Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **34** (2008), 39–54

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

WEIGHTED DISPERSIVE ESTIMATES FOR SOLUTIONS OF THE SCHRÖDINGER EQUATION*

Fernando Cardoso, Claudio Cuevas, Georgi Vodev

Communicated by L. Stoyanov

Dedicated to Vesselin Petkov on the occasion of his 65th birthday

Abstract. We obtain $\langle x\rangle^s L^1 \to \langle x\rangle^{-s} L^\infty$ time decay estimates for the Schrödinger group $e^{it(-\Delta+V)}$, where $V \in L^\infty(\mathbf{R}^n)$, $n \geq 3$, is a real-valued potential satisfying $V(x) = O\left(\langle x\rangle^{-n+1/2-\epsilon}\right)$, $\epsilon > 0$.

1. Introduction and statement of results. In the present paper we will be interested in studying the decay properties of the Schrödinger group e^{itG} as $|t| \gg 1$, where G is the self-adjoint realization of $-\Delta + V(x)$ on $L^2(\mathbf{R}^n)$, $n \geq 3$. Here $V \in L^\infty(\mathbf{R}^n)$ is a real-valued potential satisfying

$$|V(x)| \leq C\langle x\rangle^{-\delta}, \quad \forall x \in \mathbf{R}^n, \tag{1.1}$$

2000 *Mathematics Subject Classification:* 35L15, 35B40, 47F05.
Key words: potential, dispersive estimates.
*The authors have been supported by the agreement Brazil-France in Mathematics – Proc. 69.0014/01-5. The first two authors have also been partially supported by the CNPq-Brazil.

with constants $C > 0$, $\delta > (n+2)/2$. Denote also by G_0 the self-adjoint realization of the operator $-\Delta$ on $L^2(\mathbf{R}^n)$. It is well-known that the following dispersive estimate holds for the free Schrödinger group:

$$\left\| e^{itG_0} \right\|_{L^1 \to L^\infty} \le C|t|^{-n/2}, \quad t \neq 0. \tag{1.2}$$

Given any $a > 0$, set $\chi_a(\sigma) = \chi_1(\sigma/a)$, where $\chi_1 \in C^\infty(\mathbf{R})$, $\chi_1(\sigma) = 0$ for $\sigma \le 1$, $\chi_1(\sigma) = 1$ for $\sigma \ge 2$. A difficult interesting problem is to find the largest possible class of potentials for which the following dispersive estimate holds true:

$$\left\| e^{itG} \chi_a(G) \right\|_{L^1 \to L^\infty} \le C|t|^{-n/2}, \quad t \neq 0. \tag{1.3}$$

While in the case of $n = 2$ and $n = 3$ there exist quite optimal results (see [7], [2], [6], [1], [11], [8]), when $n \ge 4$ there are very few ones. In this case (1.3) is proved in [4] for potentials satisfying (1.1) with $\delta > n$, the condition $\widehat{V} \in L^1$ and an extra technical condition which turns out not to be essential and therefore can be removed. Indeed, (1.3) has been recently proved in [5] under (1.1) with $\delta > n-1$ and $\widehat{V} \in L^1$, only. It is also shown in [5] that if we additionally suppose that zero is a regular point for G (that is, zero is neither an eigenvalue nor a resonance of G), then we have

$$\left\| e^{itG} P_{ac} \right\|_{L^1 \to L^\infty} \le C|t|^{-n/2}, \quad t \neq 0, \tag{1.4}$$

where P_{ac} denotes the spectral projection onto the absolutely continuous spectrum of G. Note that (1.4) is proved in [4] for a much smaller class of potentials. On the other hand, it is shown in [3] that when $n \ge 4$ there exist compactly supported potentials $V \in C^k(\mathbf{R}^n)$, $\forall k < (n-3)/2$, for which (1.3) does not hold. In other words, one needs to control at least $(n-3)/2$ derivatives of V in order that (1.3) could hold, so one expects that one could replace the condition $\widehat{V} \in L^1$ in [4] by a less restrictive one. For potentials satisfying (1.1) only, it has been recently obtained in [9] dispersive estimates with a loss of $(n-3)/2$ derivatives, and this seems to be the best one could do under this condition. However, if one replaces the spaces L^1 and L^∞ by similar ones with weights, one could overcome the loss of derivatives as well as get a better time decay. Indeed, for the free Schrödinger group we have the following weighted dispersive estimate (which is an easy consequence of the estimate (2.1) below):

$$\left\| \langle x \rangle^{-s} e^{itG_0} \chi_a(G_0) \langle x \rangle^{-s} \right\|_{L^1 \to L^\infty} \le C_s |t|^{-n/2-s}, \quad |t| \ge 1,\ s \ge 0. \tag{1.5}$$

It turns out that such an estimate holds for the perturbed Schrödinger group as well, provided s is taken big enough. More precisely, we have the following

Theorem 1.1. *Let V satisfy* (1.1) *with $\delta > n - 1/2$. Then, for every $a > 0$, $(n-3)/2 < s < \delta - (n+2)/2$, we have the estimate*

$$\left\|\langle x\rangle^{-s} e^{itG}\chi_a(G)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty} \le C|t|^{-n/2-s}, \quad |t|\ge 1. \tag{1.6}$$

If in addition zero is a regular point for G, then we have

$$\left\|\langle x\rangle^{-s} e^{itG}P_{ac}\langle x\rangle^{-s}\right\|_{L^1\to L^\infty} \le C|t|^{-n/2}, \quad |t|\ge 1. \tag{1.7}$$

Moreover, for every $2(n-1)/(n-3) \le p < +\infty$ and $(n-1)/2 - \alpha^{-1} < s < \delta - (n+2)/2$, we have

$$\left\|\langle x\rangle^{-\alpha s} e^{itG}\chi_a(G)\langle x\rangle^{-\alpha s}\right\|_{L^{p'}\to L^p} \le C|t|^{-\alpha(n/2+s)}, \quad |t|\ge 1, \tag{1.8}$$

$$\left\|\langle x\rangle^{-\alpha s} e^{itG}P_{ac}\langle x\rangle^{-\alpha s}\right\|_{L^{p'}\to L^p} \le C|t|^{-\alpha n/2}, \quad |t|\ge 1, \tag{1.9}$$

where $1/p + 1/p' = 1$ and $\alpha = 1 - 2/p$. We also have for all $2 \le p \le +\infty$, $\alpha(n-3)/2 < s < \delta - (n+2)/2$,

$$\left\|\langle x\rangle^{-s} e^{itG}\chi_a(G)\langle x\rangle^{-s}\right\|_{L^{p'}\to L^p} \le C|t|^{-\alpha n/2-s}, \quad |t|\ge 1. \tag{1.10}$$

Remark 1. We conjecture that (1.6) holds true for potentials satisfying (1.1) with $\delta > n-1$ and $(n-3)/2 < s < \delta - (n+1)/2$.

Remark 2. We expect that (1.6) holds with $s = (n-3)/2$ as well.

Note that (1.7) and (1.9) are a direct consequence of (1.6) and (1.8), respectively, and the low frequency dispersive estimates proved in [5].

It is natural also to expect that one could overcome the loss of derivatives when one keeps the space L^∞ but replaces the space L^1 by a suitable subspace. Indeed, it was proved in [9] that we have the following modified dispersive estimate under (1.1) only:

$$\left\|e^{itG}\chi_a(G)f\right\|_{L^\infty} \le C_\epsilon|t|^{-n/2}\left\|\langle x\rangle^{n/2+\epsilon}f\right\|_{L^2}, \quad t\ne 0, \tag{1.11}$$

for all $0 < \epsilon \ll 1$. The subspace $\langle x\rangle^{-n/2-\epsilon}L^2$, however, is not optimal and can be improved. We will prove in the present paper the following

Theorem 1.2. *Let $n \ge 4$ and let V satisfy* (1.1)*. Then, for every $a > 0$, $0 < \epsilon \ll 1$, we have the estimate*

$$\left\|e^{itG}\chi_a(G)f\right\|_{L^\infty} \le C_\epsilon|t|^{-n/2}\left\|\langle x\rangle^{\frac{(n+\epsilon')(n-3)}{2(n-2)}}f\right\|_{L^{\frac{2+2(n-3)(1+\epsilon)}{n-1}}}, \quad t\ne 0, \tag{1.12}$$

with some $0 < \epsilon' = O(\epsilon) \ll 1$. *If in addition zero is a regular point for* G, *then we have*

$$(1.13)\qquad \left\|e^{itG}P_{ac}f\right\|_{L^\infty} \le C_\epsilon |t|^{-n/2}\left\|\langle x\rangle^{\frac{(n+\epsilon')(n-3)}{2(n-2)}} f\right\|_{L^{\frac{2+2(n-3)(1+\epsilon)}{n-1}}},\quad t\neq 0.$$

More generally, given any $0 \le q \le (n-3)/2$, *we have the estimates*

$$(1.14)\quad \left\|e^{itG}G^{-q/2}\chi_a(G)f\right\|_{L^\infty} \le C_\epsilon |t|^{-n/2}\left\|\langle x\rangle^{\frac{(n+\epsilon')(n-3-2q)}{2(n-2-2q)}} f\right\|_{L^{\frac{2+2(n-3-2q)(1+\epsilon)}{n-1-2q}}},\ t\neq 0,$$

$$(1.15)\quad \left\|e^{itG}\langle G\rangle^{-q/2}P_{ac}f\right\|_{L^\infty} \le C_\epsilon |t|^{-n/2}\left\|\langle x\rangle^{\frac{(n+\epsilon')(n-3-2q)}{2(n-2-2q)}} f\right\|_{L^{\frac{2+2(n-3-2q)(1+\epsilon)}{n-1-2q}}},\ t\neq 0.$$

Remark 3. We conjecture that (1.12) and (1.14) hold true for potentials satisfying (1.1) with $\delta > (n+1)/2$.

Remark 4. The estimate (1.14) with $q = (n-3)/2$ is proved in [9].

Note that (1.13) and (1.15) follow from (1.12) and (1.14), respectively, and the low frequency dispersive estimates proved in [5].

To prove the estimates (1.6), (1.8), (1.10) and (1.14) we follow the semi-classical approach developed in [9]. To this end, we need to generalize the key semi-classical dispersive estimates proved in [9]. We believe that this approach could allow to get $L^1 \to L^\infty$ dispersive estimates with a loss of $(n-3)/2 - k$ derivatives, $0 \le k \le (n-3)/2$, for potentials $V \in C^k(\mathbf{R}^n)$ with a suitable decay at infinity. When $0 < k \le (n-3)/2$, this problem turns out to be quite hard and to our best knowledge it is not solved even for compactly supported potentials.

2. Proof of Theorem 1.1. We will first show that (1.6), (1.8) and (1.10) follow from the following

Proposition 2.1. *Let* $\psi \in C_0^\infty((0,+\infty))$. *For every* $s \ge 0$, $0 < h \le 1$, $t \neq 0$, *we have*

$$(2.1)\qquad \left\|\langle x\rangle^{-s} e^{itG_0}\psi(h^2G_0)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty} \le Ch^s|t|^{-n/2-s}.$$

If V *satisfies (1.1), then for every* $0 \le s < \delta - (n+2)/2$, $0 < h \le 1$, $t \neq 0$, *we have*

$$(2.2)\qquad \left\|\langle x\rangle^{-s} e^{itG}\psi(h^2G)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty} \le Ch^{s-(n-3)/2}|t|^{-n/2-s}.$$

Writing the function χ_a as follows

$$\chi_a(\sigma) = \int_0^1 \psi(\sigma\theta)\frac{d\theta}{\theta},$$

where $\psi(\sigma) = \sigma\chi_a'(\sigma) \in C_0^\infty((0,+\infty))$, we obtain from (2.2),

$$\left\|\langle x\rangle^{-s}e^{itG}\chi_a(G)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty} \le \int_0^1 \left\|\langle x\rangle^{-s}e^{itG}\psi(\theta G)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty}\frac{d\theta}{\theta}$$

$$\le C|t|^{-n/2-s}\int_0^1 \theta^{-1+(2s-n+3)/4}d\theta \le C|t|^{-n/2-s},$$

provided $s > (n-3)/2$. To prove (1.8) observe that an interpolation between the bound

$$\left\|\langle x\rangle^{-s}\left(e^{itG}\psi(h^2G) - e^{itG_0}\psi(h^2G_0)\right)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty} \le Ch^{s-(n-3)/2}|t|^{-n/2-s},$$

and the following estimate proved in [9]

$$\left\|e^{itG}\psi(h^2G) - e^{itG_0}\psi(h^2G_0)\right\|_{L^2\to L^2} \le Ch,$$

yields

$$\left\|\langle x\rangle^{-\alpha s}\left(e^{itG}\psi(h^2G) - e^{itG_0}\psi(h^2G_0)\right)\langle x\rangle^{-\alpha s}\right\|_{L^{p'}\to L^p} \le$$
$$\le Ch^{1+\alpha(s-(n-1)/2)}|t|^{-\alpha(n/2+s)}.$$

Hence

$$\left\|\langle x\rangle^{-\alpha s}\left(e^{itG}\chi_a(G) - e^{itG_0}\chi_a(G_0)\right)\langle x\rangle^{-\alpha s}\right\|_{L^{p'}\to L^p}$$
$$\le \int_0^1 \left\|\langle x\rangle^{-\alpha s}\left(e^{itG}\psi(\theta G) - e^{itG_0}\psi(\theta G_0)\right)\langle x\rangle^{-\alpha s}\right\|_{L^{p'}\to L^p}\frac{d\theta}{\theta}$$
$$\le C|t|^{-\alpha(n/2+s)}\int_0^1 \theta^{-1/2+\alpha(2s-n+1)/4}d\theta \le C|t|^{-\alpha(n/2+s)},$$

provided $s > (n-1)/2-\alpha^{-1}$, which clearly implies (1.8). To prove (1.10) observe that an interpolation between the estimates (2.2) and (2.9) below yields

$$\left\|\langle x\rangle^{-s}e^{itG}\psi(h^2G)\langle x\rangle^{-s}\right\|_{L^{p'}\to L^p} \le Ch^{s-\alpha(n-3)/2}|t|^{-\alpha n/2-s},$$

for every $2 \le p \le +\infty$. Hence

$$\left\|\langle x\rangle^{-s} e^{itG}\chi_a(G)\langle x\rangle^{-s}\right\|_{L^{p'}\to L^p} \le \int_0^1 \left\|\langle x\rangle^{-s} e^{itG}\psi(\theta G)\langle x\rangle^{-s}\right\|_{L^{p'}\to L^p}\frac{d\theta}{\theta}$$

$$\le C|t|^{-\alpha n/2-s}\int_0^1 \theta^{-1+(2s-\alpha(n-3))/4}d\theta \le C|t|^{-\alpha n/2-s},$$

provided $s > \alpha(n-3)/2$.

P r o o f o f P r o p o s i t i o n 2.1. To prove (2.1) we will make use of the fact that the kernel of the operator $e^{itG_0}\psi(h^2G_0)$ is of the form $K_h(|x-y|,t)$, where

$$K_h(\sigma,t) = \frac{\sigma^{-2\nu}}{(2\pi)^{\nu+1}}\int_0^\infty e^{it\lambda^2}\psi(h^2\lambda^2)\mathcal{J}_\nu(\sigma\lambda)\lambda d\lambda = h^{-n}K_1(\sigma h^{-1}, th^{-2}),$$

where $\mathcal{J}_\nu(z) = z^\nu J_\nu(z)$, $J_\nu(z) = (H_\nu^+(z) + H_\nu^-(z))/2$ being the Bessel function of order $\nu = (n-2)/2$. It is shown in [9] that the function K_h satisfies

$$|K_1(\sigma,t)| \le C|t|^{-s-1/2}\langle\sigma\rangle^{s-(n-1)/2}, \quad s\ge 0, \sigma>0, t\ne 0.$$

Hence, for all $s \ge 0$, $\sigma > 0$, $t \ne 0$, $0 < h \le 1$, we have

$$|K_h(\sigma,t)| \le Ch^s|t|^{-s-n/2}\langle\sigma\rangle^s. \tag{2.3}$$

Clearly, (2.1) follows from (2.3) and the bound

$$\langle x\rangle^{-s}\langle x-y\rangle^s\langle y\rangle^{-s} \le C, \quad \forall x,y\in\mathbf{R}^n.$$

To prove (2.2), it suffices to study the difference

$$\Psi(t,h) = e^{itG}\psi(h^2G) - e^{itG_0}\psi(h^2G_0).$$

As in [9] one can deduce from Duhamel's formula the identity

$$\Psi(t;h) = \sum_{j=1}^2 \Psi_j(t;h), \tag{2.4}$$

where

$$\Psi_1(t;h) =$$
$$= \psi_1(h^2G_0)e^{itG_0}\left(\psi(h^2G)-\psi(h^2G_0)\right) + \left(\psi_1(h^2G)-\psi_1(h^2G_0)\right)e^{itG}\psi(h^2G),$$

$$\Psi_2(t;h) = i\int_0^t \psi_1(h^2G_0)e^{i(t-\tau)G_0}Ve^{i\tau G}\psi(h^2G)d\tau,$$

where $\psi_1 \in C_0^\infty((0,+\infty))$, $\psi_1 = 1$ on $\operatorname{supp}\psi$.

Proposition 2.2. *If V satisfies* (1.1), *then for every* $0 \le s < \delta - (n+2)/2$, $0 < \epsilon \ll 1$, $1-\epsilon/2 \le \mu \le 1+\epsilon/2$, $0 < h \le 1$, $t \neq 0$, *we have*

$$\left\|\psi(h^2G_0)e^{itG_0}\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^\infty} \le C_\epsilon h^{-(n-2)/2-\epsilon}|t|^{-\mu}, \tag{2.5}$$

$$\left\|\Psi(t;h)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^\infty} \le C_\epsilon h^{-(n-4)/2-\epsilon}|t|^{-\mu}, \tag{2.6}$$

$$\left\|\langle x\rangle^{-s}\Psi(t;h)\langle x\rangle^{-s-n/2-\epsilon}\right\|_{L^2\to L^\infty} \le C_\epsilon h^{s+1-\epsilon}|t|^{-n/2-s}. \tag{2.7}$$

Proof. The estimates (2.5), (2.6) and (2.7) with $s = 0$ are proved in [9] (Propositions 2.1 and 4.1). To prove (2.7) with $0 < s < \delta - (n+2)/2$, observe that (2.1) implies

$$\left\|\langle x\rangle^{-s}\Psi_1(t;h)\langle x\rangle^{-s-n/2-\epsilon}f\right\|_{L^\infty}$$

$$\le O(h^2)\left\|\langle x\rangle^{-s}\Psi(t;h)\langle x\rangle^{-s-n/2-\epsilon}f\right\|_{L^\infty} + O(h^{s+2})|t|^{-n/2-s}\|f\|_{L^2}, \tag{2.8}$$

where we have also used the bounds (see Appendix 1 of [5])

$$\left\|\langle x\rangle^{-s}\left(\psi(h^2G)-\psi(h^2G_0)\right)\langle x\rangle^{s}\right\|_{L^\infty\to L^\infty} \le Ch^2,$$

$$\left\|\langle x\rangle^{-\delta}\left(\psi(h^2G)-\psi(h^2G_0)\right)\langle x\rangle^{\delta}\right\|_{L^2\to L^2} \le Ch^2.$$

To deal with the operator Ψ_2 we need the following uniform estimates on weighted L^2 spaces proved in [9] (Theorem 3.3). □

Proposition 2.3. *If V satisfies* (1.1), *then for every* $0 \le s < \delta - 1$, $0 < \epsilon \ll 1$, $0 < h \le 1$, $\forall t$, *we have*

$$\left\|\langle x\rangle^{-s}e^{itG}\psi(h^2G)\langle x\rangle^{-s}\right\|_{L^2\to L^2} \le C_\epsilon\langle t/h\rangle^{-s}. \tag{2.9}$$

Using (2.1), (2.5) and (2.9), we get

$$\left\|\langle x\rangle^{-s}\Psi_2(t;h)\langle x\rangle^{-s-n/2-\epsilon}\right\|_{L^2\to L^\infty}$$

$$\leq C\int_0^{t/2}\left\|\langle x\rangle^{-s}\psi_1(h^2G_0)e^{i(t-\tau)G_0}\langle x\rangle^{-s-n/2-\epsilon}\right\|_{L^2\to L^\infty}\times$$
$$\times\left\|\langle x\rangle^{-1-\epsilon}e^{i\tau G}\psi(h^2G)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^2}d\tau$$

$$+C\int_0^{t/2}\left\|\psi_1(h^2G_0)e^{i\tau G_0}\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^\infty}\times$$
$$\times\left\|\langle x\rangle^{-s-n/2-\epsilon}e^{i(t-\tau)G}\psi(h^2G)\langle x\rangle^{-s-n/2-\epsilon}\right\|_{L^2\to L^2}d\tau$$
$$\leq Ch^s|t|^{-n/2-s}\int_0^\infty\langle\tau/h\rangle^{-1-\epsilon/2}d\tau+Ch^{s+1-\epsilon}|t|^{-n/2-s}\int_0^\infty\tau^{-\mu}d\tau$$

$$\leq Ch^{s+1-\epsilon}|t|^{-n/2-s}. \tag{2.10}$$

Combining (2.4), (2.8) and (2.10), we obtain

$$\left\|\langle x\rangle^{-s}\Psi(t;h)\langle x\rangle^{-s-n/2-\epsilon}f\right\|_{L^\infty}$$

$$\leq O(h^2)\left\|\langle x\rangle^{-s}\Psi(t;h)\langle x\rangle^{-s-n/2-\epsilon}f\right\|_{L^\infty}+O(h^{s+1-\epsilon})|t|^{-n/2-s}\|f\|_{L^2}. \tag{2.11}$$

Hence, there exists a constant $0<h_0\leq 1$ so that for $0<h\leq h_0$ we have

$$\left\|\langle x\rangle^{-s}\Psi(t;h)\langle x\rangle^{-s-n/2-\epsilon}f\right\|_{L^\infty}\leq O(h^{s+1-\epsilon})|t|^{-n/2-s}\|f\|_{L^2}. \tag{2.12}$$

Let now $h_0\leq h\leq 1$. Without loss of generality we may suppose $h=1$. In view of (2.9) we have

$$\left\|\langle x\rangle^{-s}\left(\psi_1(G)-\psi_1(G_0)\right)e^{itG}\psi(G)\langle x\rangle^{-s-n/2-\epsilon}f\right\|_{L^\infty}$$

$$\leq C\left\|\langle x\rangle^{-s-n/2-\epsilon}e^{itG}\psi(G)\langle x\rangle^{-s-n/2-\epsilon}f\right\|_{L^2}\leq C|t|^{-n/2-s}\|f\|_{L^2},$$

which clearly implies (2.12) in this case. □

In view of (2.1) we have

$$\left\|\langle x\rangle^{-s}\Psi_1(t;h)\langle x\rangle^{-s}f\right\|_{L^\infty}$$

$$\leq O(h^2)\left\|\langle x\rangle^{-s}\Psi(t;h)\langle x\rangle^{-s}f\right\|_{L^\infty}+O(h^{s+2})|t|^{-n/2-s}\|f\|_{L^1}. \tag{2.13}$$

Furthermore, we decompose Ψ_2 as $\Psi_3 + \Psi_4$, where

$$\Psi_3(t;h) = i\int_0^t \psi_1(h^2G_0)e^{i(t-\tau)G_0}Ve^{i\tau G_0}\psi(h^2G_0)d\tau.$$

Using (2.1), (2.6) and (2.7), we obtain

$$\left\|\langle x\rangle^{-s}\Psi_4(t;h)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty}$$

$$\leq C\int_0^{t/2}\left\|\langle x\rangle^{-s}\psi_1(h^2G_0)e^{i(t-\tau)G_0}\langle x\rangle^{-s-n/2-\epsilon}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-1-\epsilon}\Psi(\tau;h)\right\|_{L^1\to L^2}d\tau$$

$$+C\int_0^{t/2}\left\|\psi_1(h^2G_0)e^{i\tau G_0}\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-s-n/2-\epsilon}\Psi(t-\tau;h)\langle x\rangle^{-s}\right\|_{L^1\to L^2}d\tau$$

$$\leq Ch^{s-(n-4)/2-2\epsilon}|t|^{-n/2-s}. \tag{2.14}$$

Proposition 2.4. *If V satisfies (1.1) with $\delta > (n+1)/2$, then for every $0 \leq s < \delta - (n+1)/2$, $0 < h \leq 1$, $t \neq 0$, we have*

$$\left\|\langle x\rangle^{-s}\Psi_3(t;h)\langle x\rangle^{-s}\right\|_{L^1\to L^\infty} \leq Ch^{s-(n-3)/2}|t|^{-n/2-s}. \tag{2.15}$$

Proof. It is easy to see that the kernel of the operator Ψ_3 is of the form

$$\int_{\mathbf{R}^n} U_h(|x-\xi|, |y-\xi|; t)V(\xi)d\xi,$$

where

$$\begin{aligned} U_h(\sigma_1,\sigma_2;t) &= i\int_0^t \widetilde{K}_h(\sigma_1, t-\tau)K_h(\sigma_2,\tau)d\tau = \\ &= h^{-2n+2}U_1(\sigma_1h^{-1}, \sigma_2h^{-1}; th^{-2}), \end{aligned} \tag{2.16}$$

where $\widetilde{K}_h$ is defined by replacing in the definition of K_h the function ψ by ψ_1. Clearly, (2.15) follows from the bounds

$$\begin{aligned} |U_h(\sigma_1,\sigma_2;t)| &\leq Ch^{s-(n-3)/2}|t|^{-n/2-s}\times \\ &\times\left(\sigma_1^{-n+1} + \sigma_1^{-(n-1)/2} + \sigma_2^{-n+1} + \sigma_2^{-(n-1)/2}\right)(1+\sigma_1+\sigma_2)^s, \end{aligned} \tag{2.17}$$

and

$$\langle x\rangle^{-s}(\langle x-\xi\rangle+\langle y-\xi\rangle)^s\langle y\rangle^{-s}\le C\langle\xi\rangle^s,\quad \forall x,y,\xi\in\mathbf{R}^n.$$

On the other hand, in view of (2.16), it suffices to prove (2.17) with $h=1$. The function U_1 is of the form $U_1^{(1)}-U_1^{(2)}$, where

$$U_1^{(j)}(\sigma_1,\sigma_2;t)=$$
$$=\frac{(\sigma_1\sigma_2)^{-2\nu}}{(2\pi)^n}\int_0^\infty\int_0^\infty e^{it\lambda_j^2}\psi_1(\lambda_1^2)\psi(\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)\frac{\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2}d\lambda_1 d\lambda_2.$$

The function $\mathcal{J}_\nu$ is of the form $\mathcal{J}_\nu(z)=e^{iz}b_\nu^+(z)+e^{-iz}b_\nu^-(z)$, with functions $b_\nu^\pm$ satisfying (e.g. see Appendix 2 of [9])

$$\left|\partial_z^j b_\nu^\pm(z)\right|\le C\langle z\rangle^{(n-3)/2-j},\quad \forall z>0,\ 0\le j\le n-3, \tag{2.18}$$

$$\left|\partial_z^j b_\nu^\pm(z)\right|\le Cz^{-\epsilon}\langle z\rangle^{-(n-1)/2+\epsilon},\quad \forall z>0,\ j=n-2, \tag{2.19}$$

$$\left|\partial_z^j b_\nu^\pm(z)\right|\le C_j z^{n-2-j}\langle z\rangle^{-(n-1)/2},\quad \forall z>0,\ j\ge n-1. \tag{2.20}$$

The function $\mathcal{J}_\nu$ also satisfies

$$\left|\partial_z^j \mathcal{J}_\nu(z)\right|\le Cz^{n-2-j}\langle z\rangle^{j-(n-1)/2},\quad \forall z>0,\ 0\le j\le n-2, \tag{2.21}$$

$$\left|\partial_z^j \mathcal{J}_\nu(z)\right|\le C_j\langle z\rangle^{(n-3)/2},\quad \forall z>0,\ j\ge n-1. \tag{2.22}$$

It is shown in [9] that the function $U_1^{(1)}$ is of the form $W_1^{(1)}+L_1^{(1)}$, where

$$W_1^{(1)}(\sigma_1,\sigma_2;t)=$$
$$=\mathrm{Const}(\sigma_1\sigma_2)^{-2\nu}\sum_\pm\pm\int_0^\infty e^{it\lambda_1^2\pm i\sigma_2\lambda_1}\psi_1(\lambda_1^2)\mathcal{J}_\nu(\sigma_1\lambda_1)b_\nu^\pm(\sigma_2\lambda_1)\lambda_1 d\lambda_1,$$

$$L_1^{(1)}(\sigma_1,\sigma_2;t)=$$
$$=\mathrm{Const}(\sigma_1\sigma_2)^{-2\nu}\sum_\pm\int_0^\infty e^{it\lambda_1^2}\psi_1(\lambda_1^2)\mathcal{J}_\nu(\sigma_1\lambda_1)A^\pm(\lambda_1,\sigma_2)\lambda_1 d\lambda_1,$$

$$A^\pm(\lambda_1,\sigma_2)=\int_{-\infty}^\infty e^{\pm i\sigma_2\lambda_2}a^\pm(\lambda_1,\lambda_2;\sigma_2)d\lambda_2,$$

$$a^{\pm}(\lambda_1,\lambda_2;\sigma_2)=(\lambda_1-\lambda_2)^{-1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\psi(\lambda_2^2)b_\nu^{\pm}(\sigma_2\lambda_2)-\frac{1}{2}\psi(\lambda_1^2)b_\nu^{\pm}(\sigma_2\lambda_1)\right).$$

We will bound the functions $W_1^{(1)}$ and $L_1^{(1)}$ by using the following well known inequality

$$\left|\int e^{it\lambda^2+i\sigma\lambda}\varphi(\lambda)d\lambda\right|\le C_m|t|^{-m-1/2}\sum_{j=0}^{m}\sigma^{m-j}\left\|\widehat{\partial_\lambda^j\varphi}\right\|_{L^1}$$

$$\le C_m'|t|^{-m-1/2}\sum_{j=0}^{m}\sigma^{m-j}\sum_{\ell=0}^{1}\sup_{\lambda}\langle\lambda\rangle\left|\partial_\lambda^{j+\ell}\varphi(\lambda)\right|,\quad \forall t\neq 0,\ \sigma\in\mathbf{R}, \tag{2.23}$$

for every integer $m\ge 0$ with a constant $C_m'>0$ independent of t, σ and φ, where $\varphi\in C_0^\infty(\mathbf{R})$. By (2.18)–(2.22), we have (for $\lambda_1^2\in\operatorname{supp}\psi_1$)

$$\left|\partial_{\lambda_1}^k\left(\mathcal{J}_\nu(\sigma_1\lambda_1)b_\nu^{\pm}(\sigma_2\lambda_1)\right)\right|\le C_k\sum_{j=0}^{k}\sigma_1^{k-j}\sigma_2^j\left|\left(\partial_z^{k-j}\mathcal{J}_\nu\right)(\sigma_1\lambda_1)\right|\left|\left(\partial_z^j b_\nu^{\pm}\right)(\sigma_2\lambda_1)\right|$$

$$\le C_k\sigma_1^{n-2}\langle\sigma_1\rangle^{k-(n-1)/2}\langle\sigma_2\rangle^{(n-3)/2}, \tag{2.24}$$

for every integer $k\ge 0$. Let $0<\sigma_1\le 1$. By (2.23) with $\sigma=\pm\sigma_2$ and (2.24) we get

$$\left|W_1^{(1)}(\sigma_1,\sigma_2;t)\right|\le C_m|t|^{-m-1/2}\sigma_2^{-n+2}\langle\sigma_2\rangle^{m+(n-3)/2}. \tag{2.25}$$

Let now $\sigma_1\ge 1$. Then, by (2.18)-(2.20), we have (for $\lambda_1^2\in\operatorname{supp}\psi_1$)

$$\left|\partial_{\lambda_1}^k\left(b_\nu^{\pm}(\sigma_1\lambda_1)b_\nu^{\pm}(\sigma_2\lambda_1)\right)\right|\le C_k\langle\sigma_1\rangle^{(n-3)/2}\langle\sigma_2\rangle^{(n-3)/2}, \tag{2.26}$$

for every integer $k\ge 0$. Thus, using (2.23) with $\sigma=\pm\sigma_1\pm\sigma_2$ together with (2.26) we get

$$\left|W_1^{(1)}(\sigma_1,\sigma_2;t)\right|\le \tag{2.27}$$

$$\le C_m|t|^{-m-1/2}\sigma_1^{-(n-1)/2}\sigma_2^{-n+2}\langle\sigma_2\rangle^{(n-3)/2}\left(\langle\sigma_1\rangle+\langle\sigma_2\rangle\right)^m.$$

By (2.25) and (2.27), we conclude

$$\begin{aligned}(2.28)\quad & \left|W_1^{(1)}(\sigma_1,\sigma_2;t)\right| \le \\ & \le C_m|t|^{-m-1/2}\langle\sigma_1\rangle^{-(n-1)/2}\sigma_2^{-n+2}\langle\sigma_2\rangle^{(n-3)/2}\left(\langle\sigma_1\rangle+\langle\sigma_2\rangle\right)^m,\end{aligned}$$

for every integer $m \ge 0$ and all $t \ne 0$, $\sigma_1,\sigma_2 > 0$. Hence, (2.28) holds for all real $m \ge 0$ and in particluar for $m = (n-1)/2 + s$, $s \ge 0$. Thus, we obtain

$$\begin{aligned}(2.29)\quad & \left|W_1^{(1)}(\sigma_1,\sigma_2;t)\right| \le \\ & \le C_s|t|^{-n/2-s}\left(\sigma_1^{-(n-1)/2}+\sigma_2^{-n+2}+\sigma_2^{-(n-1)/2}\right)\left(\langle\sigma_1\rangle+\langle\sigma_2\rangle\right)^s.\end{aligned}$$

The function $L_1^{(1)}$ can be bounded in the same way. Indeed, it is shown in [9] that the functions $A^\pm$ satisfy (for $\lambda^2 \in \operatorname{supp}\psi_1$)

$$(2.30)\qquad \left|\partial_\lambda^j A^\pm(\lambda,\sigma)\right| \le C_j\sigma^{-1}, \quad \forall\sigma > 0,$$

for every integer $j \ge 0$. By (2.21), (2.22) and (2.30), we have (for $\lambda_1^2 \in \operatorname{supp}\psi_1$)

$$(2.31)\qquad \left|\partial_{\lambda_1}^k\left(\mathcal{J}_\nu(\sigma_1\lambda_1)A^\pm(\lambda_1,\sigma_2)\right)\right| \le C_k\sigma_2^{-1}\sigma_1^{n-2}\langle\sigma_1\rangle^{k-(n-1)/2},$$

for every integer $k \ge 0$ and all $\sigma_1,\sigma_2 > 0$. As above, consider first the case $0 < \sigma_1 \le 1$. By (2.23) with $\sigma = 0$ and (2.31) we get

$$(2.32)\qquad \left|L_1^{(1)}(\sigma_1,\sigma_2;t)\right| \le C_m|t|^{-m-1/2}\sigma_2^{-n+1}.$$

When $\sigma_1 \ge 1$, by (2.18)-(2.20) and (2.30), we have (for $\lambda_1^2 \in \operatorname{supp}\psi_1$)

$$(2.33)\qquad \left|\partial_{\lambda_1}^k\left(b_\nu^\pm(\sigma_1\lambda_1)A^\pm(\lambda_1,\sigma_2)\right)\right| \le C_k\sigma_2^{-1}\sigma_1^{(n-3)/2}.$$

By (2.23) with $\sigma = \pm\sigma_1$ and (2.33) we get

$$(2.34)\qquad \left|L_1^{(1)}(\sigma_1,\sigma_2;t)\right| \le C_m|t|^{-m-1/2}\sigma_1^{-(n-1)/2+m}\sigma_2^{-n+1}.$$

By (2.32) and (2.34), we conclude

$$(2.35)\qquad \left|L_1^{(1)}(\sigma_1,\sigma_2;t)\right| \le C_m|t|^{-m-1/2}\langle\sigma_1\rangle^{-(n-1)/2+m}\sigma_2^{-n+1},$$

for every integer $m \geq 0$ and all $t \neq 0$, $\sigma_1, \sigma_2 > 0$. Hence, (2.35) holds for all real $m \geq 0$ and in particluar for $m = (n-1)/2 + s$, $s \geq 0$. We have

$$\left|L_1^{(1)}(\sigma_1, \sigma_2; t)\right| \leq C_s |t|^{-n/2-s} \langle \sigma_1 \rangle^s \sigma_2^{-n+1}. \tag{2.36}$$

It follows from (2.29) and (2.36) that the function $U_1^{(1)}$ satisfies (2.17) with $h = 1$. Clearly, the function $U_1^{(2)}$ can be treated in precisely the same way. Thus, we conclude that the function U_1 satisfies (2.17) with $h = 1$. □

Summing up (2.13), (2.14) and (2.15), we obtain

$$\left\|\langle x \rangle^{-s} \Psi(t; h) \langle x \rangle^{-s} f\right\|_{L^\infty}$$

$$\leq O(h^2) \left\|\langle x \rangle^{-s} \Psi(t; h) \langle x \rangle^{-s} f\right\|_{L^\infty} + O(h^{s-(n-3)/2}) |t|^{-n/2-s} \|f\|_{L^1}. \tag{2.37}$$

Hence, there exists a constant $0 < h_0 < 1$ so that for $0 < h \leq h_0$ we can absorb the first term in the RHS of (2.37), which in turn implies (2.2) for these values of h. Let now $h_0 \leq h \leq 1$. Without loss of generality we may suppose $h = 1$. In view of (2.7) we have

$$\left\|\langle x \rangle^{-s} \left(\psi_1(G) - \psi_1(G_0)\right) e^{itG} \psi(G) \langle x \rangle^{-s} f\right\|_{L^\infty}$$

$$\leq C \left\|\langle x \rangle^{-s-n/2-\epsilon} e^{itG} \psi(G) \langle x \rangle^{-s} f\right\|_{L^2} \leq C |t|^{-n/2-s} \|f\|_{L^1},$$

which implies (2.2) in this case. □

3. Proof of Theorem 1.2. Given any $1 \leq p \leq 2$, denote by $\Lambda^p \subset L^1$ the space $\langle x \rangle^{-(n+\epsilon')(p-1)/p} L^p$, $0 < \epsilon' \ll 1$, equipped with the norm

$$\|f\|_{\Lambda^p} = \left\|\langle x \rangle^{(n+\epsilon')(p-1)/p} f\right\|_{L^p}.$$

In what follows we keep the same notations as in the previous section. The key point in the proof of (1.14) is the following

Proposition 3.1. *If V satisfies* (1.1) *with $\delta > (n+2)/2$, then for every $0 \leq q \leq (n-3)/2$, $0 < h \leq 1$, $t \neq 0$, $0 < \epsilon \ll 1$, we have*

$$\|\Psi(t; h)\|_{\Lambda^p \to L^\infty} \leq C_\epsilon h^{\epsilon_1} |t|^{-n/2}, \tag{3.1}$$

where

$$p = \frac{2 + 2(n-3-2q)(1+\epsilon)}{n-1-2q}, \tag{3.2}$$

and $0 < \epsilon_1 = O(\epsilon) \ll 1$.

Proof. We may suppose $0 \le q < (n-3)/2$ since for $q = (n-3)/2$ the estimate (1.14) is proved in [9]. Writing the estimates (2.2) and (2.7) with $s = 0$ in the form

$$\|\Psi(t;h)\|_{\Lambda^1 \to L^\infty} \le C h^{-(n-3)/2} |t|^{-n/2},$$

$$\|\Psi(t;h)\|_{\Lambda^2 \to L^\infty} \le C_{\epsilon_2} h^{1-\epsilon_2} |t|^{-n/2},$$

for every $0 < \epsilon_2 \ll 1$, we conclude by a standard interpolation argument that for every $1 \le p \le 2$, $0 < \epsilon_2 \ll 1$, we have

$$\|\Psi(t;h)\|_{\Lambda^p \to L^\infty} \le C'_{\epsilon_2} h^{\beta(p)} |t|^{-n/2}, \tag{3.3}$$

where

$$\beta(p) = -(2-p)(n-3)/2 + (p-1)(1-\epsilon_2).$$

Now, given any $0 < \epsilon \ll 1$ define p by (3.2). It is easy to see that one can choose $0 < \epsilon_2 = O(\epsilon) \ll 1$ such that $\epsilon_1 := \beta(p) = O(\epsilon) > 0$, which clearly implies (3.1). $\square$

As at the beginning of the previous section we obtain from (3.1)

$$\left\| e^{itG}\chi_a(G) - e^{itG_0}\chi_a(G_0) \right\|_{\Lambda^p \to L^\infty} \le \int_0^1 \left\| \Psi(t;\sqrt{\theta}) \right\|_{\Lambda^p \to L^\infty} \frac{d\theta}{\theta}$$

$$\le C|t|^{-n/2} \int_0^1 \theta^{-1+\epsilon_1/2} d\theta \le C'|t|^{-n/2}. \tag{3.4}$$

Now, (1.14) follows from (3.4) and the bounds

$$\left\| e^{itG_0}\chi_a(G_0) \right\|_{\Lambda^p \to L^\infty} \le C_1 \left\| e^{itG_0}\chi_a(G_0) \right\|_{L^1 \to L^\infty} \le C_2 \left\| e^{itG_0} \right\|_{L^1 \to L^\infty} \le C|t|^{-n/2}.$$

$\square$

REFERENCES

[1] M. Goldberg. Dispersive bounds for the three dimensional Schrödinger equation with almost critical potentials. *Geom. Funct. Anal.* **16**, *3* (2006), 517–536.

[2] M. Goldberg and W. Schlag. Dispersive estimates for Schrödinger operators in dimensions one and three. *Comm. Math. Phys.* **251** (2004), 157–178.

[3] M. Goldberg, M. Visan. A conterexample to dispersive estimates for Schrödinger operators in higher dimensions. *Comm. Math. Phys.* **266** (2006), 211–238.

[4] J.-L. Journé, A. Soffer, C. Sogge. Decay estimates for Schrödinger operators. *Comm. Pure Appl. Math.* **44** (1991), 573–604.

[5] S. Moulin, G. Vodev. Low frequency dispersive estimates for the Schrödinger group in higher dimensions. *Asymptot. Anal.* **55** (2007), 49–71.

[6] I. Rodnianski, W. Schlag. Time decay for solutions of Schrödinger equations with rough time-dependent potentials. *Invent. Math.* **155** (2004), 451–513.

[7] W. Schlag. Dispersive estimates for Schrödinger operators in two dimensions. *Comm. Math. Phys.* **257** (2005), 87–117.

[8] G. Vodev. Dispersive estimates of solutions to the Schrödinger equation. *Ann. H. Poincaré* **6**, *6* (2005), 1179–1196.

[9] G. Vodev. Dispersive estimates of solutions to the Schrödinger equation in dimensions $n \geq 4$. *Asymptot. Anal.* **49** (2006), 61–86.

[10] K. Yajima. The $W^{k,p}$ continuity of wave operators for Schrödinger operators. *J. Math. Soc. Japan* **47** (1995), 551–581.

[11] K. Yajima. Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue. *Comm. Math. Phys.* **259** (2005), 475–509.

F. Cardoso
Universidade Federal de Pernambuco
Departamento de Matemática
CEP. 50540-740 Recife-Pe, Brazil
e-mail: `fernando@dmat.ufpe.br`

C. Cuevas
Universidade Federal de Pernambuco
Departamento de Matemática
CEP. 50540-740 Recife-Pe, Brazil
e-mail: `cch@dmat.ufpe.br`

G. Vodev
Université de Nantes
Département de Mathématiques
UMR 6629 du CNRS
2, rue de la Houssinière, BP 92208
44332 Nantes Cedex 03, France
e-mail: georgi.vodev@math.univ-nantes.fr

Received June 11, 2007