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A GENERAL APPROACH TO METHODS WITH A SPARSE JACOBIAN FOR SOLVING NONLINEAR SYSTEMS OF EQUATIONS

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Abstract. Here we give methodological survey of contemporary methods for solving nonlinear systems of equations in R^n. The reason of this review is that many authors in present days rediscovered such classical methods. In particular, we consider Newton's-type algorithms with sparse Jacobian. Method for which the inverse matrix of the Jacobian is replaced by the inverse matrix of the Vandermondian is proposed. A number of illustrative numerical examples are displayed. We demonstrate Herzberger's model with fixed-point relations to the some discrete versions of Halley's and Euler-Chebyshev's methods for solving such kind of systems.

1. Introduction. The multivariant Newton's method for solving equation

$$f(x)=0, \quad f=(f_1,\dots,f_n), \quad x=(x_1,\dots,x_n) \tag{1}$$

is described by the iteration formula

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$$x^{k+1} = x^k - \left(f'(x)\right)^{-1} f(x^k), \quad k = 0, 1, 2, \dots, \tag{2}$$

where

$$f'(x) = J(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right), \quad i, j = 1, 2, \dots, n,$$

and suppose that $f'(x)^{-1}$ exists in neighbourhood of solution $\xi(\xi_1, \xi_2, \dots, \xi_n)$.

In a number of cases, the calculation of the Jacobian in (2) is quite difficult. Readers interested in some modifications of Newton's-type iterative schemes are referred to the books by Ortega and Rheinboldt [13] and by Petkov and Kyurkchiev [15].

Let $\theta_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ solves

$$f_i(x_1, \dots, x_{i-1}, \theta_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) = 0, \quad i = 1, \dots, n. \tag{3}$$

Following Herzberger [5], we present a model for a class of iterative methods for the determination of the roots $\xi_1, \dots, \xi_n$ of a nonlinear system defined by some based fixed-point relations

$$\varphi_i(x_1, \dots, x_{i-1}, \xi_i, x_{i+1}, \dots, x_n) = \xi_i, \quad i = 1, \dots, n, \tag{4}$$

i.e. the i-th iteration function is stationary for $x_i = \xi_i$.

Our last assumption restrict the methods described here in our model to a class of simultaneous methods by demanding

$$\varphi_i(\xi_1, \dots, \xi_{i-1}, \theta_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \xi_{i+1}, \dots, \xi_n) = \theta_i, \quad i = 1, \dots, n. \tag{5}$$

We demonstrate our model for the method

$$\varphi_i(x_1, \dots, x_n) = x_i - \frac{f_i(x_1, \dots, x_n)}{\frac{\partial f_i(x_1, \dots, x_n)}{\partial x_i}} \tag{6}$$

and define the iteration methods by

$$x_i^{k+1} = \varphi_i(x_1^k, \dots, x_n^k) \qquad \text{(total-step method)}, \tag{7}$$

$$x_i^{k+1} = \varphi_i(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x_i^k, \dots, x_n^k) \qquad \text{(single-step method)} \tag{8}$$

for given initial values $x_1^0, \dots, x_n^0$.

2. Main results. We prove the following theorem:

Theorem 1. *For every member of the class of iteration methods defined by properties* (4) *and* (5) *the convergence of the total-step iteration* (7) *is locally linear.*

Proof. In order to estimate the order of convergence for the methods defined by the iteration functions φ_i, we derive an error recursion for these functions.

By virtue of (4), (5), we have

$$\begin{aligned} &\varphi_i(\xi_1,\ldots,\xi_n) = \xi_i, \quad i = 1,\ldots,n \\ &\frac{\partial\varphi_i(\xi_1,\ldots,\xi_n)}{\partial x_i} = 0, \quad i = 1,\ldots,n \\ &\frac{\partial\varphi_i(\xi_1,\ldots,\xi_n)}{\partial x_j} = \frac{\partial\theta_i(\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_n)}{\partial x_j}, \quad j \neq i. \end{aligned} \tag{9}$$

We expand φ_i in a Taylor series at the point $(\xi_1,\ldots,\xi_n)$

$$\varphi_i(\xi_1+\epsilon_1,\ldots,\xi_n^*+\epsilon_n) = \varphi_i(\xi_1,\ldots,\xi_n) + \sum_{j=1}^{n}\frac{\partial\varphi_i}{\partial x_j}(\xi_1+\eta\epsilon_1,\ldots,\xi_n+\eta\epsilon_n)\epsilon_j, \quad 0<\eta<1.$$

On the other hand

$$\frac{\partial\varphi_i}{\partial x_j}(\xi_1+\eta\epsilon_1,\ldots,\xi_n+\eta\epsilon_n) = \frac{\partial\varphi_i}{\partial x_j}(\xi_1,\ldots,\xi_n) + \sum_{l=1}^{n}\frac{\partial^2\varphi_i}{\partial x_j\partial x_l}(\xi_1+\sigma\epsilon_1,\ldots,\xi_n+\sigma\epsilon_n)\eta\epsilon_l, \quad 0<\sigma<1$$

and

$$\begin{aligned}\varphi_i(\xi_1+\epsilon_1,\ldots,\xi_n+\epsilon_n) = {} & \varphi_i(\xi_1,\ldots,\xi_n^)) + \frac{\partial\varphi_i}{\partial x_i}(\xi_1,\ldots,\xi_n)\epsilon_i + \\ & + \sum_{j\neq i}^{n}\frac{\partial\varphi_i}{\partial x_j}(\xi_1,\ldots,\xi_n)\epsilon_j + \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\partial^2\varphi_i}{\partial x_j\partial x_l}(\xi_1+\sigma\epsilon_1,\ldots,\xi_n+\sigma\epsilon_n)\eta\epsilon_l\epsilon_j.\end{aligned}$$

Setting $\epsilon_i = x_i^k - \xi_i$, $e_i^k = |x_i^k - \xi_i|$ we get from (9) in view of the total-step method (7) the error recursion:

$$
\begin{aligned}
e_i^{k+1} &= |x_i^{k+1} - \xi_i| = |\varphi_i(x_1^k, \dots, x_n^k) - \xi_i| = \\
&= |\varphi_i(\xi_1 + \epsilon_1, \dots, \xi_n + \epsilon_n) - \xi_i| \le \\
&\le \sum_{j \ne i}^{n} \left| \frac{\partial \theta_i}{\partial x_j}(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n) \right| e_j^k + \\
&\quad + \sum_{j=1}^{n} \sum_{l=1}^{n} \left| \frac{\partial^2 \varphi_i}{\partial x_j \partial x_l}(\xi_1 + \sigma\epsilon_1, \dots, \xi_n + \sigma\epsilon_n) \right| \eta e_j^k e_l^k.
\end{aligned} \tag{10}
$$

Obviously from this recursion that the convergence is linear.

Remark 1. Some methods, known from the literature occur as special cases on the family (4) ((9)).

In [22], Wegge proposed method (6). Here the inverse matrix of the Jacobian in well-known Newton's method (2) is replaced by the inverse of a matrix which consists of the main diagonal of the Jacobian.

In this case G. Iliev [8] shows that $\{x_i^k\}$ from (7) converges quadratic, when in (9)

$$
\frac{\partial \theta_i(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n)}{\partial x_j} = 0; \quad i, j = 1, \dots, n; \quad j \ne i.
$$

This result follows immediately from our recursion (10).

Other results can be found in the papers by Scheurle [17], Schubert [18] and Taiwo [21].

Extensions and applications in Banach spaces were given by Kantorovič [11], Altman [2], Stein [20], Ben-Israel [3], Yamamoto and Chen [23], Iliev [7] and others.

Note that (6) is a Jacoby-Newton-like iteration, but it possible to use a Gauss-Seidel procedure as follows:

$$
\begin{aligned}
x_i^{k+1} &= \varphi_i(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x_i^k, \dots, x_n^k) \\
&= x_i^k - \frac{f_i}{\frac{\partial f_i}{\partial x_i}}(x_1^{k+1}, \dots, x_{i-1}^{k+1}, x_i^k, \dots, x_n^k), \\
&\qquad i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots.
\end{aligned} \tag{11}
$$

For other SOR (successive overrelaxation) methods with parameter see the paper by Ishihara, Muroya and Yamamoto [9].

We give some discrete versions of Halley's and Euler-Chebyshev's methods for solving nonlinear systems of equations.

First, we define the following iteration (total-step Halley's method (HM)):

$$x_i^{k+1} = x_i^k - \frac{f_i(x^k)}{\frac{\partial f_i(x^k)}{\partial x_i}} \left(1 + \frac{1}{2} \frac{f_i(x^k) \frac{\partial^2 f_i(x^k)}{\partial x_i^2}}{\left(\frac{\partial f_i(x^k)}{\partial x_i} \right)^2} \right)^{-1}, \tag{12}$$

$$i = 1, \ldots, n; \quad k = 0, 1, \ldots.$$

This method obviously is improvement of the method (6).

We note that, in (12) we use the main diagonal of the Hessean

$$H := \left\| f_i^{mn} = \frac{\partial^2 f_i}{\partial x_m \partial x_n} \right\|.$$

There are many methods based on the fixed-point relations (see, the books by Alefeld and Herzberger [1], Petkovic [16], Sendov, Andreev and Kyurkchiev [19]) and Kyurkchiev [11].

These algorithms can be arranged for solving nonlinear equations in several variables.

For example, we define the following iteration (total-step Euler-Chebyshev's method):

$$x_i^{k+1} = x_i^k - \frac{f_i(x^k)}{\frac{\partial f_i(x^k)}{\partial x_i}} \left(1 - \frac{1}{2} \frac{f_i(x^k) \frac{\partial^2 f_i(x^k)}{\partial x_i^2}}{\left(\frac{\partial f_i(x^k)}{\partial x_i} \right)^2} \right), i = 1, \ldots, n; \ k = 0, 1, \ldots. \tag{13}$$

Numerical example (G.Iliev [8]). Let

$$\left| \begin{array}{l} f_1(x_1, x_2) = (x_2 - 0.07)x_1^2 - 0.3x_1x_2 - 0.15x_2^2 = 0 \\ f_2(x_1, x_2) = (x_1 - 0.15)x_2^2 - 0.14x_1x_2 - 0.07x_1^2 = 0 \end{array} \right.$$

and $x_1^0 = x_2^0 = 1$.

The solutions of this system are:

$$\left| \begin{array}{l} x_1 = 0.504939015319191968 \\ x_2 = 0.344939015319191968. \end{array} \right.$$

For the numerical determination of these roots, we apply Newton-Raphson method, method (6) and method (12).

The results are shown in Table 1, Table 2 and Table 3.

The presented numerical example was realized in precision arithmetic (about 18 significant digits).

Table 1. Solution by Newton's method

k	x_1^k	x_2^k
0	1.	1.
1	0.770320656226696495	0.695749440715883669
2	0.626370174040210892	0.505673164801260020
3	0.545377788002368922	0.398768875062025654
4	0.511666539615088034	0.353997741004263732
5	0.505175082853654779	0.345262185297826159
6	0.504939323674820933	0.344939445744945562
7	0.504939015319723450	0.344939015319950318
8	0.504939015319191968	0.344939015319950318

Table 2. Solution by method (6)

k	x_1^k	x_2^k
0	1.	1.
1	0.692307692307692308	0.589743589743589744
2	0.555123173090900958	0.409350643541659678
3	0.510754813273147686	0.352251613156086770
4	0.505035199913144336	0.345057597334594785
5	0.504939042018383390	0.344939047675290015
6	0.504939015319193995	0.344939015319194388
7	0.504939015319191968	0.344939015319191968

In [6], Ibidapo-Obe, Asaolu and Badiri developed a new technique for solving nonlinear system of equations (1). This is achieved by the iterative solution of a parametric linear system coupled with a nonlinear single variable equation.

Let $\mathbf{x}_0$ be an approximate solution to (1). The authors construct a new vector $\mathbf{y} = (y_1, \ldots, y_{n-1}, t)$ such that y_j, $j = 1, \ldots, n-1$ are distinct elements of

Table 3. Solution by method (12)

k	x_1^k	x_2^k
0	1.	1.
1	0.623188405797101449	0.471634208298052498
2	0.515759913152825672	0.352702914879984042
3	0.504992271181411001	0.344986324556096995
4	0.504939017246729918	0.344939016459300214
5	0.504939015319191969	0.344939015319191969
6	0.504939015319191968	0.344939015319191968

$\mathbf{x}_0$ (though y_j does not necessarily map into x_j); t is variable correspond to the $\mathbf{x}_0$ and renumber the system of equations in (1) such that y_j corresponds to x_j. Assuming $\mathbf{y}_0$ is an approximate solution but with t fixed and unknown implies (1) becomes

$$\mathbf{f}_j(\mathbf{y}_0) + \sum_{k=1}^{n-1} \frac{\partial \mathbf{f}_j(\mathbf{y}_0)}{\partial y_k} \xi_k = 0, \ \ j = 1, 2, \dots, n; \ (\xi_k = dy_k).$$

The updated solution to (1) becomes $\mathbf{y}_1 = (y_{01} + \xi_1, y_{02} + \xi_2, \dots, t)$. The proposed method [6] is a convergent method for a good initial guess.

Evidently, this procedure yields the exact solution $\mathbf{x}$, when applied to a linear system since the Jacobian then becomes the constant coefficient matrix.

Remark 2. Note that the iteration $\mathbf{y}_j$, $j = 0, 1, 2, \dots$ has been mentioned by N. Obreshkoff in [12].

Remark 3. The convergence order can be increased by the following way.

Let

$$\begin{aligned} \mathbf{f}_j(\mathbf{x}) &= \mathbf{f}_j(\mathbf{y}_0 + \xi) = \\ &= \mathbf{f}_j(\mathbf{y}_0) + \sum_{k=1}^{n-1} \frac{\partial \mathbf{f}_j(\mathbf{y}_0)}{\partial y_k} \xi_k + \frac{1}{2} \sum_{l=1}^{n-1} \sum_{k=1}^{n-1} \frac{\partial^2 \mathbf{f}_j(\mathbf{y}_0)}{\partial y_l \partial y_k} \xi_l \xi_k = 0, \\ & \qquad j = 1, \dots, n. \end{aligned} \tag{14}$$

The simultaneous system of equations given by (14) is quadratic in the $\xi_k's$.

Convergence to the solution is attained when $\mathbf{f}(y_{l+1}) \approx 0$ and $\|y_{l+1} - y_l\| < \eta$, for an arbitrary chosen η, depending on the accuracy desired ($y_{l+1} = (y_{l1} + \xi_1, y_{l2} + \xi_2, \dots, t)$).

The new procedure (based on (14)) yields the exact solution $\mathbf{x}$, when applied to quadratic nonlinear system.

Numerical example (Paterson [14], [6]). Consider the following system

$$\text{(A)}\qquad \left|\begin{array}{l} 4x_1 + x_2^2 + x_3 - 11 = 0,\\ x_1 + 4x_2 + x_3^2 - 18 = 0,\\ x_1^2 + x_2 + 4x_3 - 15 = 0 \end{array}\right.$$

with $\mathbf{x}_0 = (1,1,1)^T$. The system (A) has the different roots: $\mathbf{x} = (1,2,3)^T$. The method by Ibidapo-Obe, Asaolu, Badiru [6], with $\mathbf{y}_0 = (t, x_{20}, x_{30})^T$ gives

$$\text{(B)}\qquad \left|\begin{array}{l} 4t + x_{20}^2 + x_{30} - 11 + 2x_{20}\xi_1 + \xi_2 = 0,\\ t + 4x_{20} + x_{30}^2 - 18 + 4\xi_1 + 2x_{30}\xi_2 = 0,\\ t^2 + x_{20} + 4x_{30} - 15 + \xi_1 + 4\xi_2 = 0. \end{array}\right.$$

In particular, with $y_0 = (t,1,1)^T$, we have

$$\text{(C)}\qquad \left|\begin{array}{l} 4t - 9 + 2\xi_1 + \xi_2 = 0,\\ t - 13 + 4\xi_1 + 2\xi_2 = 0,\\ t^2 - 10 + \xi_1 + 4\xi_2 = 0. \end{array}\right.$$

From the last two equations in (C) we find

$$\xi_1 = \frac{32 - 4t + 2t^2}{14}, \quad \xi_2 = \frac{-4t^2 + t + 27}{14}.$$

Substituting in the first equation (C) yields

$$49t - 35 = 0, \quad \text{or } t = 0.7145;\ \xi_1 = 2.1543;\ \xi_2 = 1.834,$$

i.e. $\mathbf{y}_1 = (0.7145,\ 3.1543,\ 2.834)^T = (t, 1+\xi_1, 1+\xi_2)^T$.

Finally

$$\mathbf{x} = \mathbf{y}_5 = \mathbf{y}_6 = (1.0000, 2.0000, 3.0000)^T.$$

The new method (based on (14), with $\mathbf{y}_0 = (t,1,1)^T$ gives

$$\text{(B}'\text{)}\qquad \left|\begin{array}{l} 4t + x_{20}^2 + x_{30} - 11 + 2x_{20}\xi_1 + \xi_2 + \xi_1^2 = 0,\\ t + 4x_{20} + x_{30}^2 - 18 + 4\xi_1 + 2x_{30}\xi_2 + \xi_2^2 = 0,\\ t^2 + x_{20} + 4x_{30} - 15 + \xi_1 + 4\xi_2 = 0 \end{array}\right.$$

and

$$\text{(C}'\text{)}\qquad \left|\begin{array}{l} 4t - 9 + 2\xi_1 + \xi_2 + \xi_1^2 = 0,\\ t - 13 + 4\xi_1 + 2\xi_2 + \xi_2^2 = 0,\\ t^2 - 10 + \xi_1 + 4\xi_2 = 0. \end{array}\right.$$

From (C′), we have

$$\xi_2 = \frac{321 + 12t - 42t^2 - t^4}{137 + 8t^2}; \;\; \xi_1 = 10 - t^2 - 4\xi_2.$$

Substituting in the second equation (C′) yields

$$t - 13 + 4\xi_1(t) + 2\xi_2(t) + \xi_2^2(t) = 0. \tag{D}$$

Evidently, for $t = 1$, we have $\xi_2 = 2, \xi_1 = 1$, i.e. $y_1 = (1, 2, 3)^T$.

Here we give a method for which the inverse matrix of the Jacobian $J(x)$ is replaced by the inverse matrix of the Vandermondian $V(x)$. The notation $V(x_1^k, \dots, x_n^k)$ will denote a Vandermonde matrix:

$$V(x_1^k, \dots, x_n^k) = \begin{pmatrix} (x_1^k)^{n-1} & (x_1^k)^{n-2} & \dots & x_1^k & 1 \\ (x_2^k)^{n-1} & (x_2^k)^{n-2} & \dots & x_2^k & 1 \\ \dots & \dots & \dots & \dots & \dots \\ (x_n^k)^{n-1} & (x_n^k)^{n-2} & \dots & x_n^k & 1 \end{pmatrix}$$

of dimension n, and $|V(x_1^k, \dots, x_n^k)|$ is its determinant and is equal to $\prod_{i<j}(x_i^k - x_j^k) \neq 0$ $(x_i^k \neq x_j^k,\ i \neq j,\ i, j = 1, 2, \dots, n)$. Let

$$V^{-1} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix} = \frac{1}{|V(x_1^k, x_2^k, \dots, x_n^k)|} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Let us consider the following method

$$x_i^{k+1} = x_i^k - \sum_{j=1}^{n} c_{ij} f_j\left(x_1^k, x_2^k, \dots, x_n^k\right), \quad i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots, \tag{15}$$

or

$$x^{k+1} = x^k - V^{-1}(x^k) f(x^k), \quad k = 0, 1, 2, \dots. \tag{16}$$

Let $x_i^k = x_i + \epsilon_i$, $i = 1, 2, \dots, n$ and f has a continuous derivative in the neighborhood of the root x. Then

$$f_i\left(x_1^k, x_2^k, \dots, x_n^k\right) = f_i\left(x_1 + \epsilon_1, x_2 + \epsilon_2, \dots, x_n + \epsilon_n\right) =$$

$$= f_i(x_1, x_2, \dots, x_n) + \sum_{j=1}^{n} \frac{\partial f_i(x_1, x_2, \dots, x_n)}{\partial x_j} \epsilon_j + o(||\epsilon||),$$

$$x_i^{k+1} - x_i = x_i^k - x_i - \sum_{j=1}^{n} c_{ij} \left(f_j \left(x_1^k, x_2^k, \dots, x_n^k \right) - f_j(x_1, x_2, \dots, x_n) \right) =$$

$$= x_i^k - x_i - \sum_{j=1}^{n} c_{ij} \left(\sum_{l=1}^{n} \frac{\partial f_j(x_1, x_2, \dots, x_n)}{\partial x_l} \left(x_l^k - x_l \right) + o(\|\epsilon\|) \right).$$

Let $\|x - x^k\| \longrightarrow 0$, then

$$x^{k+1} - x = (x^k - x) \left(I - V^{-1}(x^k) \frac{\partial f(p^k)}{\partial x} \right),$$

with p^k on the line $l_{x^k x} = 0$, where $l_{x^k x}$ denotes the equation of the straight-line through the points $x^k = \left(x_1^k, x_2^k, \dots, x_n^k \right)$ and $x = (x_1, x_2, \dots, x_n)$. Let $\epsilon, \delta < 1$ be positive constants such that for all $k = 0, 1, 2, \dots$.

$$\left\| I - V^{-1}(x^k) \frac{\partial f(p^k)}{\partial x} \right\| \le \delta < 1,$$

$$\rho(x^k, x) = ||x^k - x|| \le \epsilon.$$

Then the sequence $\{x_i^{k+1}\}_{i=1}^n$ converges to a solution of $f(x) = 0$ and

$$\rho(x^{k+1}, x) < \epsilon. \tag{17}$$

For $k = 0$, we have

$$\rho(x^1, x) = \|x^1 - x\| = \left\| I - V^{-1}(x^0) \frac{\partial f(p^0)}{\partial x} \right\| \|x^0 - x\| \le \delta\epsilon < \epsilon.$$

Assuming (17) it is true for all subscripts $\le k$, we prove (17) for $k + 1$. Indeed,

$$\rho(x^{k+1}, x) = \|x^{k+1} - x\| \le \delta \|x^k - x\| \le \delta\epsilon < \epsilon,$$

$$\rho(x^{k+1}, x) \le \delta^{k+1} \rho(x^0, x). \tag{18}$$

But $\delta < 1$, the right-hand side of (18) tends to zero as $k \longrightarrow \infty$. So that we now frame a theorem

Theorem 2. *Let f be defined in the closed ball $B\left(\|x^0 - x\| \le \epsilon\right)$ and have continuous Frechét derivative. Moreover, suppose that*

$$\left\| I - V^{-1}\left(x^0\right) \frac{\partial f\left(x^0\right)}{\partial x} \right\| \le \delta < 1,$$

$$\rho\left(x^0 - x\right) = \left\|x^0 - x\right\| \le \varepsilon, \quad \varepsilon > 0.$$

Then the sequence $\left\{x^k\right\}$ generated by (16), *starting with x^0 converges to a solution of the equation $f(x) = 0$.*

Remark 4. The standard implementation of the method (16) is as follows:

$$(19) \qquad x^{k+1} = x^k - \frac{\left.\begin{pmatrix} a_{11} \ldots a_{1n} \\ \vdots \\ a_{n1} \ldots a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}\right|_{x=x^k}}{\left|V\left(x_i^k, x_1^k, \ldots, x_{i-1}^k, x_{i+1}^k, \ldots, x_n^k\right)\right|}, \quad k = 0, 1, 2, \ldots$$

Remark 5. For each $1 \le j \le n$, we have

$$(20) \qquad \begin{cases} c_{1j} = (-1)^{j+1} \dfrac{\left|V\left(x_1^k, \ldots, x_{j-1}^k, x_{j+1}^k, \ldots, x_n^k\right)\right|}{\left|V\left(x_1^k, \ldots, x_n^k\right)\right|}, \\ c_{2j} = (-1)^{j+2} \dfrac{\left|V\left(x_1^k, \ldots, x_{j-1}^k, x_{j+1}^k, \ldots, x_n^k\right)\right|}{\left|V\left(x_1^k, \ldots, x_n^k\right)\right|} \displaystyle\sum_{i \ne j}^{n} x_i^k, \\ \ldots \\ c_{nj} = (-1)^{j+n} \dfrac{\left|V\left(x_1^k, \ldots, x_{j-1}^k, x_{j+1}^k, \ldots, x_n^k\right)\right|}{\left|V\left(x_1^k, \ldots, x_n^k\right)\right|} \displaystyle\prod_{i \ne j}^{n} x_i^k. \end{cases}$$

Remark 6. The convergence order can be increased by calculating the new approximations x_i^{k+1}, in (19) using the already calculated approximations $x_1^{k+1}, \ldots, x_{i-1}^{k+1}$ (the so called Gauss-Seidel approach).

Remark 7. An interesting modification of (6) is to use so-called Weierstrass' correction:

$$(21) \qquad x_i^{k+1} = x_i^k - \frac{f_i\left(x_1^k, \ldots, x_n^k\right)}{\prod\limits_{j \ne i} \left(x_i^k - x_j^k\right)}, \quad i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots.$$

Remark 8. Wide area of problems and practical tasks in electrodynamics, and physics are reduced to the problem of solving a system of nonlinear algebraic equations with some constraint conditions. For example, from Kirchhoff law, we have system of nonlinear algebraic equations for determining the bus voltages (see, Cai and Chen [4]).

In this case, equations (15) can be rewritten in the following form:

$$x_1^{k+1} = x_1^k - \sum_{j=1}^{n} \frac{f_j\left(x_1^k, \ldots, x_n^k\right)}{\prod\limits_{s \neq j}^{n} \left(x_j^k - x_s^k\right)},$$

$$x_2^{k+1} = x_2^k - \sum_{j=1}^{n} \frac{\sum\limits_{s \neq j}^{n} x_s^k f_j\left(x_1^k, \ldots, x_n^k\right)}{\prod\limits_{s \neq j}^{n} \left(x_j^k - x_s^k\right)},$$

$$\ldots$$

$$x_n^{k+1} = x_n^k - \sum_{j=1}^{n} \frac{\prod\limits_{s \neq j}^{n} x_s^k f_j\left(x_1^k, \ldots, x_n^k\right)}{\prod\limits_{s \neq j}^{n} \left(x_j^k - x_s^k\right)}$$

and iteration algorithm can be refined using Euler's formula:

$$\sum_{i=1}^{n} \frac{\left(x_i^k\right)^t}{\prod\limits_{j \neq i}^{n} \left(x_i^k - x_j^k\right)} = \begin{cases} \sum\limits_{i=1}^{n} x_i^k, \ t = n \\ 1, \quad t = n-1 \\ 0, \quad 0 \leq t \leq n-2. \end{cases}$$

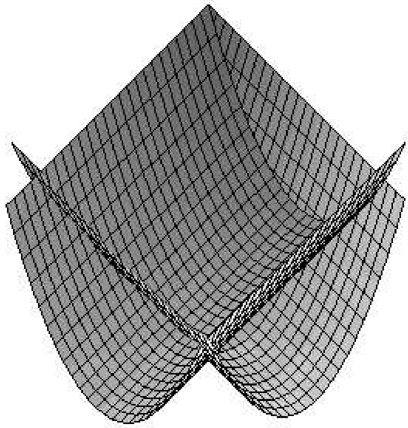
3. Numerical examples. To illustrate the method (19) we present the following simple example (see Ortega and Rheinboldt [13]):

Let

$$\left| \begin{array}{l} f_1\left(x_1, x_2\right) = \quad x_1^2 - x_2 - 1 = 0 \\ f_2\left(x_1, x_2\right) = -x_1 + x_2^2 - 1 = 0 \end{array} \right.$$

The solutions of this system are: $(x_1 = 0, x_2 = -1)$; $(x_1 = -1, x_2 = 0)$; $\left(x_1 = x_2 = \dfrac{1+\sqrt{5}}{2}\right)$; $\left(x_1 = x_2 = \dfrac{1-\sqrt{5}}{2}\right)$.

Compute simultaneously, the solution $(0, -1)$.



The choice of initial guess $\left(x_1^0 = -0.2,\ x_2^0 = -1.25\right)$ is critical for the Newton's method.

Indeed,

$$J(x) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & -1 \\ -1 & 2x_2 \end{pmatrix}, \quad \det J\left(x^0\right) = 4x_1^0 x_2^0 - 1 = 0\ !$$

We illustrate the behavior of Newton's method (2), starting from initial guess: $x_1^0 = -0.19,\ x_2^0 = -1.24$.

Table 4. Solution by Newton's method

k	x_1^k	x_2^k
0	–0.19	–1.24
1	0.5543055555555555	–1.246736111111111111
2	0.187353268299377751	–1.0995527339800818
3	0.036885904446349925	–1.02127985763805421
4	0.002021592710723367	–1.0012114333957109
5	0.000006662067951401	–1.00000405990111203
6	0.000000000072281936	–1.00000000004438219
7	0	–1

The components x_1^7 and x_2^7 tend to solution $(0, -1)$. For numerical determination of $x_i^{k+1},\ i = 1, 2$, we apply method (19), with $x_1^0 = -0.2$ and

$x_2^0 = -1.25$:

$$\left| \begin{array}{l} x_1^{k+1} = x_1^k - \frac{f_1(x_1^k, x_2^k) - f_2(x_1^k, x_2^k)}{x_1^k - x_2^k} \\ x_1^{k+1} = x_1^k - \frac{x_1^k f_2(x_1^k, x_2^k) - x_2^k f_1(x_1^k, x_2^k)}{x_2^k - x_1^k} \end{array} \right.$$

Table 5. Solution by method (19)

k	x_1^k	x_2^k
0	–0.2	–1.25
1	0.25	–1.05
2	0.05	–0.9875
3	–0.0125	–1
4	0	–1

The components x_1^4 and x_2^4 tend to exact solution $(0, -1)$. The method (21) leads to

Table 6. Solution by method (21)

k	x_1^k	x_2^k
0	–0.19	–1.24
1	–0.452952380952380952	–0.547047619047619048
2	2.180406400616926930	–3.18040640061692694
3	0.88683791957015072	–1.88683791957015072
4	0.283552059144475374	–1.28355205914447537
5	0.051305953003849593	–1.051305953003849590
6	0.002387332115035988	–1.00238733211503599
7	0.000005672271435954	–1.00000567227143595
8	0.000000000032174298	–1.0000000000321743
9	0	–1

We receive the exact solution with accuracy of 18 decimal digits after 9 iterations.

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