-2k} = 0. \tag{5}$$

Here and in what follows we set $\binom{n}{k} := 0$ whenever either $k < 0$ or $k > n$. Let

$$\begin{aligned} A_{n-3}(\varphi) &= \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{2k+1}\binom{n-3}{2k}(\tan\varphi)^{2k}, \\ B_{n-3}(\varphi) &= \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{2k+1}\binom{n-3}{2k-1}(\tan\varphi)^{2k}, \\ C_{n-3}(\varphi) &= \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{2k+1}\binom{n-3}{2k-2}(\tan\varphi)^{2k}. \end{aligned}$$

The equality (5) may be represented in the form

(6) $$a^2 + (b - c_n)^2 = r_n^2,$$

with

$$c_n = -\nu_n \frac{B_{n-3}(\pi/n)}{A_{n-3}(\pi/n)}.$$

Since the polynomial $g_n(z)$, defined by (4), is extremal for R_n, then the circumference (6) passes through $i\nu_n$. Thus, $r_n = \nu_n - c_n$. It is easy to see that

$$A_n(\varphi) = \frac{\sin(n+1)\varphi}{(n+1)\sin\varphi\cos^n\varphi}.$$

Hence, setting $\varphi = \pi/n$ in the latter, we obtain

(7)
$$A_{n-3}(\pi/n) = \frac{2}{n-2}\cos^{4-n}\frac{\pi}{n}, \quad A_{n-2}(\pi/n) = \frac{1}{n-1}\cos^{2-n}\frac{\pi}{n}, \quad A_{n-1}(\pi/n) = 0.$$

On the other hand, the binomial identity

$$\binom{n-3}{2k-1} = \binom{n-2}{2k} - \binom{n-3}{2k}$$

yields

(8) $$B_{n-3}(\varphi) = A_{n-2}(\varphi) - A_{n-3}(\varphi).$$

Setting $\varphi = \pi/n$ in this identity and using (7), we obtain

$$B_{n-3}(\pi/n) = -\frac{1 + (n-1)\cos\frac{2\pi}{n}}{(n-1)(n-2)\cos^2\frac{\pi}{n}}.$$

Finally, we obtain

$$r_n = \frac{n-2}{n-1}\,\frac{1}{\sin(2\pi/n)}, \qquad c_n = \cot\frac{\pi}{n} - r_n. \tag{9}$$

Thus, we have proved the following:

Statement 2. *Let c_n and r_n be defined by* (9). *Then*

$$D(-ic_n; r_n) \cup D(ic_n; r_n) \subset R_n.$$

Now we study the diameter of R_n over the real axis. According to Theorem 4, this diameter is included in the segment $[-1, 1]$. Consider the polynomial $p(z)$ with $p'(z) = (z+a)(z-a)^{n-2}$, where a is real. The condition $p(-1) = p(1)$ is equivalent to

$$\left(\frac{a-1}{a+1}\right)^{n-1} = \frac{(n+1)a - n + 1}{(n+1)a + n - 1}. \tag{10}$$

Equation (10) has only one real positive root a_n. Moreover,

$$a_n = 1 - \frac{2}{n+1} + O(n^{-n+1}) \quad \text{and} \quad \lim_{n\to\infty} a_n = 1. \tag{11}$$

The polynomial $f(z)$ with $f'(z) = (z + a_n)(z - a_n)^{n-2}$ is probably extremal in R_n. This is part of the Conjecture 1. Next we consider the polynomial $q(z)$ with

$$q'(z) = (z + a_n)(z - u)(z - \overline{u})(z - a_n)^{n-4}, \tag{12}$$

where $u = x + iy$ and $|u|^2 = x^2 + y^2 = a_n^2$. The condition $q(-1) = q(1)$ can be represented as

$$(x - d_n)^2 + y^2 = \rho_n^2,$$

where

$$d_n = \frac{V_n}{U_n}, \qquad \rho_n = a_n - \frac{V_n}{U_n}, \tag{13}$$

and

$$U_n = \int_{-1}^{1} (z + a_n)(z - a_n)^{n-4}\,dz, \qquad V_n = \int_{-1}^{1} z(z + a_n)(z - a_n)^{n-4}\,dz.$$

Calculating these integrals explicitly and having in mind (11), we obtain

$$\lim_{n\to\infty}\frac{V_n}{U_n}=0. \tag{14}$$

We may formulate the following:

Statement 3. *For d_n and ρ_n defined by (13), we have*

$$D(-d_n;\rho_n)\cup D(d_n;\rho_n)\subset R_n.$$

Conjecture 1. *For every natural $n\geq 2$, the equality*

$$R_n=D(-ic_n;r_n)\cup D(ic_n;r_n)\cup D(-d_n;\rho_n)\cup D(d_n;\rho_n)$$

holds.

3. Proof of Conjecture 1 for small n. **For** $n=2$, Conjecture 1 is trivial. **For** $n=3$, from (3), we get $z_1z_2+1/3=0$, or $R_3=D(0;1/\sqrt{3})$. The result coincide with this of Grace-Heawood theorem. Observe, that from Theorem 2 follows, that the smallest Rolle's interval for real polynomials is $(-1/\sqrt{3},1/\sqrt{3})$, the diameter of R_3. **For** $n=4$, from (3), we have

$$z_1z_2z_3+\frac{1}{3}(z_1+z_2+z_3)=0. \tag{15}$$

In what follow, we denote by $DD\,(i\alpha;r)$ the union of he disks $D\,(i\alpha;r)$ and $D\,(-i\alpha;r)$.

Theorem 5. *With this notation, we have $R_4=DD\,(i/3;2/3)$.*

P r o o f. It follows from Statement 2 that $R_4\supset DD\,(i/3;2/3)$. To prove the inclusion $R_4\subset DD\,(i/3;2/3)$, suppose exist $z_1,z_2,z_3\notin DD\,(-1/3;2/3)$, that is,

$$\left|z_k-i\frac{\varepsilon_k}{3}\right|>\frac{2}{3};\quad k=1,2,3,$$

where $\varepsilon_k=\pm1;\ k=1,2,3,$ that obey equality (15). Since every such z_k is nonzero, it is equivalent to the fact that there are complex numbers $\zeta_k=1/z_k,\ k=1,2,3$ such that

$$\zeta_k\in\Upsilon:=D(i,2)\cap D(-i,2),\ k=1,2,3,$$

and satisfy $\zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1 = -3$. The latter equality is equivalent to

$$\frac{\zeta_3 - \sqrt{3}}{\zeta_3 + \sqrt{3}} = \frac{\zeta_1 + \sqrt{3}}{\zeta_1 - \sqrt{3}} \frac{\zeta_2 + \sqrt{3}}{\zeta_2 - \sqrt{3}}. \tag{16}$$

Since the Möbius transformations $w = (z - \sqrt{3})/(z + \sqrt{3})$ and $w = (z - \sqrt{3})/(z + \sqrt{3})$ both take the domain Υ onto the angular domain $\Delta := \{w \ : \ |\arg w - \pi| < \pi/3\}$ and the products of any two complex number from Δ lie outside Δ, we conclude that (16) cannot hold. We have already proved that

$$DD(ic_n; r_n) = R_n \tag{17}$$

for $n = 2, 3, 4$. The relation (17) is not true for $n \geq 5$. In Table 1, the values of c_n, r_n and l_n for several n are listed, where $[-l_n, l_n]$ is the segment of the real axis in $DD(c_n; r_n)$.

n	c_n	r_n	l_n
2	0	0	0
3	0	$1/\sqrt{3}$	$1/\sqrt{3}$
4	$1/3$	$2/3$	$1/\sqrt{3} = 0.5773\ldots$
5	$0.58778\ldots$	$0.78859\ldots$	$0.5257\ldots$
6	$7\sqrt{3}/15 = 0.80829\ldots$	$8\sqrt{3}/15 = 0.92376\ldots$	$1\sqrt{5} = 0.4472\ldots$
7	$1.01064\ldots$	$1.06587\ldots$	$0.33865\ldots$
8	$1 + \sqrt{2}/7 = 1.20203\ldots$	$6\sqrt{2}/7 = 1.212183\ldots$	$0.14655\ldots$
9	$1.4260\ldots$	$1.3612\ldots$	–
100	$15.79\ldots$	$15.65\ldots$	–
1000	$159.31\ldots$	$158.99\ldots$	–
10000	$1591.7083\ldots$	$1591.3903\ldots$	–

Table 1

From Table 1 we have that $l_4 > l_5$, hence for $n \geq 5$, the domain $DD(ic_n; r_n)$ is strictly smaller than R_n. Observe that for $n \geq 9$ the double disk $DD(ic_n; r_n)$ consists of two disjoint disks. In Table 2, the values of a_n, d_n and ρ_n for several n are listed.

n	a_n	d_n	ρ_n
5	0.66874030...	0	0.66874030...
6	0.71410133...	0.23803378...	0.47606755...
7	0.75001275...	0.16096471...	0.58904804...
8	0.77777704...	0.14345435...	0.63432269...
9	0.80000004...	0.12495000...	0.67505004...
10	0.81818182...	0.11111357...	0.70706725...
100	0.98019802...	0.01010101...	0.97009701...
1000	0.99800200...	0.00119796...	0.99700106...

Table 2

4. The domain *R*. From (9), we have

(18) $$c_n - r_n = \frac{2}{n-1}\sin^{-1}\frac{2\pi}{n} - \tan\frac{\pi}{n} < \frac{1}{\pi} \quad \text{and} \quad \lim_{n\to\infty}(c_n - r_n) = \frac{1}{\pi}.$$

It follows from (18):

Statement 4. *The inclusion*

$$I_\pi = \left\{z:\ |Im(z)| > \pi^{-1}\right\} \subset R$$

holds.

Let $(\gamma_n, 1/\pi)$ be a point of intersection of the circle $x^2 + (y - c_n)^2 = r_n^2$ with the line $y = 1/\pi$. From (9) we calculate that

(19) $$\gamma_n < \gamma_{n+1} \quad \text{and} \quad \lim_{n\to\infty}\gamma_n = \sqrt{1-\pi^{-2}}.$$

Observe that $\left(\sqrt{1-\pi^{-2}}, 1/\pi\right)$ is also a point of intersection of the circle $x^2+y^2 = 1$ with the line $y = 1/\pi$. Consider the polynomial

$$p(z) = (z - e^{i\varphi})(z+1)(z-1)^{n-2}.$$

The critical points of this polynomial are the zeros $z_1^{(n)}$, $z_2^{(n)}$ of the polynomial

$$z^2 + \left(\frac{n-3}{n} - \frac{n-1}{n}e^{i\varphi}\right)z - \frac{1}{n} - \frac{n-3}{n}e^{i\varphi}$$

and $z_3 = z_4 = \cdots = z_{n-1} = 1$. As

$$\lim_{n\to\infty} z_1^{(n)} = e^{i\varphi}, \quad \lim_{n\to\infty} z_2^{(n)} = -1,$$

we obtain:

Statement 5. *The open disk $D_1 = \{z : |z| < 1\}$ belongs to R.*

Statements 4 and 5 imply that $R \supset I_\pi \cup D_1$.

Conjecture 2 (Rolle's theorem for complex polynomials). *If $p(z)$ is a complex polynomial and $p(-1) = p(1)$, then at least one critical point of $p(z)$ is in the domain $I_\pi \cup D_1$ and $I_\pi \cup D_1$ is the smallest domain with this property, i. e., $R = I_\pi \cup D_1$.*

We formulate a possible generalization of Theorem 1 (Lagguerre-Cesàro):

Theorem 6. *If $p(x)$ is a polynomial of degree $n \geq 2$ with at most one non real zero and $p(-1) = p(1)$, then at least one zero of $p'(x)$ is in the disk $D(0, 1-2/n)$. The disk $D(0, 1-2/n)$ is the smallest segment with this property.*

5. A theorem of L. Tschakaloff. L. Tschakaloff [2] studied a more general problem, but we shall consider only the case of the Rolle theorem for real polynomials. Let

$$P_0(z) = 1, \;\; P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m; \quad m = 1, 2, \ldots \tag{20}$$

be the Legendre polynomials that are orthogonal on the interval $[-1, 1]$. Then, for every real polynomial $p(x)$ of degree $< m$, we have

$$\int_{-1}^{1} p(x) P_m(x)\, dx = 0. \tag{21}$$

Let $x_{m,1} < x_{m,2} < \cdots < x_{m,m} = \alpha_n$ be the zeros of $P_m(x)$. It is known that they are real, distinct, all belong to $(-1, 1)$, and are symmetric with respect to the origin,

$$x_{m,k} = -x_{m,m-k+1}; \;\; k = 1, 2, \ldots, m. \tag{22}$$

Moreover, the zeros of two consecutive Legendre polynomials interlace. In particular, we have

$$\alpha_1 = 0 < \alpha_2 = \frac{1}{\sqrt{3}} < \alpha_3 = \sqrt{\frac{3}{5}} < \alpha_4 < \cdots < 1. \tag{23}$$

P r o o f o f t h e t h e o r e m o f T s c h a k a l o f f. First we prove the first statement of the theorem. Recall that the Gaussian quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=1}^{m} A_{m,k} f(x_{m,k}) \tag{24}$$

has nodes at the zeros of $P_m(x)$ and is precise for every real polynomial of degree $2m-1$. Moreover the Cotes numbers $A_{m,k}$ are all positive and symmetric, $A_{m,k} = A_{m,m-k+1}$. Thus, if $p(x)$ is any real polynomial of degree $2m$ with $p(-1) = p(1)$, applying (24) to $p'(x)$, we obtain

$$0 = \int_{-1}^{1} p'(x)\,dx = \sum_{k=1}^{m} A_{m,k} p'(x_{m,k}).$$

Therefore, the convex combination of $p'(x_{m,k}),\ k = 1,\dots,m$, is equal to zero in either of the cases:

- $p'(x_{m,k}) = 0$ for every $k = 1,\dots,m$;
- $m \geq 2$, there exist indexes $i < j$ such that $p'(x_{m,i})p'(x_{m,j}) < 0$ and thus there is $\xi \in (x_{m,i}, x_{m,j})$ with $p'(\xi) = 0$.

In order to prove that $(x_{m,1}, x_{m,m})$ is the smallest interval that contain a zero of $p'(x)$, we investigate some specific polynomials. For even $n = 2m$, consider the polynomial $p(x)$ with

$$p'(x) = (x-\xi)\left[C + \frac{P_m^2(x)}{(x-\alpha_m)^2}\right], \quad C > 0.$$

This polynomial has only one real critical point, equal to ξ. From the condition $p(-1) = p(1)$ and (21), we get

$$-2\xi C + (\alpha_m - \xi)\int_{-1}^{1} \frac{P_m^2(x)}{(x-\alpha_m)^2}\,dt = 0.$$

The proof of the theorem for $n = 2m$ follows from the last equality as it holds if and only if $\xi \in (-\alpha_m, \alpha_m)$. For even $n = 2m-1$, consider the polynomial $p(x)$ with

$$p'(x) = (x-\xi)\left[C + \frac{P_m^2(x)}{(x-\alpha_1)(x-\alpha_m)^2}\right], \quad C > 0.$$

This polynomial has only one real critical point ξ provided C is sufficiently large. The conditions $p(-1) = p(1)$ and (21) imply

$$-2\xi C + (\alpha_m - \xi) \int_{-1}^{1} \frac{P_m^2(x)}{(x-\alpha_1)(x-\alpha_m)^2}\, dt = 0.$$

The proof for $n = 2m - 1$ follows from the latter equality as it is possible if and only if $\xi \in (-\alpha_m, \alpha_m)$. $\square$

If in Theorem 1 is drooped the condition that -1 and 1 are two consecutive zeros of $p(z)$, then $2/n$ may be replaced by a smaller number. We formulate, without proof:

Theorem 7. *If $p(x)$ is a polynomial of degree $n \geq 2$ with only real zeros and $p(-1) = p(1)$, then at least one zero of $p'(x)$ is in the segment $[-a_n, a_n]$, where a_n is the zero of (10) . The segment $[-a_n, a_n]$ is the smallest segment with this property.*

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