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GLOBAL WAVES WITH NON-POSITIVE ENERGY IN GENERAL RELATIVITY

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Abstract. The theory of the waves equations has a long history since M. Riesz and J. Hadamard. It is impossible to cite all the important results in the area, but we mention the authors related with our work: J. Leray [34] and Y. Choquet-Bruhat [9] (Cauchy problem), P. Lax and R. Phillips [33] (scattering theory for a compactly supported perturbation), L. Hörmander [27] and J-M. Bony [7] (microlocal analysis). In all these domains, V. Petkov has made fundamental contributions, mainly in microlocal analysis, scattering theory, dynamical zeta functions (see in particular the monography [42]).

In this paper we present a survey of some recent results on the global existence and the asymptotic behaviour of waves, when the conserved energy is not definite positive. This unusual situation arises in important cosmological models of the General Relativity where the gravitational curvature is very strong. We consider the case of the closed time-like curves (violation of the causality) [1], and the charged black-holes (superradiance) [3].

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1. Violation of Causality. There are few works on the global hyperbolic problems on the *non* globally hyperbolic spacetimes. Nevertheless the global hyperbolicity is an extremely strong hypothesis, which is not satisfied by a lot of solutions of the (in)homogeneous Einstein equations. The origin of the loss of global hyperbolicity can be a non trivial topology, an elementary example is $S^1_t \times \mathbb{R}^3_x$ endowed with the Minkowski metric. Other examples are the lorentzian wormholes [18], [51], but since they lead to violations of the local energy conditions, these models are somewhat exotic. A deeper raison is linked with the non linearity of the Einstein equations that can create some singularities of curvature, and also some closed time-like geodesics. In particular, the violation of the causality can be due to a fast rotation of the space-time that tilts over the light cones so strongly that some closed causal curves appear. This phenomenon is present in several important Einstein manifolds: the Van Stockum space-time [47], the Gödel universe [22], the Kerr black-hole (third Boyer-Lindquist block and fast Kerr) [32], the spinning cosmic string [15]. These lorentzian manifolds belong to a wide range of stationnary, axisymmetric spacetimes that are described by the Papapetrou metric [41] on some 3D+1 manifold $\mathcal{M}$

$$\begin{aligned} g_{\mu,\nu}dx^\mu dx^\nu = &A(r,z)\left[dt - C(r,z)d\varphi\right]^2 \\ &-\frac{1}{A(r,z)}\left[r^2d\varphi^2 + B(r,z)\left(dr^2+dz^2\right)\right], \quad 0 < A, B, \quad 0 \le C. \end{aligned}$$

Our model consists by choosing $\mathcal{M} = \mathbb{R}^4$, $A = B = 1$, and for simplicity we assume that C is compactly supported. When we allow that $C(r,z) > r$ (resp. $C(r,z) = r$) for some (r,z), some *closed* time-like (resp. null) curves appear and this spacetime has the same properties that the previous Einstein manifolds of point of view of the causality. We investigate the wave equation

$$(1.1)\quad \mid \det g \mid^{-\frac{1}{2}} \partial_\mu \left(\mid \det g\mid^{\frac{1}{2}} g^{\mu,\nu}\partial_\nu\right) u = \left(1-\frac{C^2}{r^2}\right)\partial_t^2 u - \Delta_x u - 2\frac{C}{r^2}\partial_t\partial_\varphi u = 0.$$

Obviously the study of the solutions is difficult because of the presence of closed timelike/null curves: there exists no global Cauchy hypersurface. We can see how much intricated is the situation by formally expanding a solution of (1.1) in Fourier series with respect to φ:

$$u(t,\varphi,r,z) = \sum_{m\in\mathbb{Z}} r^{-\frac{1}{2}} u_m(t,r,z)e^{im\varphi}.$$

Then u_m is solution of a changing type equation:

$$\left(1-\frac{C^2}{r^2}\right)\partial_t^2 u_m - \left(\partial_r^2+\partial_z^2\right)u_m - 2im\frac{C}{r^2}\partial_t u_m + \frac{m^2}{r^2}u_m = 0,$$

which is hyperbolic on $\{C < r\}$, elliptic on $\mathbb{T} := \{C > r\}$, and of Schrödinger type on $\Sigma := \{C = r\}$. In particular, $M_{t_0} := \{t = t_0\} \times \mathbb{R}^3_x$ is not a Cauchy hypersurface for (1.1) when Σ is not empty. Another crucial point is that since ∂_t is a Killing vector field, there exists a conserved current for the sufficiently smooth solutions of (1.1):

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^3} \left(1 - \frac{C^2}{r^2}\right) \mid \partial_t u(t,x) \mid^2 + \mid \nabla u(t,x) \mid^2 dx.$$

But this energy is *not* a positive form when the manifold is not chronological ($\mathbb{T} \neq \emptyset$).

1.1. Geometrical Framework. We consider the topologically trivial manifold

$$\mathcal{M} := \mathbb{R}^4_{(x^0,x^1,x^2,x^3)} = \mathbb{R}_t \times \mathbb{R}^3_x \tag{1.2}$$

endowed with a lorentzian metric g which is equal to the Minkowski metric outside a torus

$$\mathbb{R}_t \times \left\{(x^1,x^2,x^3); 0 < r_-^2 <\mid x^1 \mid^2 + \mid x^2 \mid^2 < r_+^2,\ z_- < x^3 < z_+\right\}.$$

We choose a particular case of the Papapetrou metric:

$$g_{\mu,\nu}dx^\mu dx^\nu = dt^2 - \left[r^2 - C^2(r,z)\right] d\varphi^2 - 2C(r,z)dtd\varphi - dr^2 - dz^2, \tag{1.3}$$

where we have used the cylindrical coordinates $(t,\varphi,r,z) \in \mathbb{R} \times [0,2\pi[\times[0,\infty[\times\mathbb{R}$ given by

$$x^1 = r\cos\varphi,\ x^2 = r\sin\varphi,\ x^3 = z. \tag{1.4}$$

We assume that C satisfies

$$0 \leq C(r,z),\ \ C \in C^2(\mathbb{R}^2),\ \ (r,z) \notin [r_-,r_+] \times [z_-,z_+] \Rightarrow C(r,z) = 0, \tag{1.5}$$

and our geometrical framework is given by (1.2), (1.3), (1.5). We note that t is a timelike coordinate and $(\mathcal{M}, g)$ is naturally time oriented by the continuous, nowhere vanishing, timelike (and Killing) vector field ∂_t. Moreover r and z are spacelike coordinates. The interesting fact is that the nature of the Killing vector field ∂_φ is ambiguous: the crucial point is that φ is a *timelike* coordinate when $C > r$, thus we introduce

$$\mathbb{T} := \mathbb{R}_t \times \mathbb{T}_0,\ \ \mathbb{T}_0 := S^1 \times \{(r,z);\ \ C(r,z) > r\}, \tag{1.6}$$

$$\Sigma := \mathbb{R}_t \times \Sigma_0, \quad \Sigma_0 := S^1 \times \{(r,z); \quad C(r,z) = r > 0\}. \tag{1.7}$$

We shall need the hypersurfaces

$$M_t := \{t\} \times \mathbb{R}^3. \tag{1.8}$$

Its causal structure is complex. Since its normal is dt, the nature of M_t is locally given by the sign of

$$g^{tt} = 1 - \frac{C^2}{r^2},$$

hence $M_t \cap (\mathcal{M} \setminus (\mathbb{T} \cup \Sigma))$ is spacelike, $M_t \cap \Sigma$ is null, and $M_t \cap \mathbb{T}$ is timelike.

We shall be mainly concerned by the case where Σ is not empty. In this situation the causality is violated in a severe way: given $m_0 = (t_0, \varphi_0, r_0, z_0)$, the path

$$\tau \in \mathbb{R} \longmapsto m(\tau) = (t_0, \varphi_0 - \tau, r_0, z_0) \in \mathcal{M}, \tag{1.9}$$

is a future directed closed null curve if $m_0 \in \Sigma$, and a future directed closed timelike curve if $m_0 \in \mathbb{T}$ since:

$$g\left(\frac{dm}{d\tau}, \frac{dm}{d\tau}\right) = C^2(r_0, z_0) - r_0^2, \quad g\left(\frac{dm}{d\tau}, \frac{\partial}{\partial t}\right) = 2C(r_0, z_0) > 0.$$

More precisely, the causal structure of $\mathcal{M}$ is described by the following:

Proposition 1.1. *Let $(\mathcal{M}, g)$ be the lorentzian manifold defined by* (1.2), (1.3), (1.5).

(1) *If $\Sigma = \emptyset$, $(\mathcal{M}, g)$ is globally hyperbolic: M_t is a Cauchy hypersurface for any $t \in \mathbb{R}$.*

(2) *If $\mathbb{T} = \emptyset$ and $\Sigma \neq \emptyset$, $(\mathcal{M}, g)$ is chronological but non causal: there exists no closed timelike curve, but there exists a closed null geodesic.*

(3) *If $\mathbb{T} \neq \emptyset$, $(\mathcal{M}, g)$ is totally vicious i.e. given $m_0, m_1 \in \mathcal{M}$, there exists a timelike future-pointing curve from m_0 to m_1.*

The previous proposition explains why, in the physical litterature (see e.g. [21], [51]), $\mathbb{T}$ and Σ are respectively called, *time machine*, and *velocity-of-light surface*. This last term is somewhat misleading since $\partial(\mathcal{M} \setminus \mathbb{T}) \subset \Sigma$, but it can happen that $\partial(\mathcal{M} \setminus \mathbb{T}) \neq \Sigma$ and Σ is not necessarily a hypersurface. If there exists no (r_0, z_0) satisfying (1.11), the theorem of implicit functions immediately assures that Σ is a C^2-hypersurface that is timelike because its normal $N = (\partial_r C - 1)dr + \partial_z C dz$ is spacelike since $g^{\mu,\nu} N_{\mu,\nu} = -(\partial_r C - 1)^2 - (\partial_z C)^2 < 0$. Moreover, this is a sufficient and necessary condition on C for a geometrical property of non-trapping type:

Proposition 1.2. *Let $m \in C^2(\mathbb{R}_\tau; \mathcal{M})$ be a path. Then the following assertions are equivalent:*

(i) m is a null geodesic and for some $T > 0$:

$$m(\mathbb{R}) \subset [-T, +T]_t \times \Sigma_0, \tag{1.10}$$

(ii) there exists $(t_0, \varphi_0, r_0, z_0)$, $\lambda \in \mathbb{R}^*$, such that:

$$\left\{ \begin{array}{l} C(r_0, z_0) = r_0 > 0, \;\; \partial_r C(r_0, z_0) = 1, \;\; \partial_z C(r_0, z_0) = 0, \\ m(\tau) = (t_0, \varphi_0 + \lambda\tau, r_0, z_0) \,. \end{array} \right. \tag{1.11}$$

We say that Σ_0 is *Non-Confining* if there exists no null geodesic included in $\{t_0\} \times \Sigma_0$ for some t_0. Following the previous result, a necessary and sufficient condition is

$$C(r_0, z_0) = r_0 > 0 \Longrightarrow (\partial_r C(r_0, z_0), \partial_z C(r_0, z_0)) \neq (1, 0), \tag{1.12}$$

and in this case Σ is a C^2 timelike hypersurface.

1.2. The Wave Equation. We consider the compacty supported, scalar perturbations of the massless wave equation, invariant with respect to the both Killing vector fields ∂_t, ∂_φ:

$$L := \Box_g + V, \tag{1.13}$$

where

$$V \in C_0^0(\mathbb{R}_x^3; \mathbb{R}), \;\; \partial_\varphi V = 0. \tag{1.14}$$

These assumptions are fulfilled in the important case of the conformally invariant wave equation for which $V = \frac{1}{6} R_g$ where R_g is the scalar curvature of $(\mathcal{M}, g)$. We know that the D'Alembertian on a lorentzian curved space-time is strictly hyperbolic in a local sense (see e.g. [17]). The global hyperbolicity is more delicate. We denote

$$P_2(m, \xi) := g^{\mu,\nu}(m) \xi_\mu \xi_\nu, \;\; m \in \mathcal{M}, \;\; \xi \in T_m^* \mathcal{M}, \tag{1.15}$$

the principal symbol of L.

Proposition 1.3.

(1) *Let α be in $\mathbb{R}$. Then, $P_2(m,.)$ is (strictly) hyperbolic with respect to the covector $dt+\alpha d\varphi$ iff α satisfies:*

$$-C(m)-r<\alpha<r-C(m). \tag{1.16}$$

(2) *If $\Sigma\neq\emptyset$, there does not exist $F\in C^1(\mathcal{M};\mathbb{R})$ such that L is hyperbolic with respect to the level surfaces of F.*

The previous result implies in particular that in the interesting case where $\mathbb{T}\neq\emptyset$, the initial value problem for L with data specified on $M_{t_0}=\{t_0\}\times\mathbb{R}^3$ is not well posed. (1.16) shows that the failure of the global hyperbolicity is due to the very fast rotation of the torus. Nevertheless, since ∂_t is a Killing vector field, it will be interesting to investigate the solutions of $Lu=0$ as some distributions on $\mathbb{R}_t$, valued in some spaces of distributions on $\mathbb{R}^3_x$. In order to choose the functional framework, it is useful to note that since the time translation leaves the wave equation invariant, the Noether's theorem assures the existence of a conserved current. We formally obtain the conserved energy

$$\begin{aligned} E(u;t)& \\ :=\frac{1}{2}\int_{\mathbb{R}^3}&\left(1-\frac{C^2}{r^2}\right)\mid\partial_t u(t,x)\mid^2+\mid\nabla u(t,x)\mid^2+V(x)\mid u(t,x)\mid^2 dx.\end{aligned} \tag{1.17}$$

When $\mathbb{T}$ is not empty, this quadratic form is not definite positive. It is natural to look for the solutions of

$$Lu=0,\ \ u\in L^2_{loc}\left(\mathbb{R}_t;W^1(\mathbb{R}^3_x)\right), \tag{1.18}$$

where $W^1(\mathbb{R}^3_x)$ is the Beppo-Levi space defined as the completion of $C_0^\infty(\mathbb{R}^3_x)$ with respect to the norm:

$$\parallel f\parallel^2_{W^1}=\int_{\mathbb{R}^3}\mid\nabla f(x)\mid^2 dx,\ \ \nabla:={}^t(\partial_{x^1},\partial_{x^2},\partial_z). \tag{1.19}$$

The choice of the regularity of $\partial_t u$ is less clear when $\mathcal{M}$ is not globally hyperbolic since $\left(1-\frac{C^2}{r^2}\right)$ is negative on $\mathbb{T}_0$ and the energy is not a positive form. We introduce the space:

$$L^2_C(\mathbb{R}^3_x):=L^2\left(\mathbb{R}^3_x,\mid 1-\frac{C^2}{r^2}\mid dx\right), \tag{1.20}$$

and we investigate the solutions u of (1.18) satisfying:

$$\partial_t u \in L^2_{loc}\left(\mathbb{R}_t; L^2_C(\mathbb{R}^3_x)\right). \tag{1.21}$$

If Σ_0 is non-confining, we prove that u is much more regular by using the results of J-M. Bony [7]:

Proposition 1.4. *We assume that Σ_0 is Non-Confining. Let u be such that*

$$u \in L^2_{loc}\left(\mathbb{R}_t; W^1(\mathbb{R}^3_x)\right), \quad Lu \in L^2_{loc}\left(\mathbb{R}_t; L^2(\mathbb{R}^3_x)\right).$$

Then we have:

$$u \in C^0\left(\mathbb{R}_t; H^{\frac{1}{2}}_{loc}(\mathbb{R}^3_x)\right), \quad \left(1-\frac{C}{r}\right)\partial_t u \in C^0\left(\mathbb{R}_t; H^{-\frac{1}{2}}_{loc}(\mathbb{R}^3_x)\right). \tag{1.22}$$

Thanks to the result of continuity stated in Proposition 1.4, we may investigate the uniqueness of a possible solution of $Lu = 0$ for data specified on M_{t_0}. First we prove that $u = 0$ on $\mathcal{M}$ when $u = (C-r)\partial_t u = 0$ on M_{t_0}. This result is neither a consequence of the uniqueness theorem for the strictly hyperbolic operators ([27], Theorem 23.2.7) because the level surfaces M_t are not non-characteristic since $P_2(m, dt) = 0$ on Σ, nor a direct application of the conservation of the energy since $E(u)$ is not definite positive. Moreover, when $\mathcal{M}$ is totally vicious, i.e. $\mathbb{T} \neq \emptyset$, and the Non-Confining Condition is fullfiled, we would like that $u = 0$ on $\mathcal{M}$ when $u = 0$ on $\mathbb{T}$. Unfortunately, although Σ is non-characteristic, we cannot use the classical results of unique continuation: on the one hand, 0 is a double real root of $P_2(m, dt + \tau N) = 0$ for $m \in \Sigma$, $N = (\partial_r C(m) - 1)dr + \partial_z C(m)dz$, hence we cannot apply the Calderon Theorem ([27], theorem 28.1.8). On the other hand, we have for $m \in \Sigma$:

$$\{P_2, \{P_2, C-r\}\}(m, dt) = -4\left(\mid \partial_r C(m) - 1 \mid^2 + \mid \partial_z C(m) \mid^2\right) < 0,$$

hence Σ is nowhere strongly pseudo-convex, and we can no more use the uniqueness theorems for second order operators of real principal type due to N. Lerner and L. Robbiano (see [27], Theorem 28.4.3) to deduce that $u = 0$ on $\mathcal{M}$, from $u = 0$ on $\mathbb{T}$. A key ingredient is the following result involving the Aronszajn-Cordes theorem: We assume that Σ_0 is Non-Confining. Let u satisfying (1.18) and such that for some $t_0 \in \mathbb{R}$ $u = \partial_t u = 0$ on $\{t_0\} \times \mathbb{T}_0$. Then $u = 0$ on $\mathbb{T}$.

Theorem 1.5. *We assume that Σ_0 is Non-Confining and $\mathbb{T}_0 \neq \emptyset$. Let u be satisfying* (1.18) *and one of the following conditions for some $t_0 \in \mathbb{R}$: $u = \left(1-\frac{C}{r}\right)\partial_t u = 0$ on M_{t_0}. Then $u = 0$ on $\mathcal{M}$.*

The sequel of this work deals with the problem of the existence of global solutions, that is not obvious when the manifold is not causal. We introduce the vector space

$$\mathcal{E} := \left\{ u \in C^0\left(\mathbb{R}_t; W^1(\mathbb{R}^3_x)\right);\ \ Lu = 0,\ \ \partial_t u \in C^0\left(\mathbb{R}_t; L^2_C(\mathbb{R}^3_x)\right)\right\}, \tag{1.23}$$

and the space of the admissible Cauchy data:

$$\mathcal{H} := \left\{ (f,g) \in W^1(\mathbb{R}^3_x) \times L^2_C(\mathbb{R}^3_x);\ \ \exists u \in \mathcal{E},\ \ \mathbf{u}(0) = (f,g)\right\}, \tag{1.24}$$

where for $v \in C^1\left(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}^3_x)\right)$, we put

$$\mathbf{v} := \begin{pmatrix} v \\ \partial_t v \end{pmatrix}. \tag{1.25}$$

A priori, when $\mathbb{T} \neq \emptyset$, $\mathcal{H}$ is not an Hilbert space for the norm of $W^1 \times L^2_C$. The previous Theorem assures that the family of maps

$$U(t):\ \mathbf{u}(0) \in \mathcal{H} \longmapsto \mathbf{u}(t) \in \mathcal{H}. \tag{1.26}$$

is a strongly continuous group of linear operators on $\mathcal{H}$. In the following parts we construct global solutions u with $E(u) = 0$ or $E(u) > 0$. We let open the problem of the existence of global solution with negative energy.

1.3. The Resonant States. In this section, we investigate the global solutions $u \in H^1_{loc}(\mathcal{M})$ by separation of the variable t:

$$u(t,x) = e^{\lambda t} v(x), \tag{1.27}$$

with $\lambda \in \mathbb{C}$ and v is a distribution on $\mathbb{R}^3_x$. Then u is solution of

$$Lu = 0 \ \text{ in } \ \mathcal{M}, \tag{1.28}$$

iff $v \in L^2_{loc}(\mathbb{R}^3_x)$ is solution of the homogeneous reduced wave equation:

$$\Delta v + \frac{2C\lambda}{r^2}\partial_\varphi v - \lambda^2\left(1 - \frac{C^2}{r^2}\right) v - Vv = 0 \ \text{ on } \ \mathbb{R}^3. \tag{1.29}$$

By the standard result of elliptic regularity, $v \in H^2_{loc}(\mathbb{R}^3)$ and $v \in C^\infty$ for $\mid x \mid$ large enough, since C and V are continuous and compactly supported. (1.29) is similar to the acoustic wave equation in an inhomogeneous medium (see e.g. [11], [29], [43], [50]); the crucial difference is that $1 - r^{-2}C^2$ that plaies the role of the refractive index, is null on Σ_0 and negative in $\mathbb{T}_0$. We start by proving a result of Rellich type, stating that there exists no t-periodic, non constant, solution of $Lu = 0$ satisfying some natural constraint at the space infinity.

Lemma 1.6. *Let v be a solution of* (1.29) *for $\lambda \in i\mathbb{R}^*$, satisfying one of the following condition:*

$$v \in L^2(\mathbb{R}^3) \cup W^1(\mathbb{R}^3); \tag{1.30}$$

$$\frac{x}{\mid x \mid}.\nabla v + \lambda v = O\left(\frac{1}{\mid x \mid^2}\right), \quad \mid x \mid \to \infty; \tag{1.31}$$

$$\frac{x}{\mid x \mid}.\nabla v - \lambda v = O\left(\frac{1}{\mid x \mid^2}\right), \quad \mid x \mid \to \infty; \tag{1.32}$$

Then $v = 0$.

For $\lambda = 0$ the result is well known: for non negative potential V, the conclusion of the Lemma is valid; for general potential V, since the form $v \mapsto \int V \mid v \mid^2$ is compact on $H^1_{loc}(\mathbb{R}^3)$, the space of solutions of (1.29) with $\lambda = 0$ is of finite dimension.

Lemma 1.6 shows that we have to look for the non trivial solutions of the homogeneous reduced wave equation, for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. We adapt at our problem the concept of outgoing (resp. incoming) solution by Lax-Phillips [33]. Given $\lambda \in \mathbb{C}$, $f \in \mathcal{E}'$, the space of the compactly supported distributions, a solution $v_\lambda^{+(-)}$ of

$$\Delta v + \frac{2C\lambda}{r^2}\partial_\varphi v - \lambda^2\left(1 - \frac{C^2}{r^2}\right)v - Vv = f \ \ on \ \ \mathbb{R}^3, \tag{1.33}$$

is said to be λ-*outgoing* (resp. λ-*incoming)* if

$$v_\lambda^{+(-)} = \gamma_\lambda^{+(-)} * \left[f - \frac{2C\lambda}{r^2}\partial_\varphi v_\lambda^{+(-)} - \lambda^2\frac{C^2}{r^2}v_\lambda^{+(-)} + Vv_\lambda^{+(-)}\right], \tag{1.34}$$

where

$$\gamma_\lambda^{+(-)}(x) := -\frac{e^{-(+)\lambda|x|}}{4\pi \mid x \mid}. \tag{1.35}$$

It is well known that in the case $\lambda \in i\mathbb{R}$, the λ-outgoing (resp. λ-incoming) condition is equivalent to the Sommerfeld radiation condition (1.31) (resp. (1.32)

A complex number λ is an *outgoing resonance (resp. incoming resonance)*, if there exists a non null λ-outgoing (resp. λ-incoming) solution $v_\lambda^{+(-)}$ of (1.29), called *resonant state*. We remark that when a resonant state v_λ has a finite energy, i.e. $v_\lambda \in H^1(\mathbb{R}^3)$, the total energy of the time dependant solution $u_\lambda(t,x) = e^{\lambda t} v_\lambda(x)$ is zero:
(1.36)

$$E(u_\lambda) = \frac{1}{2} e^{2\Re(\lambda)t} \int_{\mathbb{R}^3} \mid \lambda \mid^2 \left(1 - \frac{C^2}{r^2}\right) \mid v_\lambda \mid^2 + \mid \nabla v_\lambda \mid^2 + V \mid v_\lambda \mid^2 dx = 0.$$

We denote $\mathcal{R}^{+(-)}$ the set of the outgoing (incoming) resonances. Because C and V are real axisymmetric, and since we may take $v_\lambda^+(x^1, -x^2, z) = v_{-\lambda}^-(x^1, x^2, z)$, it is easy to see that:

$$\lambda \in \mathcal{R}^+ \Longleftrightarrow \overline{\lambda} \in \mathcal{R}^+, \tag{1.37}$$

$$\lambda \in \mathcal{R}^+ \Longleftrightarrow -\lambda \in \mathcal{R}^-. \tag{1.38}$$

Hence we shall consider only the set of the outgoing resonances, simply called "resonances", and we omit the superscript $+$: $\mathcal{R} := \mathcal{R}^+$, $v_\lambda := v_\lambda^+$.

Theorem 1.7. *$\mathcal{R}$ is a discrete subset of $\mathbb{C}$, and when $\mathbb{T}_0 \neq \emptyset$ then $Card(\mathcal{R} \cap]0, \infty[) = \infty$.*

This last result can be physically interpreted as follows: in the framework of the studies of the stability of the manifolds of the General Relativity, the existence of an infinite set of resonant states with finite energy means that we cannot prove the possible stability of the metric (1.3) by a method of perturbation (see e.g. the works of Y. Choquet-Bruhat, A. Fischer, J. Marsden); hence we can suspect that the manifold is actually nonlinearly instable in a suitable set of solutions of inhomogeneous Einstein equations. This agrees with the "conjecture of chronological protection" by S. Hawking [25], that states that any universe with closed timelike curve is instable.

1.4. Scattering States. When $\mathbb{T}$ is not empty, the manifold is totally vicious, hence there exists no Cauchy hypersurface. Nevertheless the global Cauchy problem is well posed for regular data specified at the past null infinity, and these solutions are asymptotically free at the future null infinity (*Scattering States*). Furthermore, the Scattering Operator S is well defined for any free wave with finite energy, but, unlike the usual situations, the wave operators are *not* causal. As regards the mathematical tools, we keep the features of the scattering theory, that involve neither the positivity of the energy, nor the existence of a unitary group: we use the generalised eigenfunctions method.

We start with a uniqueness result for the solutions with some given asymptotic behaviour. We recall some basic notations for the wave equation on the Minkowski space-time:

$$L_0 u_0 := \partial_t^2 u_0 - \Delta_x u_0 = 0, \ \ (t,x) \in \mathbb{R} \times \mathbb{R}^3. \tag{1.39}$$

The Cauchy problem is solved in $\mathcal{D}'(\mathbb{R}^3_x)$ by the group $U_0(t)$:

$$U_0(t)\mathbf{u}_0(0) = \mathbf{u}_0(t). \tag{1.40}$$

We introduce: the spaces associated with the finite energy waves,

$$\mathcal{E}_0 := \left\{ u_0 \in C^0\left(\mathbb{R}_t; W^1(\mathbb{R}^3_x)\right); \ \ L_0 u_0 = 0, \ \ \partial_t u_0 \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}^3_x)\right)\right\}, \tag{1.41}$$
$$\mathcal{H}_0 := W^1(\mathbb{R}^3_x) \times L^2(\mathbb{R}^3_x),$$

which are Hilbert spaces for the energy norm

$$\| u_0 \|^2_{\mathcal{E}_0} = \| \mathbf{u}_0(t) \|^2_{\mathcal{H}_0} := \frac{1}{2}\int_{\mathbb{R}^3} | \partial_t u_0(t,x) |^2 + | \nabla u_0(t,x) |^2 \, dx, \tag{1.42}$$

and $U_0(t)$ is a strongly continuous unitary group on $\mathcal{H}_0$. We denote $\mathcal{E}_0^\infty$ the space of the *regular wave packets* that are the smooth solutions u_0 of (1.39) such that

$$\begin{aligned} \hat{u}_0(0,\xi) &:= \int e^{-ix.\xi} u_0(0,x)dx, \ \ \partial_t \hat{u}_0(0,\xi) \\ &:= \int e^{-ix.\xi} \partial_t u_0(0,x)dx \in C_0^\infty\left(\mathbb{R}^3_\xi \setminus \{0\}\right). \end{aligned} \tag{1.43}$$

Theorem 1.8. *Given $u_0^- \in \mathcal{E}_0^\infty$, there exists a unique $u \in \mathcal{E}$ such that $\partial_t u \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}^3_x)\right)$ and satisfying*

$$\| \mathbf{u}(t) - \mathbf{u}_0^-(t) \|_{\mathcal{H}_0} \to 0, \ \ t \to -\infty. \tag{1.44}$$

Moreover there exists a unique $u_0^+ \in \mathcal{E}_0$ such that:

$$\| \mathbf{u}(t) - \mathbf{u}_0^+(t) \|_{\mathcal{H}_0} \to 0, \ \ t \to +\infty, \tag{1.45}$$

and we have:

$$\| u_0^- \|^2_{\mathcal{E}_0} = E(u) = \| u_0^+ \|^2_{\mathcal{E}_0}, \tag{1.46}$$

$$u_0^+ \in \mathcal{E}_0^\infty. \tag{1.47}$$

2. Superradiance of the Charged Black-Holes. The asymptotic behaviours of classical fields on several important curved space-times of General Relativity, have been the subject of numerous studies. We can mention the works on the scalar equations by the author [1], [2], D. Häfner [23], [24], J-P. Nicolas [39], and on the Dirac system by F. Melnyk [38], [37], J-P. Nicolas [40]. As regards the propagation of the energy, there exists a deep difference between the bosons and the fermions: the L^2 norm of a field with half-integral spin, is conserved, while the conserved energy of the Klein-Gordon field on a curved background is not necessarily positive. In such cases of indefinite conserved energy, the field is allowed to extract energy from a particular region of space-time, for instance the ergosphere of a Kerr black-hole, or the dyadosphere of a charged black-hole. This phenomenon has been described, for the first time, by R. Penrose who proved that a classical particle can enter the ergosphere of a rotating black hole, and come out again with more energy than it originally had. The corresponding effect for integral spin fields is called superradiance [21], [46]. To our knowledge, a rigorous mathematical analysis is missing, and the present study is a first step in this direction since we can apply the results of the previous sections to the superradiant scattering of charged Klein-Gordon fields by a charged black-hole in an expanding universe.

The spin 0 field with mass $m \geq 0$, and charge $e \in \mathbb{R}$, on a lorentzian manifold $(\mathcal{M}, g)$ endowed with an electromagnetic potential $A_\mu dx^\mu$, obeys the Klein-Gordon equation

$$(2.1) \qquad \left(\nabla_\mu - ieA_\mu\right)\left(\nabla^\mu - ieA^\mu\right)\Phi + m^2\Phi + \xi R\Phi = 0,$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, and $\xi \in \mathbb{R}$ is a numerical factor. We are concerned with the 3+1 dimensional, spherically symmetric space-time $\mathbb{R}_t \times I_r \times S^2_\omega$, I being a real open interval, that describes a black hole in an expanding universe. In this case the metric can be written as:

$$(2.2) \qquad g_{\mu\nu}dx^\mu dx^\nu = F(r)dt^2 - [F(r)]^{-1}dr^2 - r^2 d\omega^2,$$

where $F \in C^2([r_-, r_+])$, $0 < r_- < r_+ < \infty$, is called the lapse function, and satisfies:

$$(2.3) \quad F(r_-) = F(r_+) = 0, \;\; r_- < r < r_+ \Rightarrow 0 < F(r), \qquad 0 < F'(r_-), \;\; F'(r_+) < 0.$$

r_- is the radius of the Horizon of the Black-Hole, r_+ is the radius of the Cosmological Horizon. The Ricci scalar is given by

$$R = F'' + \frac{4}{r}F' + \frac{2}{r^2}(F-1).$$

We assume that the electromagnetic potential is electrostatic and also spherically symetric

$$A_\mu dx^\mu = A_t(r)dt,\ \ A_t \in C^1([r_-, r_+]),\ \ A_t(r_-) \neq A_t(r_+). \tag{2.4}$$

These hypotheses are satisfied, for a suitable choice of the physical parameters, in the important case of a charged black-hole in an expanding universe, for which the DeSitter-Reissner-Nordstrøm metric, and the Maxwell connection, are given by:

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2,\quad A_t(r) = \frac{Q}{r}. \tag{2.5}$$

Here $0 < M$ and $Q \in \mathbb{R}$ are the mass and the charge of the black-hole, $\Lambda > 0$ is the cosmological constant (see e.g. [38]).

It is convenient to push the horizons away to infinity by putting:

$$x = \frac{1}{F'(r_-)}\left\{\ln \mid r - r_- \mid - \int_{r_-}^{r}\left[\frac{1}{r-r_-} - \frac{F'(r_-)}{F(r)}\right]dr\right\}.$$

Then $u = r\Phi$ is solution of

$$\left(\partial_t - iA(x)\right)^2 u - \partial_x^2 u - B(x)\Delta_{S^2} u + C(x)u = 0,\ \ t \in \mathbb{R},\ x \in \mathbb{R},\ \ \omega \in S^2,$$

with

$$A(x) = eA_t(r),\ \ B(x) = \frac{1}{r^2}F(r),$$
$$C(x) = \left(\xi F''(r) + \frac{4\xi + 1}{r}F'(r) + \frac{2\xi}{r^2}F(r) - \frac{2\xi}{r^2} + m^2\right)F(r).$$

The conserved energy is given by:

$$E(u) := \int_{\mathbb{R}\times S^2_\omega} \big(\mid \partial_t u(t,.) \mid^2 + \mid \partial_x u(t,.) \mid^2$$
$$+ B(x) \mid \nabla_\omega u(t,.) \mid^2 + \left[C(x) - A^2(x)\right] \mid u(t,.) \mid^2\big) dx d\omega.$$

The *dyadosphere* $\mathcal{D}_{e,m}$ is defined as the region outside the black hole horizon where the electrostatic energy, associated with the charge e of the field, exceeds the gravitational interacting energy associated with the mass m of the field:

$$\mathcal{D}_{e,m} := \left\{x \in \mathbb{R};\ A^2(x) > C(x)\right\} \times S^2_\omega.$$

We remark that, because of the existence of the cosmological horizon, unlike the case of the asymptotically flat space-time for which $F(r) \to 1$ as $r \to +\infty$, $\mathcal{D}_{e,m}$ is never empty, whatever the mass of the field and the gauge transform on A. Furthermore, if $\mid e \mid$ is large enough, we can have $\mathcal{D}_{e,m} = \mathbb{R}_x \times S^2_\omega$.

Taking advantage of the spherical symmetry, we expand $u(t,x,.)$ on the basis of spherical harmonics $Y_{l,m}$ of $L^2\left(S^2_\omega\right)$:

$$u(t,x,\omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} u_{l,m}(t,x)Y_{l,m}(\omega).$$

Finally $u_{l,m}$ is solution of the gyroscopic Klein-Gordon equation

$$(\partial_t - iA(x))^2 u_{l,m} - \partial_x^2 u_{l,m} - V(x)u_{l,m} = 0, \;\; t \in \mathbb{R}, \; x \in \mathbb{R}, \tag{2.6}$$

with

$$V(x) = l(l+1)B(x) + C(x).$$

Since A and V satisfy

$$\mid A(x) - eA_t(r_\pm) \mid + \mid A'(x) \mid + \mid V(x) \mid \leq Ce^{F'(r_\pm)x}, \;\; x \to \pm\infty,$$

Therfore we deal with the gyroscopic Klein-Gordon equation

$$(\partial_t - iA(x))^2 u - \partial_x^2 u + V(x)u = 0, \;\; t \in \mathbb{R}, \;\; x \in \mathbb{R}, \tag{2.7}$$

with the main hypotheses

$$V(x) \longrightarrow 0, \;\; A(x) \longrightarrow a_\pm, \;\; x \longrightarrow \pm\infty, \;\; a_- \neq a_+. \tag{2.8}$$

Other one-dimensional field equations with step-like perturbations have been studied: the existence of a Scattering Operator that is unitary, was established for the Dirac system by S. N. M. Ruijsenaars and P. J. M. Bongaarts [46], and for the Schrödinger equation by E. B. Davies and B. Simon [13]. The key point for both these equations is the conservation of the L^2 norm. The situation drastically differs for the Klein-Gordon equation (2.7) since the conserved energy

$$E(u,t) := \int \mid \partial_t u(t,x) \mid^2 + \mid \partial_x u(t,x) \mid^2 + \left[V(x) - A^2(x)\right] \mid u(t,x) \mid^2 dx \tag{2.9}$$

is not always positive. In particular, when A satisfies the steplike hypothesis (2.8), the set of modes is finite dimensional, but there exists no finite codimensional subspace of Cauchy data, on which this energy is positive. This is the root of the so called *Klein paradox*. Nevertheless we shall be able to describe the asymptotic behaviours of the solutions of (2.7), and to prove the existence of a Scattering Operator the norm of which is always *strictly larger* than one: this is the *superradiance*. Furthermore, in some situations, there exist solutions polynomially increasing in time (*hyperradiant* modes). Recently, the γ-bursts have been attributed to the superradiance of the charged black-holes (R. Ruffini [45]). Since we cannot use this energy to get some asymptotic estimates, we construct the spectral representation for the harmonic equation, then we establish the existence of the Scattering Operator the symbol of which has a norm strictly larger than 1.

2.1. Spectral Decomposition. We investigate the harmonic Klein-Gordon equation:

$$\frac{d^2}{dx^2}y + [k - A(x)]^2\, y - V(x)y = 0, \quad x \in \mathbb{R}, \quad k \in \mathbb{C}. \tag{2.10}$$

We assume that the potentials satisfy:

$$A \in L^\infty(\mathbb{R};\mathbb{R}), \quad V \in L^\infty(\mathbb{R};\mathbb{R}), \tag{2.11}$$

and there exist $\alpha > 0$, $a_\pm \in \mathbb{R}$, $a_- < a_+$, such that:

$$\int_{\mathbb{R}^\pm} (\mid A(x) - a_\pm \mid + \mid V(x) \mid)\, e^{\alpha|x|}dx + \sup_{0<|h|<1} \int_{-\infty}^{\infty} \left| \frac{A(x+h) - A(x)}{h} \right| e^{\alpha|x|}dx < \infty. \tag{2.12}$$

We start by constructing suitable Jost functions, taking the different asymptotics as $x \to \pm\infty$, into account. For any $k \in \mathbb{C}$, $\Im k > -\frac{\alpha}{2}$ (resp. $\Im k < \frac{\alpha}{2}$), there exists unique functions $f^\pm_{in(out)}(k;x) \in C^1(\mathbb{R}_x)$, solutions of (2.10) and satisfying $\lim_{x\to\pm\infty} f^\pm_{in(out)}(k;x) - e^{\pm(\mp)i(k-a_\pm)x} = 0$. Moreover, for each $x \in \mathbb{R}$, they are analytic functions of $k \in \mathbb{C}$, $\Im k > -\frac{\alpha}{2}$ (resp. $\Im k < \frac{\alpha}{2}$). The following Wronskians do not depend of x:

$$W_{in(out)}(k) := [f^+_{in(out)}, f^-_{in(out)}](k).$$

Since W_{in} is an analytic function of $k \in \mathbb{C}$, $\Im k > -\alpha/2$, the set of its zeros is locally finite, and each of them is of finite multiplicity. We introduce

$$\sigma_p := \{k \in \mathbb{C};\ \ \Im k > 0,\ \ W_{in}(k) = 0\},$$

$$\sigma_{ss} := \{k \in \mathbb{R}; \quad W_{in}(k) = 0\},$$
$$\mathcal{R} := \{k \in \mathbb{C}; \ -\frac{\alpha}{2} < \Im k < 0, \ \ W_{in}(k) = 0\}.$$

The elements of σ_p are the *eigenvalues* or *normal modes*, and the elements of $\mathcal{R}$ are the *resonances* or *quasinormal modes.* The *Klein zone* is the open interval $I_K :=]a_-, a_+[$. We shall see that the asymptotic behaviour of the solutions of the Klein-Gordon equation with step-like potential A is very peculiar, and justifies to call *superradiant modes* the real frequencies in $I_K \setminus \sigma_{ss}$, and *hyperradiant modes* the elements of σ_{ss}, that play the role of the *spectral singularities* of the quadratic pencils with short range complex potential [5]. In the simple example of the step potential $A_0(x) = a\mathbf{1}_{]-\infty,0]}(x)$, $a \in \mathbb{R}^*$, $V = 0$, we easily find $W_{in}(k) = i(2k-a)$, hence $\sigma_p = \mathcal{R} = \emptyset$, $\sigma_{ss} = \left\{\frac{a}{2}\right\}$. There exists cases where there is no hyperradiant mode. For instance if we choose $A_1(x) = 1 - \tanh(x)$ or $A_2(x) = 1$ when $x < 0$, $A_2(x) = 0$ when $x > 1$, $A_2(x) = 1 - x$ when $0 \leq x \leq 1$, and $V(x) = 0$, we can compute $W_{in}(\kappa)$ by using some formal calculus system. We get frightful combinations of hypergeometric functions for A_1, and Bessel functions for A_2 and the investigation of the possible roots of the equation $W_{in}(\kappa) = 0$ seems to be rather delicate. A numerical evaluation of $\mid W_{in}(\kappa) \mid$ using the Maple system, clearly shows that $\sigma_{ss} = \emptyset$ for both these potentials. In the general case, we prove that σ_p and σ_{ss} are finite sets and

$$\sigma_{ss} \subset I_K. \tag{2.13}$$

For the Schrödinger equation, i.e. $A = 0$, we know that the multiplicity of the zeros of W_{in} is simple. This is proved in [15] for the short range potentials V, and in [11] for the steplike case. Unlike this situation, when $A \neq 0$, the multiplicity $m(k) \in \mathbb{N}^*$ of $k \in \mathbb{C}$, defined by

$$\frac{d^l}{dk^l} W_{in}(k) = 0, \ \ 0 \leq l \leq m(k) - 1, \ \ \frac{d^{m(k)}}{dk^{m(k)}} W_{in}(k) \neq 0,$$

can be strictly larger that 1. As an example, we choose $A_3(x) = \mathbf{1}_{]-\infty,0]\cup[\frac{\pi}{3},\frac{2\pi}{3}]}(x)$, $V = 0$. By tedious but elementary calculations, we check that $W_{in}\left(\frac{1}{2}\right) = W'_{in}\left(\frac{1}{2}\right) = 0$. We put:

$$\nu := \max_{\kappa \in \sigma_{ss}} (m(\kappa)) \ \text{ if } \ \sigma_{ss} \neq \emptyset, \ \ \nu := 0 \ \ if \ \ \sigma_{ss} = \emptyset. \tag{2.14}$$

We introduce the transmission coefficients $T^{\pm}(\kappa)$ and the reflection coefficients $R^{\pm}(\kappa)$, defined for $\kappa\in\mathbb{R}\setminus\sigma_{ss}$, by:

$$\kappa\neq a_{\pm},\Rightarrow T^{\pm}(\kappa):=\frac{1}{\tau^{\pm}_{out}(\kappa)},\quad R^{\pm}(\kappa):=\frac{\rho^{\mp}_{out}(\kappa)}{\tau^{\mp}_{out}(\kappa)},$$
$$R^{\pm}(a_{\pm})=-1,\quad T^{\pm}(a_{\mp})=0. \tag{2.15}$$

These quantities describe the propagation of the field as $x\to\pm\infty$:

$$f^{\pm}_{in}=T^{\mp}f^{\mp}_{out}-R^{\pm}f^{\pm}_{out}.$$

$R^{\pm}(\kappa)$ and $T^{\pm}(\kappa)$ are analytic functions on $\mathbb{R}_{\kappa}\setminus\sigma_{ss}$ and satisfy

$$\frac{\kappa-a_{\pm}}{\kappa-a_{\mp}}\mid T^{\pm}(\kappa)\mid^{2}+\mid R^{\pm}(\kappa)\mid^{2}=1, \tag{2.16}$$

$$\mid T^{+}(\kappa)T^{-}(\kappa)-R^{+}(\kappa)R^{-}(\kappa)\mid=1, \tag{2.17}$$

$$\kappa\in\mathbb{R}\setminus I_{K}\Longrightarrow\mid R^{\pm}(\kappa)\mid\leq 1, \tag{2.18}$$

$$\kappa\in I_{K}\setminus\sigma_{ss}\Longrightarrow\mid R^{\pm}(\kappa)\mid>1, \tag{2.19}$$

$$\kappa\to\kappa_{j}\in\sigma_{ss}\Longrightarrow\mid R^{\pm}(\kappa)\mid,\mid T^{\pm}(\kappa)\mid\to\infty. \tag{2.20}$$

We emphasize that when κ is outside the Klein zone, the reflection coefficient is not greater than one as in the usual case of the decaying potential (i.e. $a_{\pm}=0$). But when κ is a superradiant mode, $\mid R^{\pm}(\kappa)\mid$ is strictly larger than one, but finite: this is the phenomenon of *superradiance* of the Klein-Gordon fields (2.19). At last $T^{\pm}$ and $R^{\pm}$ diverge at the hyperradiant modes. The situation differs for the Dirac or Schrödinger equations, for which the reflection is total in the Klein zone (*i.e.* $T=0,\ R=1$, see [13], [46]).

We construct the distorded Fourier transforms. Given $f\in C^{\infty}_{0}(\mathbb{R}_{x})$, $\varphi\in C^{\infty}_{0}(\mathbb{R}_{\kappa})$ we put:

$$F^{\pm}_{in(out)}(f)(k):=\int_{-\infty}^{\infty}f^{\pm}_{in(out)}(k;x)f(x)dx,\quad k\in\mathbb{C},\ \ +(-)\Im k\geq 0. \tag{2.21}$$

$F^{\pm}_{in(out)}$ is well defined from $C^{\infty}_{0}(\mathbb{R}_{x})$ to $L^{2}(\mathbb{R}_{\kappa})$, but, unlike the short range case, $a_{\pm}=0$, a problem arises for the low frequencies $\kappa=a_{\pm}$, with loss of regularity, when we want to define $F^{\pm}_{in(out)}$ on $L^2(\mathbb{R}_x)$. To overcome this difficulty, it is necessary to use the weighted L^2-spaces, $L^2_s(\mathbb{R}) := L^2(\mathbb{R}_y, (1+y^2)^s dy)$, $s \in \mathbb{R}$.

Proposition 2.1. *$F^{\pm}_{in(out)}$ that is defined from $C^\infty_0(\mathbb{R}_x)$ to $\cap_n \left[H^n \cap L^2_n\right](\mathbb{R}_\kappa)$, has a continuous extension:*

1. *from $L^2_{\frac{1}{2}+\delta}(\mathbb{R}_x)$ to $H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_\kappa) \cap \mathcal{E}'(\mathbb{R}_\kappa) + H^{\frac{1}{2}+\delta}(\mathbb{R}_\kappa)$, for any $\delta, \varepsilon > 0$;*
2. *from $L^2_1(\mathbb{R}_x)$ to $L^2(\mathbb{R}_\kappa)$;*
3. *from $H^1 \cap L^2_1(\mathbb{R}_x)$ to $L^2_1(\mathbb{R}_\kappa)$.*

There exists no continuous extension from $L^2_{\frac{1}{2}}(\mathbb{R}_x)$ to $\mathcal{D}'(\mathbb{R}_\kappa)$. For any $\delta > 0$ there exists no continuous extension from $L^2_{1-\delta}(\mathbb{R}_x)$ to $L^2(\mathbb{R}_\kappa)$.

We now introduce the inverse distorded Fourier transforms, defined for $\varphi \in C^\infty_0(\mathbb{R}_\kappa)$ by:

$$\Phi^{\pm}_{in(out)}(\varphi)(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f^{\pm}_{in(out)}(\kappa; x)\varphi(\kappa)d\kappa, \quad x \in \mathbb{R}. \tag{2.22}$$

Lemma 2.2. *There exists a constant $C > 0$, a function $C(s)$, such that for all $s \in \mathbb{R}$, $-1 \leq p \leq 1$, $\varphi \in C^\infty_0(\mathbb{R}_\kappa)$, we have*

$$\| \Phi^{\pm}_{in(out)}(\varphi) \|_{H^p(\mathbb{R}^{\pm}_x)} \leq C \| \varphi \|_{L^2_p(\mathbb{R}_\kappa)}, \quad \| \Phi^{\pm}_{in(out)}(\varphi) \|_{L^2_s(\mathbb{R}^{\pm}_x)} \leq C(s) \| \varphi \|_{H^s(\mathbb{R}_\kappa)} .$$

Moreover $\Phi^{\pm}_{in(out)}$ is a bounded operator from $\mathcal{E}'(\mathbb{R}_\kappa)$ to $H^2_{loc}(\mathbb{R}_x)$.

We are now ready to state the resolution of the identity.

Theorem 2.3. *There exists complex numbers $c_{\lambda,l}$, for $\lambda \in \sigma_p$, $0 \leq l \leq m(\lambda) - 1$, with $c_{\lambda,m(\lambda)-1} \neq 0$, such that for all $f \in L^2_s(\mathbb{R}_x)$, $s > \max(\frac{1}{2}, \nu - \frac{1}{2})$, where ν is defined by (2.14), we have for $p = 0, 1$:*

$$\begin{aligned} pf = {} & \Phi^{\pm}_{in}\left(\frac{i\kappa^p}{W_{in}(\kappa + i0)} F^{\mp}_{in}(f)\right) - \Phi^{\pm}_{out}\left(\frac{i\kappa^p}{W_{out}(\kappa - i0)} F^{\mp}_{out}(f)\right) \\ & + \sum_{\lambda\in\sigma_p} \sum_{l=0}^{m(\lambda)-1} c_{\lambda,l}\partial^l_k \left(k^p f^{\pm}_{in}(k; x) F^{\mp}_{in}(f)\right)(k = \lambda) \\ & + \overline{c_{\lambda,l}}\partial^l_k \left(k^p f^{\pm}_{out}(k; x) F^{\mp}_{out}(f)\right)(k = \overline{\lambda}). \end{aligned}$$

2.2. Scattering. In this section, we investigate the asymptotic behaviours in time of the solutions of the charged Klein-Gordon equation with the assumptions (2.11), (2.12):

$$(\partial_t - iA(x))^2 u - \partial_x^2 u + V(x)u = 0, \ \ t \in \mathbb{R}, \ x \in \mathbb{R}, \tag{2.23}$$

$$u(t=0,x) = u_0(x), \ \ \partial_t u(t=0,x) = u_1(x). \tag{2.24}$$

It is convenient to introduce the following Hilbert space:

$$H_s^1(\mathbb{R}) := \left\{ f \in L_s^2(\mathbb{R}), \ \ f' \in L_s^2(\mathbb{R}) \right\}, \ \ s \in \mathbb{R}, \ \ \| f \|_{H_s^1}^2 := \| f \|_{L_s^2}^2 + \| f' \|_{L_s^2(I)}^2 .$$

The Cauchy problem is well solved: For any $u_0 \in H_s^1(\mathbb{R})$, $u_1 \in L_s^2(\mathbb{R})$, $s \in \mathbb{R}$, there exists a unique solution $u \in C^0\left(\mathbb{R}_t; H_s^1(\mathbb{R}_x)\right) \cap C^1\left(\mathbb{R}_t; L_s^2(\mathbb{R}_x)\right)$ of (2.23), (2.24). To give a representation of the solution involving the distorded Fourier transforms, we introduce the operators

$$E_{in(out)}^{\pm} : (u_0, u_1) \mapsto E_{in(out)}^{\pm}(u_0, u_1)(k) := kF_{in(out)}^{\pm}(u_0) - iF_{in(out)}^{\pm}(u_1 - 2iAu_0),$$

and the Hilbert space of initial data, where ν is defined by (2.14):

$$X := H^1_{\max(\nu,1)}(\mathbb{R}_x) \times L^2_{\max(\nu,1)}(\mathbb{R}_x). \tag{2.25}$$

Proposition 2.4. *For any $(u_0, u_1) \in X$, the solution u is expressed by:*

$$\begin{aligned} u(t) =& \Phi_{in}^{\mp}\left(\frac{ie^{i\kappa t}}{W_{in}(\kappa + i0)} E_{in}^{\pm}(u_0, u_1) \right) - \Phi_{out}^{\mp}\left(\frac{ie^{ikt}}{W_{out}(\kappa - i0)} E_{out}^{\pm}(u_0, u_1) \right) \\ &+ \sum_{\lambda \in \sigma_p} \sum_{l=0}^{m(\lambda)-1} c_{\lambda,l} \partial_k^l \left(e^{ikt} f_{in}^{\mp}(k;x) E_{in}^{\pm}(u_0, u_1) \right)(k = \lambda) \\ &+ \overline{c_{\lambda,l}} \partial_k^l \left(e^{ikt} f_{out}^{\mp}(k;x) E_{out}^{\pm}(u_0, u_1) \right)(k = \overline{\lambda}) \end{aligned}$$

where the constants $c_{\lambda,l}$ are defined in Theorem 2.3.

We denote $\langle t \rangle := (1 + t^2)^{\frac{1}{2}}$. The energy estimates for the solutions are the following:

Theorem 2.5. *There exist $C>0$, $N\in\mathbb{N}$, such that for any $(u_0,u_1)\in X$, we have:*

$$
\begin{aligned}
&\parallel (u(t),\partial_t u(t)) \parallel_{H^1\times L^2}\\
&\quad\leq C\Big(\parallel (u_0,u_1)\parallel_X + \sum_{\kappa\in\sigma_{ss}}\sum_{l=0}^{m(\kappa)-1}\langle t\rangle^{m(\kappa)-l+\frac{1}{2}}\sum_{\substack{\sharp=+,-\\ \flat=in,out}}\left|\frac{d^l}{dk^l}E_\flat^\sharp(u_0,u_1)(\kappa)\right|\\
&\qquad+\sum_{\lambda\in\sigma_p}\sum_{l=0}^{m(\lambda)-1}\langle t\rangle^{m(\lambda)-l-1}\sum_{\sharp=+,-}\left|\frac{d^l}{dk^l}E_{in}^\sharp(u_0,u_1)(\lambda)\right|e^{-\Im(\lambda)t}\\
&\qquad+\left|\frac{d^l}{dk^l}E_{out}^\sharp(u_0,u_1)(\overline{\lambda})\right|e^{\Im(\lambda)t}\Big),
\end{aligned}
$$

$$\parallel (u(t),\partial_t u(t))\parallel_X\leq C<t>^N e^{\gamma|t|}\parallel (u_0,u_1)\parallel_X,\ \ \gamma:=\max_{\lambda\in\sigma_p}\Im\lambda.$$

By a microlocalization near the hyperradiant modes, we now construct solutions of finite energy with polynomial behaviour in time.

Theorem 2.6. *For all $\kappa\in\sigma_{ss}$, $l\leq m(\kappa)-1$, there exist $u_0,u_1\in C_0^\infty(\mathbb{R}_x)$, such that for any $x\in\mathbb{R}$, we have:*

$$u(t,x)=t^{m(\kappa)-l-1}e^{i\kappa t}f_{in}^{\mp}(\kappa;x)+o\left(t^{m(\kappa)-l-1}\right),\ t\to-\infty;$$

$$u(t,x)=o\left(t^{m(\kappa)-l-1}\right),\ t\to+\infty.$$

To investigate the scattering states, we must avoid the usual modes and the hyperradiant ones. Hence we introduce the following subspaces of finite codimension in X:

$$
\begin{aligned}
&X_{in(out)}\\
&:=\left\{(u_0,u_1)\in X;\forall k\in\sigma_p\cup\sigma_{ss},\forall l<m(k),\ \frac{d^l}{dk^l}E_{in(out)}^{\pm}(u_0,u_1)(k(\overline{k}))=0\right\},
\end{aligned}
$$

$$X_{scatt}:=X_{in}\cap X_{out}. \tag{2.26}$$

$X_{in(out)}$ and X_{scatt} are well defined, and they are Hilbert subspaces of X, invariant under the action of the group $G(t):\ (u_0,u_1)\mapsto(u(t),\partial_t u(t))$.

We now arrive at an important result of this work: the solutions with Cauchy data in X_{scatt} are asymptotically free. We emphazise that the conserved energy of such solutions given by (2.9) can be negative. In fact we do not use this conservation law to get our scattering theory.

Theorem 2.7. *For any $(u_0, u_1) \in X_{scatt}$ there exists unique $u^{\pm}_{in(out)} \in H^1(\mathbb{R})$ such that*

$$\begin{aligned}(2.27)\quad \lim_{t\to -(+)\infty} \| u(t,x) - (e^{+(-)ia_- x} u^{-}_{in(out)}(t-(+)x) \\ + e^{-(+)ia_+ x} u^{+}_{in(out)}(t+(-)x)) \|_{H^1(\mathbb{R}_x)} \\ + \| \partial_t u(t,x) - (e^{+(-)ia_- x} (u^{-}_{in(out)})'(t-(+)x) \\ + e^{-(+)ia_+ x} (u^{+}_{in(out)})'(t+(-)x)) \|_{L^2(\mathbb{R}_x)} = 0.\end{aligned}$$

u_0, u_1, $u^{\pm}_{in(out)}$ are bound by the following relations:

$$u^{\pm}_{in(out)} = +(-)\mathcal{F}^{-1}\left(\frac{i}{W_{in(out)}(\kappa+(-)i0)} E^{\mp}_{in(out)}(u_0,u_1)\right),$$

$$u_p = \Phi^{\pm}_{in}\left((i\kappa)^p \mathcal{F}\left(u^{\pm}_{in}\right)\right) + \Phi^{\pm}_{out}\left((i\kappa)^p \mathcal{F}\left(u^{\pm}_{out}\right)\right), \quad p = 0,1,$$

$$\| u^{\pm}_{in(out)} \|_{H^1(\mathbb{R})} \leq C \| (u_0,u_1) \|_X,$$

$$(2.28)\quad \forall \kappa \in \mathbb{R}\backslash\sigma_{ss}, \begin{pmatrix} \mathcal{F}\left(u^{+}_{out}\right)(\kappa) \\ \mathcal{F}\left(u^{-}_{out}\right)(\kappa) \end{pmatrix} = \begin{pmatrix} R^+(\kappa) & T^+(\kappa) \\ T^-(\kappa) & R^-(\kappa) \end{pmatrix} \begin{pmatrix} \mathcal{F}\left(u^{+}_{in}\right)(\kappa) \\ \mathcal{F}\left(u^{-}_{in}\right)(\kappa) \end{pmatrix}.$$

We introduce the Wave Operators

$$\mathbb{W}_{in(out)} : (u_0,u_1) \longmapsto \left(u^{+}_{in(out)}, u^{-}_{in(out)}\right).$$

Corollary 2.8. *$\mathbb{W}_{in(out)}$ is a one-to-one, continuous operator from X_{scatt} onto a subspace $Y_{in(out)}$ of $H^1(\mathbb{R}_s) \times H^1(\mathbb{R}_s)$. Moreover the map*

$$\left(u^+(s), u^-(s)\right) \longmapsto \left(\overline{u^+(-s)}, \overline{u^-(-s)}\right)$$

is one-to-one from Y_{in} onto Y_{out}.

This result assures that $\mathbb{W}^{-1}_{in(out)}$ is well defined on $Y_{in(out)}$, and we define the scattering operator by:

$$\mathbb{S} = \mathbb{W}_{out} \left(\mathbb{W}_{in}\right)^{-1}.$$

Since we do not know if $Y_{in(out)}$ is closed, the question of the continuity of $\mathbb{W}^{-1}_{in(out)}$ remains open. Therefore we want to construct continuous inverse Wave Operators formally given by:

$$\mathbf{\Omega}_{in(out)} : \left(u^{+}_{in(out)}, u^{-}_{in(out)}\right) \longmapsto (u_0, u_1),$$

such that limit (2.27) is satisfied. When $\sigma_p \neq \emptyset$, the modes associated with an eigenvalue are exponentially decreasing as $t \to +(-)\infty$, hence $\mathbf{\Omega}_{in(out)}$ would be multivalued. Therefore it is natural to assume that there exists no such exponentially damped modes.

Proposition 2.9. *When $\sigma_p = \emptyset$, there exists $q \geq 1$, and bounded operators $\mathbf{\Omega}_{in(out)}$ from $\left[H^1_{\max(\nu,1)}(\mathbb{R}) \cap H^1_q(\mathbb{R}^{-(+)})\right]^2$, to $X_{in(out)} \cap D\left(\mathbb{W}_{in(out)}\right)$ such that*

$$\mathbb{W}_{in(out)}\mathbf{\Omega}_{in(out)} = Id \ \ on \ \ \left[H^1_{\max(\nu,1)}(\mathbb{R}) \cap H^1_q(\mathbb{R}^{-(+)})\right]^2.$$

When $\sigma_{ss} \neq \emptyset$, (2.28) shows that the continuity of $\mathbb{S}$ are not clear. Nevertheless we can develop a complete scattering theory when there occurs no usual or hyperradiant mode. We need a subspace of X:

Lemma 2.10. *We assume $\sigma_{ss} = \sigma_p = \emptyset$. Then given $(u_0, u_1) \in X$, $E^{+}_{in}(u_0, u_1), E^{-}_{in}(u_0, u_1)$ belong to $H^1(\mathbb{R}_\kappa)$ iff $E^{+}_{out}(u_0, u_1), E^{-}_{out}(u_0, u_1)$ belong to $H^1(\mathbb{R}_\kappa)$. We put:*

$$X_1 := \left\{(u_0, u_1) \in X;\ E^{\pm}_{in/out}(u_0, u_1) \in H^1(\mathbb{R}_\kappa)\right\},$$

$$\| (u_0, u_1) \|^2_{X_1, in(out)} := \| E^{+}_{in(out)}(u_0, u_1) \|^2_{H^1(\mathbb{R}_\kappa)} + \| E^{-}_{in(out)}(u_0, u_1) \|^2_{H^1(\mathbb{R}_\kappa)}.$$

$\| \,.\, \|_{X_1,in}$ and $\| \,.\, \|_{X_1,out}$ are two equivalent norms for which X_1 is a Hilbert space, invariant under the action of the group $G(t)$, and there exists $C > 0$ such that for all $(u_0, u_1) \in X_1$ we have:

$$\| (u_0, u_1) \|_X \leq C \| (u_0, u_1) \|_{X_1, in(out)}.$$

Moreover we have:

$$H^1 \cap \mathcal{E}'(\mathbb{R}_x) \times L^2 \cap \mathcal{E}'(\mathbb{R}_x) \subset X_1.$$

We introduce the Hilbert spaces:

$$K^{\pm} := \left\{ u \in H^1(\mathbb{R}_x);\ \ iu' + a_{\pm}u \in L^2_1(\mathbb{R}_x) \right\},$$
$$\| u \|^2_{K^{\pm}} := \| u \|^2_{H^1} + \| iu' + a_{\pm}u \|^2_{L^2_1}.$$

The scattering theory in the absence of modes, is described by the following:

Theorem 2.11. *We assume $\sigma_{ss} = \sigma_p = \emptyset$. Then $(u^+, u^-) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ belongs to $Y_{in(out)}$ iff*

$$u_p^{in(out)} := \Phi^+_{out(in)} \left(\frac{(i\kappa)^p}{\tau^+_{out(in)}(\kappa)} \mathcal{F}\left(u^-\right) \right) + \Phi^-_{out(in)} \left(\frac{(i\kappa)^p}{\tau^-_{out(in)}(\kappa)} \mathcal{F}\left(u^+\right) \right) \in H^1_1(\mathbb{R}_x), \quad p = 0, 1,$$

and in this case we have:

$$\mathbb{W}_{in(out)}(u_0^{in(out)}, u_1^{in(out)}) = (u^+, u^-).$$

Moreover we have:

$$H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R}) \subset Y_{in(out)} \subset K^+ \times K^-,$$

and $\mathbb{W}_{in(out)}$ are continuous, one-to-one, operators, from X_1 onto $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$, and from X onto $Y_{in(out)}$ endowed with the norm of $K^+ \times K^-$. The scattering operator is a continuous, one-to-one operator from $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$ onto $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$, and from Y_{in} onto Y_{out} where $Y_{in(out)}$ are endowed with the norm of $K^+ \times K^-$, or $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. This operator has the form:

$$\mathbb{S} = \mathcal{F}^{-1} \hat{S}(\kappa) \mathcal{F}, \quad \hat{S}(\kappa) := \begin{pmatrix} R^+(\kappa) & T^+(\kappa) \\ T^-(\kappa) & R^-(\kappa) \end{pmatrix}.$$

The scattering matrix $\hat{S}(k)$ is meromorphic on $\omega := \{k \in \mathbb{C};\ \mid \Im k \mid < \frac{\alpha}{2}\}$ and $k \in \omega$ is a pole of $\hat{S}$ iff $\overline{k}$ belongs to the set of resonances $\mathcal{R}$. Furthermore the scattering is superradiant for the frequencies in the Klein zone:

$$\kappa \in]a_-, a_+[\Longrightarrow 1 < \mid R^{\pm}(\kappa) \mid,\ \| \hat{S}(\kappa) \|_{\mathcal{L}(\mathbb{C}^2)},\ \| \left(\hat{S}(\kappa) \right)^{-1} \|_{\mathcal{L}(\mathbb{C}^2)}.$$

We emphasize that this extended scattering operator is of an unusual type: we do not know if the inverse wave operators $\mathbb{W}_{in(out)}^{-1}$, which are defined from $Y_{in(out)}$ onto X, can be extended from $BL^1_{(a+)}(\mathbb{R}) \times BL^1_{(a-)}(\mathbb{R})$ to $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. This situation has already been encountered in the case of space-times with causality violation [1]. The root of this phenomenon is the same: the conserved energy is not definite positive.

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