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DYNAMICAL RESONANCES AND SSF SINGULARITIES FOR A MAGNETIC SCHRÖDINGER OPERATOR

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Dedicated to Vesselin Petkov on the occasion of his 65th birthday

Abstract. We consider the Hamiltonian H of a 3D spinless non-relativistic quantum particle subject to parallel constant magnetic and non-constant electric field. The operator H has infinitely many eigenvalues of infinite multiplicity embedded in its continuous spectrum. We perturb H by appropriate scalar potentials V and investigate the transformation of these embedded eigenvalues into resonances. First, we assume that the electric potentials are dilation-analytic with respect to the variable along the magnetic field, and obtain an asymptotic expansion of the resonances as the coupling constant $\varkappa$ of the perturbation tends to zero. Further, under the assumption that the Fermi Golden Rule holds true, we deduce estimates for the time evolution of the resonance states with and without analyticity assumptions; in the second case we obtain these results as a corollary of

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suitable Mourre estimates and a recent article of Cattaneo, Graf and Hunziker [11]. Next, we describe sets of perturbations V for which the Fermi Golden Rule is valid at each embedded eigenvalue of H; these sets turn out to be dense in various suitable topologies. Finally, we assume that V decays fast enough at infinity and is of definite sign, introduce the Krein spectral shift function for the operator pair $(H+V, H)$, and study its singularities at the energies which coincide with eigenvalues of infinite multiplicity of the unperturbed operator H.

1. Introduction. In the present article we consider a magnetic Schrödinger operator H which, from a physics point of view, is the quantum Hamiltonian of a 3D non-relativistic spinless quantum particle subject to an electromagnetic field $(\mathbf{E}, \mathbf{B})$ with electric component $\mathbf{E} = -(0,0,v_0')$ where v_0 is a scalar potential depending only on the variable x_3, and magnetic component $\mathbf{B} = (0,0,b)$ where b is a positive constant. From a mathematical point of view this operator is remarkable because of the generic presence of infinitely many eigenvalues of infinite multiplicity, embedded in the continuous spectrum of H. These eigenvalues have the form $2bq + \lambda$, $q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, where $2bq$, $q \in \mathbb{Z}_+$, are the Landau levels, i.e. the infinite-multiplicity eigenvalues of the (shifted) Landau Hamiltonian, and λ is a simple eigenvalue of the 1D operator $-\frac{d^2}{dx^2} + v_0(x)$. We introduce the perturbed operator $H + \varkappa V$ where V is a H-compact multiplier by a real function, and $\varkappa \in \mathbb{R}$ is a coupling constant, and study the transition of the eigenvalues $2bq + \lambda$, $q \in \mathbb{Z}_+$, into a "cloud" of resonances which converge to $2bq + \lambda$ as $\varkappa \to 0$.

In order to perform this analysis, we assume that V is axisymmetric so that the operator $H + \varkappa V$ commutes with the x_3-component of the angular-momentum operator L. In this case $H + \varkappa V$ is unitarily equivalent to the orthogonal sum $\oplus_{m \in \mathbb{Z}} (H^{(m)} + \varkappa V)$ where $H^{(m)}$ is unitarily equivalent to the restriction of H onto $\mathrm{Ker}\,(L - m)$, $m \in \mathbb{Z}$. This allows us to reduce the analysis to a perturbation of a simple eigenvalue $2bq + \lambda$ of the operator $H^{(m)}$ with fixed magnetic quantum number m.

We apply two different approaches to the definition of resonances. First, we suppose that v_0 and V are analytic in x_3, and following the classical approach of Aguilar and Combes [2], define the resonances as the eigenvalues of the dilated non-self-adjoint operator $H(\theta) + \varkappa V_\theta$. We obtain an asymptotic expansion as $\varkappa \to 0$ of each of these resonances in the spirit of the Fermi Golden Rule (see e.g. [34, Section XII.6]), and estimate the time decay of the resonance states. A similar relation between the small-coupling-constant asymptotics of the resonance, and the exponential time decay of the resonance state has been established by Herbst [19] in the case of the Stark Hamiltonian, and later by other authors in the case of various quantum Hamiltonians (see e.g. [36], [17], [3]).

Our other approach is close to the time dependent methods developed in [35] and [12], and, above all, to the recent article by Cattaneo, Graf and Hunziker [11], where the dynamic estimates of the resonance states are based on appropriate Mourre estimates [25]. We prove Mourre type estimates for the operators $H^{(m)}$, which might be of independent interest, and formulate a theorem on the dynamics of the resonance states which can be regarded as an application of the general abstract result of [11].

Both our approaches are united by the requirement that the perturbation V satisfies the Fermi Golden Rule for all embedded eigenvalues for the operators $H^{(m)}$, $m \in \mathbb{Z}$. We establish several results which show that the set of such perturbations is dense in various topologies compatible with the assumptions of our theorems on the resonances of $H + \varkappa V$.
Further, we cancel the restriction that V is axisymmetric but suppose that it decays fast enough at infinity, and has a definite sign, introduce the Krein spectral shift function (SSF) for the operator pair $(H + V, H)$, and study its singularity at each energy $2bq + \lambda$, $q \in \mathbb{Z}_+$, which, as before, is an eigenvalue of infinite multiplicity of the unperturbed operator H. We show that the leading term of this singularity can be expressed via the eigenvalue counting function for compact Berezin-Toeplitz operators. Using the well-known results on the spectral asymptotics for such operators (see [29], [32]), we obtain explicitly the main asymptotic term of the SSF as the energy approaches the fixed point $2bq + \lambda$ for several classes of perturbations with prescribed decay rate with respect to the variables on the plane perpendicular to the magnetic field.

It is natural to associate these singularities of the SSF to the accumulation of resonances to these points because it is conjectured that the resonances are the poles of the SSF. This conjecture is justified by the Breit-Wigner approximation which is mathematically proved in other cases (see for instance [26], [27], [10], [7]).
The article is organized as follows. In Section 2 we summarize some well-known spectral properties of the operators H and $H^{(m)}$ and their perturbations, which are systematically exploited in the sequel. Section 3 is devoted to our approach based on the dilation analyticity, while Section 4 contains our results obtained as corollaries of appropriate Mourre estimates. In Section 5 we discuss the density in suitable topologies of the sets of perturbations V for which the Fermi Golden Rule holds true for every embedded eigenvalue $2bq + \lambda$ of the operator $H^{(m)}$,

$m \in \mathbb{Z}$. Finally, the asymptotic analysis of the SSF near the points $2bq + \lambda$ can be found in Section 6.

We dedicate the article to Vesselin Petkov with genuine admiration for his most significant contributions to the spectral and scattering theory for partial differential operators. In particular, we would like to mention his keystone results on the distribution of resonances, and the Breit-Wigner approximation of the spectral shift function for various quantum Hamiltonians (see [26], [27], [10]), and, especially, his recent works on magnetic Stark operators (see [13], [14]). These articles as well as many other works of Vesselin have strongly influenced and stimulated our own research.

2. Preliminaries.

2.1. In this subsection we summarize some well-known facts on the spectral properties of the 3D Schrödinger operator with constant magnetic field $\mathbf{B} = (0, 0, b)$, $b = \text{const.} > 0$. More details could be found, for example, in [4] or [9, Section 9].

Let

$$H_0 := H_{0,\perp} \otimes I_{\|} + I_{\perp} \otimes H_{0,\|}$$

where $I_{\|}$ and $I_{\perp}$ are the identity operators in $L^2(\mathbb{R}_{x_3})$ and $L^2(\mathbb{R}^2_{x_1,x_2})$ respectively,

$$H_{0,\perp} := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \quad (x_1, x_2) \in \mathbb{R}^2,$$

is the Landau Hamiltonian shifted by the constant b, and

$$H_{0,\|} := -\frac{d^2}{dx_3^2}, \quad x_3 \in \mathbb{R}.$$

The operator $H_{0,\perp}$ is self-adjoint in $L^2(\mathbb{R}^2)$, the operator $H_{0,\|}$ is self-adjoint in $L^2(\mathbb{R})$, and hence the operator H_0 is self-adjoint in $L^2(\mathbb{R}^3)$. Moreover, we have $\sigma(H_{0,\perp}) = \cup_{q=0}^{\infty}\{2bq\}$, and every eigenvalue $2bq$ of $H_{0,\perp}$ has infinite multiplicity (see e.g. [4]). Denote by p_q the orthogonal projection onto $\mathrm{Ker}\,(H_{0,\perp} - 2bq)$, $q \in \mathbb{Z}_+$. Since $\sigma(H_{0,\|}) = [0, \infty)$, we have $\sigma(H_0) = \cup_{q=0}^{\infty}[2bq, \infty) = [0, \infty)$.

Let now $m \in \mathbb{Z}$, $\varrho = (x_1^2 + x_2^2)^{1/2}$. Put

$$H_{0,\perp}^{(m)} := -\frac{1}{\varrho}\frac{\partial}{\partial \varrho}\varrho\frac{\partial}{\partial \varrho} + \left(\frac{m}{\varrho} - \frac{b\varrho}{2}\right)^2 - b.$$

The operator $H_{0,\perp}^{(m)}$ is self-adjoint in $L^2(\mathbb{R}_+;\varrho d\varrho)$, and we have $\sigma(H_{0,\perp}^{(m)}) = \cup_{q=m_-}^{\infty}\{2bq\}$ where $m_- = \max\{0,-m\}$ (see e.g. [4]). In contrast to the operator $H_{0,\perp}$, every eigenvalue $2bq$ of $H_{0,\perp}^{(m)}$ is simple. Denote by $\tilde{p}_{q,m}$ the orthogonal projection onto $\mathrm{Ker}\,(H_{0,\perp}^{(m)} - 2bq)$, $q \in \mathbb{Z}_+$, $q \geq m_-$. Put

$$\varphi_{q,m}(\varrho) := \sqrt{\frac{q!}{\pi(q+m)!}\left(\frac{b}{2}\right)^{m+1}}\,\varrho^m L_q^{(m)}\left(b\varrho^2/2\right)e^{-b\varrho^2/4}, \tag{2.1}$$
$$\varrho \in \mathbb{R}_+, \quad q \in \mathbb{Z}, \quad q \geq m_-,$$

where

$$L_q^{(m)}(s) := \sum_{l=m_-}^{q} \frac{(q+m)!}{(m+l)!(q-l)!}\frac{(-s)^l}{l!}, \quad s \in \mathbb{R}, \tag{2.2}$$

are the generalized Laguerre polynomials. Then we have

$$H_{0,\perp}^{(m)}\varphi_{q,m} = 2bq\varphi_{q,m},$$

$\|\varphi_{q,m}\|_{L^2(\mathbb{R}_+;\varrho d\varrho)} = 1$, and $\varphi_{q,m} = \overline{\varphi_{q,m}}$ (see e.g. [9, Section 9]). Moreover, $\tilde{p}_{q,m} = |\varphi_{q,m}\rangle\langle\varphi_{q,m}|$.

Set

$$H_0^{(m)} := H_{0,\perp}^{(m)} \otimes I_{\|} + \tilde{I}_{\perp} \otimes H_{0,\|}$$

where $\tilde{I}_\perp$ is the identity operator in $L^2(\mathbb{R}_+;\varrho d\varrho)$. Evidently, $\sigma(H_0^{(m)}) = [2m_-b,\infty)$. Let (ϱ,ϕ,x_3) be the cylindrical coordinates in $\mathbb{R}^3$. The operator $H_0^{(m)}$, $m \in \mathbb{Z}$, is unitarily equivalent to the restriction of H_0 onto $\mathrm{Ker}\,(L-m)$ where

$$L := -i\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) = -i\frac{\partial}{\partial\phi}$$

is the x_3-component of the angular-momentum operator, which commutes with H_0.

Moreover, the operator H_0 is unitarily equivalent to the orthogonal sum $\oplus_{m\in\mathbb{Z}}H_0^{(m)}$. More precisely, if we pass to cylindrical coordinates, and decompose $u \in \mathrm{Dom}(H_0)$ into a Fourier series with respect to ϕ, i.e. if we write

$$u(\varrho\cos\phi,\varrho\sin\phi,x_3) = \sum_{m\in\mathbb{Z}} e^{im\phi}u_m(\varrho,x_3),$$

we have

$$(H_0u)(\varrho\cos\phi, \varrho\sin\phi, x_3) = \sum_{m\in\mathbb{Z}} e^{im\phi}(H_0^{(m)}u_m)(\varrho, x_3).$$

2.2. In this subsection we perturb the operators $H_0^{(m)}$ and H_0 by a scalar potential v_0 which depends only on the variable x_3.
Let $v_0 : \mathbb{R} \to \mathbb{R}$ be a measurable function. Throughout the paper we assume that the multiplier by v_0 is $H_{0,\|}$-compact, which is ensured, for instance, by $v_0 \in L^2(\mathbb{R}) + L^\infty_\varepsilon(\mathbb{R})$. Set

$$H_\| := H_{0,\|} + v_0.$$

Then we have

$$\sigma_{\rm ess}(H_\|) = \sigma_{\rm ess}(H_{0,\|}) = [0, \infty).$$

For simplicity, throughout the article we suppose also that

$$\inf \sigma(H_\|) > -2b. \tag{2.3}$$

Evidently, (2.3) holds true if the negative part $v_{0,-}$ of the function v_0 is bounded, and we have $\|v_{0,-}\|_{L^\infty(\mathbb{R})} < 2b$.

Assume now that the discrete spectrum of $H_\|$ is not empty; this would follow, for example, from the additional conditions $v_0 \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} v_0(x)dx < 0$ (see e.g. [34, Theorem XIII.110]). Occasionally, we will impose also the assumption that the discrete spectrum of $H_\|$ consists of a unique eigenvalue; this would be implied, for instance, by the inequality $\int_{\mathbb{R}} |x|v_{0,-}(x)dx < 1$ (see e.g. [5, Chaper II, Theorem 5.1]).

Let λ be a discrete eigenvalue of the operator $H_\|$ which necessarily is simple. Then $\lambda \in (-2b, 0)$ by (2.3). Let ψ be an eigenfunction satisfying

$$H_\|\psi = \lambda\psi, \quad \|\psi\|_{L^2(\mathbb{R})} = 1, \quad \psi = \overline{\psi}. \tag{2.4}$$

Denote by $p_\| = p_\|(\lambda)$ the spectral projection onto ${\rm Ker}(H_\| - \lambda)$. We have $p_\| = |\psi\rangle\langle\psi|$.

Suppose now that v_0 satisfies

$$|v_0(x)| = O\left(\langle x\rangle^{-m_0}\right), \quad x \in \mathbb{R}, \quad m_0 > 1, \tag{2.5}$$

where $\langle x\rangle := (1 + |x|^2)^{\frac{1}{2}}$. Then the multiplier by v_0 is a relatively trace-class perturbation of $H_{0,\|}$, and by the Birman-Kuroda theorem (see e.g. [33, Theorem XI.9]) we have

$$\sigma_{\rm ac}(H_\|) = \sigma_{\rm ac}(H_{0,\|}) = [0, \infty).$$

Moreover, by the Kato theorem (see e.g. [34, Theorem XIII.58]) the operator $H_{\|}$ has no strictly positive eigenvalues. In fact, for all $E > 0$ and $s > 1/2$ the operator-norm limit

$$\langle x\rangle^{-s}(H_{\|} - E)^{-1}\langle x\rangle^{-s} := \lim_{\delta\downarrow 0}\langle x\rangle^{-s}(H_{\|} - E - i\delta)^{-1}\langle x\rangle^{-s}, \tag{2.6}$$

exists in $\mathcal{L}(L^2(\mathbb{R}))$, and for each compact subset J of $\mathbb{R}_+ = (0,\infty)$ and each $s > 1/2$ there exists a constant $C_{J,s}$ such that for each $E \in J$ we have

$$\|\langle x\rangle^{-s}(H_{\|} - E)^{-1}\langle x\rangle^{-s}\| \leq C_{J,s} \tag{2.7}$$

(see [1]).

Suppose again that (2.5) holds true, and let us consider the differential equation

$$-y'' + v_0 y = k^2 y, \quad k \in \mathbb{R}. \tag{2.8}$$

It is well-known that (2.8) admits the so-called *Jost solutions* $y_1(x;k)$ and $y_2(x;k)$ which obey

$$y_1(x;k) = e^{ikx}(1 + o(1)), \quad x \to \infty,$$
$$y_2(x;k) = e^{-ikx}(1 + o(1)), \quad x \to -\infty,$$

uniformly with respect to $k \in \mathbb{R}$ (see e.g. [5, Chapter II, Section 6] or [37]). The pairs $\{y_l(\cdot;k), y_l(\cdot;-k)\}$, $k \in \mathbb{R}$, $l = 1,2$, form fundamental sets of solutions of (2.8). Define the *transition coefficient* $\mathcal{T}(k)$ and *the reflection coefficient* $\mathcal{R}(k)$, $k \in \mathbb{R}$, $k \neq 0$, by

$$y_2(x;k) = \mathcal{T}(k)y_1(x;-k) + \mathcal{R}(k)y_1(x;k), \quad x \in \mathbb{R}.$$

It is well known that $\mathcal{T}(k) \neq 0$, $k \in \mathbb{R}\setminus\{0\}$. On the other hand, the Wronskian of the solutions $y_1(\cdot;k)$ and $y_2(\cdot;k)$ is equal to $-2ik\mathcal{T}(k)$, and hence these solutions are linearly independent for $k \in \mathbb{R}\setminus\{0\}$. For $E > 0$ set

$$\Psi_l(x;E) := \frac{1}{\sqrt{4\pi\sqrt{E}}\mathcal{T}(\sqrt{E})} y_l(x;\sqrt{E}), \quad l = 1,2.$$

Evidently, $\Psi_l(\cdot;E) \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $E > 0$, $l = 1,2$. Moreover, the real and the imaginary part of both functions $\Psi_l(\cdot;E)$, $l = 1,2$ with $E > 0$ do not vanish identically. Further, $\operatorname{Im}\langle x\rangle^{-s}(H_{\|} - E)^{-1}\langle x\rangle^{-s}$ with $E > 0$ and $s > 1/2$ is a rank-two operator with an integral kernel

$$K(x,x') = \pi\sum_{l=1,2}\langle x\rangle^{-s}\Psi_l(x;E)\overline{\Psi}_l(x';E)\langle x'\rangle^{-s}, \quad x,x' \in \mathbb{R}.$$

2.3. Fix now $m \in \mathbb{Z}$, and set

$$H^{(m)} := H_0^{(m)} + \tilde{I}_\perp \otimes v_0.$$

Since v_0 is $H_{0,\|}$-compact and the spectrum of $H_{0,\perp}^{(m)}$ is discrete, the operator $\tilde{I}_\perp \otimes v_0$ is $H_0^{(m)}$-compact. Therefore, the operator $H^{(m)}$ is well-defined on $\mathrm{Dom}(H_0^{(m)})$, and we have

$$\sigma_{\mathrm{ess}}(H^{(m)}) = \sigma_{\mathrm{ess}}(H_0^{(m)}) = [2bm_-, \infty).$$

Further, if λ is a discrete eigenvalue of $H_\|$, then $2bq + \lambda$ is a simple eigenvalue of $H^{(m)}$ for each integer $q \geq m_-$. If $q = m_-$, then this eigenvalue is isolated, but if $q > m_-$, then due to (2.3), it is embedded in the essential spectrum of $H^{(m)}$. Moreover,

$$H^{(m)}\Phi_{q,m} = (2bq + \lambda)\Phi_{q,m}, \quad q \geq m_-, \tag{2.9}$$

$\Phi_{q,m} = \varphi_{q,m} \otimes \psi$, $\varphi_{q,m}$ being defined in (2.1), and ψ in (2.4). Set

$$\mathcal{P}_{q,m} := \tilde{p}_{q,m} \otimes p_\|. \tag{2.10}$$

Then we have $\mathcal{P}_{q,m} = |\Phi_{q,m}\rangle\langle\Phi_{q,m}|$.

Finally, introduce the operator

$$H := H_0 + I_\perp \otimes v_0.$$

Even though the operator $I_\perp \otimes v_0$ is not H_0-compact (unless $v_0 = 0$), it is H_0-bounded with zero relative bound so that the operator H is well-defined on $\mathrm{Dom}(H_0)$. Evidently, the operator H is unitarily equivalent to the orthogonal sum $\oplus_{m\in\mathbb{Z}} H^{(m)}$. Up to the additive constant b, the operator H is the Hamiltonian of a quantum non-relativistic spinless particle in an electromagnetic field $(\mathbf{E}, \mathbf{B})$ with parallel electric component $\mathbf{E} = -(0, 0, v_0'(x_3))$, and magnetic component $\mathbf{B} = (0, 0, b)$.

Note that if λ is a discrete eigenvalue of $H_\|$, then $2bq + \lambda$, $q \in \mathbb{Z}_+$ is an eigenvalue of infinite multiplicity of H. If $q = 0$, this eigenvalue is isolated, and if $q \geq 1$, it lies on the interval $[0, \infty)$ which constitutes a part of the essential spectrum of H. Moreover, if (2.5) holds, then $\sigma_{\mathrm{ac}}(H) = [0, \infty)$, so that in this case $2bq + \lambda$, $q \in \mathbb{Z}_+$, is embedded in the absolutely continuous spectrum of H.

2.4. In this subsection we introduce appropriate perturbations of the operators H and $H^{(m)}$, $m \in \mathbb{Z}_+$.

Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a measurable function. Assume that V is H_0-bounded with zero relative bound. By the diamagnetic inequality (see e.g. [4]) this would follow, for instance, from $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. On $\mathrm{Dom}(H) = \mathrm{Dom}(H_0)$ define the operator $H + \varkappa V$, $\varkappa \in \mathbb{R}$.

Remark. We impose the condition that the relative H_0-bound is zero just for the sake of simplicity. If V is H_0-bounded with arbitrary finite relative bound, then we could again define $H + \varkappa V$ but only for sufficiently small $|\varkappa|$.

Occasionally, we will impose the more restrictive assumption that V is H_0-compact; this would follow from $V \in L^2(\mathbb{R}^3) + L^\infty_\varepsilon(\mathbb{R}^3)$. In particular, V is H_0-compact if it satisfies the estimate

$$|V(\mathbf{x})| = O\left(\langle X_\perp \rangle^{-m_\perp} \langle x_3 \rangle^{-m_3}\right), \ \mathbf{x} = (X_\perp, x_3), \ m_\perp > 0, \ m_3 > 0. \tag{2.11}$$

Further, assume that V is axisymmetric, i.e. V depends only on the variables (ϱ, x_3). Fix $m \in \mathbb{Z}$ and assume that the multiplier by V is $H_0^{(m)}$-bounded with zero relative bound. Then the operator $H^{(m)} + V$ is well defined on $\mathrm{Dom}(H^{(m)}) = \mathrm{Dom}(H_0^{(m)})$. Define the operator $H^{(m)} + \varkappa V$, $\varkappa \in \mathbb{R}$.

For $z \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} \,|\, \mathrm{Im}\,\zeta > 0\}$, $m \in \mathbb{Z}$, $q \geq m_-$, introduce the quantity

$$F_{q,m}(z) := \langle (H^{(m)} - z)^{-1} (I - \mathcal{P}_{q,m}) V \Phi_{q,m}, V \Phi_{q,m} \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}_+ \times \mathbb{R}; \varrho d\varrho dx_3)$, which we define to be linear with respect to the first factor. If λ is a discrete eigenvalue of $H_{||}$ we will say that *the Fermi Golden Rule* $\mathcal{F}_{q,m,\lambda}$ is valid if the limit

$$F_{q,m}(2bq + \lambda) = \lim_{\delta \downarrow 0} F_{q,m}(2bq + \lambda + i\delta), \tag{2.12}$$

exists and is finite, and

$$\mathrm{Im}\, F_{q,m}(2bq + \lambda) > 0. \tag{2.13}$$

3. Resonances via dilation analyticity.

3.1. In this subsection we will perturb $H_0^{(m)}$ by an axisymmetric potential $V(\varrho, x_3)$ so that the simple eigenvalue $2bq + \lambda$ of $H_0^{(m)}$ becomes a resonance of the perturbed operator. In order to use complex scaling, we impose an analyticity assumption. We assume that the potential v_0 extends to an analytic function in the sector

$$S_{\theta_0} = \{z \in \mathbb{C} \mid |\mathrm{Arg} z| \leq \theta_0, \text{ or } |z| \leq r_0\}$$

with $\theta_0 \in (0, \pi/2)$, which satisfies (2.5). As already used in similar situations (see e.g. [4], [38]) we introduce complex deformation in the longitudinal variable, $(\mathcal{U}(\theta)f)(\varrho, x_3) = e^{\theta/2} f(\varrho, e^{\theta} x_3)$, $f \in L^2(\mathbb{R}_+ \times \mathbb{R}; \varrho d\varrho dx_3)$, $\theta \in \mathbb{R}$. For $\theta \in \mathbb{R}$ we have

$$H^{(m)}(\theta) = \mathcal{U}(\theta) H^{(m)} \mathcal{U}^{-1}(\theta) = H_0^{(m)} \otimes I + I \otimes H_{\|}(\theta),$$

with $H_{\|}(\theta) = -e^{-2\theta} \dfrac{d^2}{dx_3^2} + v_{0,\theta}(x_3)$, and $v_{0,\theta}(x_3) := v_0(e^{\theta}\, x_3)$. By assumption, the family of operators $\{H_{\|}(\theta), \quad |\mathrm{Im}\ \theta| < \theta_0\}$, form a type (A) analytic family of m-sectorial operators in the sense of Kato (see for instance [20, Section 15.4], [2]). Then the discrete spectrum of $H_{0,\|}(\theta)$ is independent of θ and we have

$$\sigma(H^{(m)}(\theta)) = \bigcup_{q \geq m_-} \{2bq + \sigma(H_{\|}(\theta))\},$$

$$\sigma(H_{\|}(\theta)) = e^{-2\theta} \mathbb{R}_+ \cup \sigma_{\mathrm{disc}}(H_{\|}) \cup \{z_1, z_2, \dots\}$$

where $\sigma_{\mathrm{disc}}(H_{\|})$ denotes the discrete spectrum of $H_{\|}$, and $z_1, z_2, \dots.$ are (complex) eigenvalues of $H_{\|}(\theta)$ in $\{0 > \mathrm{Arg}\, z > -2\mathrm{Im}\ \theta\}$, $\mathrm{Im}\ \theta > 0$. In the sequel, we assume that $\sigma_{\mathrm{disc}}(H_{\|}) = \{\lambda\}$.

Further, we assume that V is axisymmetric, and admits an analytic extension with respect to $x_3 \in S_{\theta_0}$, which is $H_0^{(m)}$-compact (see e.g. [34, Chapter XII]. Let

$$V_\theta(\varrho, x_3) := V(\varrho, e^{\theta} x_3).$$

Then the family of operators $\{H^{(m)}(\theta) + \varkappa V_\theta, \quad |\mathrm{Im}\ \theta| < \theta_0,\ |\varkappa| \leq 1\}$, form also an analytic family of type (A) for sufficiently small $\varkappa$.

By definition, the resonances of $H^{(m)} + \varkappa V$ in

$$\mathcal{S}_{m_-}(\theta) := \bigcup_{q \geq m_-} \{z \in \mathbb{C};\ 2bq < \mathrm{Re} z < 2b(q+1),\ -2\mathrm{Im}\theta < \mathrm{Arg}(z - 2bq) \leq 0\}$$

are the eigenvalues of $H^{(m)}(\theta) + \varkappa V_\theta$, $\mathrm{Im}\theta > 0$.

For V axisymmetric and H_0-compact, we define the set of the resonances $\mathrm{Res}(H + \varkappa V, \mathcal{S}_0(\theta))$ of the operator $H + \varkappa V$ in $\mathcal{S}_0(\theta)$ by

$$\mathrm{Res}(H + \varkappa V, \mathcal{S}_0(\theta)) := \bigcup_{m \in \mathbb{Z}} \{\text{eigenvalues of } H^{(m)}(\theta) + \varkappa V_\theta\} \cap \mathcal{S}_0(\theta).$$

In other words, the set of resonances of $H + \varkappa V$ is the union with respect to $m \in \mathbb{Z}$ of the resonances of $H^{(m)} + \varkappa V$. This definition is correct since the restriction of $H + \varkappa V$ onto $\mathrm{Ker}(L - m)$ is unitarily equivalent to $H^{(m)} + \varkappa V$. Moreover using a standard deformation argument (see, for instance, [20, Chapter 16]), we can prove that these resonances coincide with singularities of the function $z \mapsto \langle (H + \varkappa V - z)^{-1} f, f\rangle$, for f in a dense subset of $L^2(\mathbb{R}^3)$.

Theorem 3.1. *Fix $m \in \mathbb{Z}$, $q > m_-$. Assume that:*

- *v_0 admits an analytic extension in S_{θ_0} which satisfies* (2.5)*;*
- *inequality* (2.3) *holds true, and $H_{\|}$ has a unique discrete eigenvalue λ;*
- *V is axisymmetric, and admits an analytic extension with respect to x_3 in S_{θ_0} which is $H_0^{(m)}$-compact.*

Then for sufficiently small $|\varkappa|$, the operator $H^{(m)} + \varkappa V$ has a resonance $w_{q,m}(\varkappa)$ which obeys the asymptotics

$$w_{q,m}(\varkappa) = 2bq+\lambda+\varkappa\langle V\Phi_{q,m}, \Phi_{q,m}\rangle-\varkappa^2 F_{q,m}(2bq+\lambda)+O_{q,m,V}(\varkappa^3),\ \varkappa \to 0, \tag{3.1}$$

the eigenfunction $\Phi_{q,m}$ being defined in (2.9)*, and the quantity $F_{q,m}(2bq+\lambda)$ being defined in* (2.12)*.*

P r o o f. Fix θ such that $\theta_0 > \mathrm{Im}\ \theta \geq 0$ and assume that $z \in \mathbb{C}$ is in the resolvent set of the operator $H^{(m)}(\theta) + \varkappa V_\theta$. Put

$$R^{(m)}_{\varkappa,\theta}(z) := (H^{(m)}(\theta) + \varkappa V_\theta - z)^{-1}.$$

By the resolvent identity, we have

$$\begin{aligned} R^{(m)}_{\varkappa,\theta}(z) = R^{(m)}_{0,\theta}(z) - \varkappa\, R^{(m)}_{0,\theta}(z)V_\theta R^{(m)}_{0,\theta}(z) \\ + \varkappa^2\, R^{(m)}_{0,\theta}(z)V_\theta R^{(m)}_{0,\theta}(z)V_\theta R^{(m)}_{0,\theta}(z) + O(\varkappa^3), \end{aligned} \tag{3.2}$$

as $\varkappa \to 0$, uniformly with respect to z in a compact subset of the resolvent sets of $H^{(m)}(\theta) + \varkappa V$ and $H^{(m)}(\theta)$.

Now note that the simple embedded eigenvalue $2bq+\lambda$ of $H^{(m)}$ is a simple isolated eigenvalue of $H^{(m)}(\theta)$. According to the Kato perturbation theory (see [22, Section VIII.2]), for sufficiently small $\varkappa$ there exists a simple eigenvalue $w_{q,m}(\varkappa)$ of $H^{(m)}(\theta) + \varkappa V_\theta$ such that $\lim_{\varkappa\to 0} w_{q,m}(\varkappa) = w_{q,m}(0) = 2bq + \lambda$. For $|\varkappa|$ sufficiently small, define the eigenprojector

$$\mathcal{P}_\varkappa(\theta) = \mathcal{P}_{\varkappa,q,m}(\theta) := \frac{-1}{2i\pi}\int_\Gamma R^{(m)}_{\varkappa,\theta}(z)dz \tag{3.3}$$

where Γ is a small positively oriented circle centered at $2bq+\lambda$. Evidently, for $u \in \operatorname{Ran}\mathcal{P}_\varkappa(\theta)$ we have $(H^{(m)}(\theta)+\varkappa V_\theta)u = w_{q,m}(\varkappa)u$; in particular, if $u \in \operatorname{Ran}\mathcal{P}_0(\theta)$, then $H^{(m)}(\theta)u = (2bq+\lambda)u$. Since $w_{q,m}(\varkappa)$ is a simple eigenvalue, we have

$$w_{q,m}(\varkappa) = \operatorname{Tr}\left(\frac{-1}{2i\pi}\int_\Gamma zR^{(m)}_{\varkappa,\theta}(z)dz\right) \tag{3.4}$$

for Γ and $\varkappa$ as above. Inserting (3.2) into (3.4), we get

$$\begin{aligned} w_{q,m}(\varkappa) = {} & 2bq+\lambda+\varkappa\operatorname{Tr}(\mathcal{P}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)) \\ & -\frac{\varkappa^2}{2i\pi}\operatorname{Tr}\left(\int_\Gamma zR^{(m)}_{0,\theta}(z)\,V_\theta\,R^{(m)}_{0,\theta}(z)\,V_\theta\,R^{(m)}_{0,\theta}(z)\,dz\right)+O(\varkappa^3) \end{aligned} \tag{3.5}$$

as $\varkappa \to 0$. Next, we have

$$\operatorname{Tr}(\mathcal{P}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)) = \operatorname{Tr}(\mathcal{P}_{q,m}\,V\,\mathcal{P}_{q,m}) = \langle V\Phi_{q,m},\Phi_{q,m}\rangle_{L^2(\mathbb{R}^+\times\mathbb{R};\varrho d\varrho dx_3)}, \tag{3.6}$$

the orthogonal projection $\mathcal{P}_{q,m}$ being defined in (2.10). For $\theta \in \mathbb{R}$ the relation is obvious since the operators $\mathcal{P}_{0,q,m}(\theta)\,V_\theta\,\mathcal{P}_{0,q,m}(\theta)$ and $\mathcal{P}_{q,m}\,V\,\mathcal{P}_{q,m}$ are unitarily equivalent. For general complex θ identity (3.6) follows from the fact the function $\theta \mapsto \operatorname{Tr}(\mathcal{P}_{0,q,m}(\theta)\,V_\theta\,\mathcal{P}_{0,q,m}(\theta))$ is analytic.

Set

$$\mathcal{Q}_0(\theta) := I-\mathcal{P}_0(\theta), \quad \tilde{H}^{(m)}(\theta) := H^{(m)}(\theta)\mathcal{Q}_0(\theta).$$

By the cyclicity of the trace, we have

$$\operatorname{Tr}\left(\int_\Gamma zR^{(m)}_{0,\theta}(z)\,V_\theta\,R^{(m)}_{0,\theta}(z)\,V_\theta\,R^{(m)}_{0,\theta}(z)dz\right) = T_1+T_2+T_3+T_4 \tag{3.7}$$

where

$$\begin{aligned} T_1 &:= \int_\Gamma z(2bq+\lambda-z)^{-3}\operatorname{Tr}\left(\mathcal{P}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)V_\theta\,\mathcal{P}_0(\theta)\right)dz, \\ T_2 &:= \int_\Gamma z(2bq+\lambda-z)^{-2}\operatorname{Tr}\Big(\mathcal{P}_0(\theta)\,V_\theta\,(\tilde{H}^{(m)}(\theta)-z)^{-1}\,\mathcal{Q}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)\Big)\,dz, \\ T_3 &:= \operatorname{Tr}\Bigg(\int_\Gamma z(\tilde{H}^{(m)}(\theta)-z)^{-1}\mathcal{Q}_0(\theta)V_\theta(\tilde{H}^{(m)}(\theta)-z)^{-1} \\ &\qquad\qquad \times\mathcal{Q}_0(\theta)V_\theta(\tilde{H}^{(m)}(\theta)-z)^{-1}\mathcal{Q}_0(\theta)dz\Bigg), \\ T_4 &:= \int_\Gamma z\,(2bq+\lambda-z)^{-1}\operatorname{Tr}\Big(\mathcal{P}_0(\theta)\,V_\theta(\tilde{H}^{(m)}(\theta)-z)^{-2}\,\mathcal{Q}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)\Big)dz. \end{aligned}$$

Since $\int_\Gamma z(2bq+\lambda-z)^{-3}dz = 0$, and the function $z \mapsto (\tilde{H}^{(m)}(\theta)-z)^{-1}$ is analytic inside Γ, we have

$$T_1 = T_3 = 0. \tag{3.8}$$

Further, using the identity

$$(2bq+\lambda-z)^{-2}\mathcal{P}_0(\theta)\,V_\theta\,(\tilde{H}^{(m)}(\theta)-z)^{-1}\,\mathcal{Q}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)+$$

$$(2bq+\lambda-z)^{-1}\mathcal{P}_0(\theta)\,V_\theta\,(\tilde{H}^{(m)}(\theta)-z)^{-2}\,\mathcal{Q}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta) =$$

$$\frac{\partial}{\partial z}\left((2bq+\lambda-z)^{-1}\mathcal{P}_0(\theta)\,V_\theta\,(\tilde{H}^{(m)}(\theta)-z)^{-1}\,\mathcal{Q}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)\right),$$

integrating by parts, and applying the Cauchy theorem, we obtain

$$T_2+T_4 = 2i\pi\,\mathrm{Tr}\left(\mathcal{P}_0(\theta)\,V_\theta\,(\tilde{H}^{(m)}(\theta)-2bq-\lambda)^{-1}\,(I-\mathcal{P}_0(\theta))\,V_\theta\,\mathcal{P}_0(\theta)\right). \tag{3.9}$$

Arguing as in the proof of (3.6), we get

$$\begin{aligned}\mathrm{Tr}\left(\mathcal{P}_0(\theta)\,V_\theta\,(\tilde{H}^{(m)}(\theta)-2bq-\lambda-i\delta)^{-1}\,\mathcal{Q}_0(\theta)\,V_\theta\,\mathcal{P}_0(\theta)\right)&\\ = F_{q,m}(2bq+\lambda+i\delta), \quad \delta>0.&\end{aligned} \tag{3.10}$$

For θ fixed such that $\mathrm{Im}\,\theta > 0$, the point $2bq+\lambda$ is not in the spectrum of $\tilde{H}^{(m)}(\theta)$. Taking the limit $\delta \downarrow 0$ in (3.10), we find that (3.9) implies

$$T_2+T_4 = 2i\pi\,F_{q,m}(2bq+\lambda). \tag{3.11}$$

Putting together (3.5) – (3.8) and (3.11), we deduce (3.1). $\square$

Remarks. (i) We will see in Section 4 that generically $\mathrm{Im}\,F_{q,m}(2bq+\lambda) > 0$ for all $m \in \mathbb{Z}$, and $q > m_-$.

(ii) Taking into account the above remark, we find that Theorem 3.1 implies that generically near $2bq+\lambda$, $q \geq 1$, there are infinitely many resonances of $H+\varkappa V$ with sufficiently small $\varkappa$, namely the resonances of the operators $H^{(m)}+\varkappa V$ with $m > -q$.

3.2. In this subsection we consider the dynamical aspect of resonances. We prove the following proposition which will be extended to non-analytic perturbations in Section 4.

Proposition 3.1. *Under the assumptions of Theorem* 3.1 *there exists a function* $g \in C_0^\infty(\mathbb{R};\mathbb{R})$ *such that* $g = 1$ *near* $2bq+\lambda$, *and*

$$\langle e^{-i(H^{(m)}+\varkappa V)t} g(H^{(m)}+\varkappa V)\Phi_{q,m}, \Phi_{q,m}\rangle = a(\varkappa)e^{-iw_{q,m}(\varkappa)t} + b(\varkappa,t), \quad t \geq 0, \tag{3.12}$$

with a *and* b *satisfying the asymptotic estimates*

$$|a(\varkappa)-1| = O(\varkappa^2),$$

$$|b(\varkappa,t)| = O(\varkappa^2(1+t)^{-n}), \quad \forall n \in \mathbb{Z}_+,$$

as $\varkappa \to 0$ *uniformly with respect to* $t \geq 0$.

In order to prove the proposition, we will need the following

Lemma 3.1. *Set*

$$\mathcal{Q}_\varkappa(\theta) := I - \mathcal{P}_\varkappa(\theta), \quad \tilde{R}^{(m)}_{\varkappa,\theta}(z) := \Big((H^{(m)}(\theta)+\varkappa V_\theta)\mathcal{Q}_\varkappa(\theta) - z\Big)^{-1}\mathcal{Q}_\varkappa(\theta),$$

the projection $\mathcal{P}_\varkappa(\theta)$ *being defined in* (3.3). *Then for* $|\varkappa|$ *small enough, there exists a finite-rank operator* $\mathcal{F}^{(m)}_{\varkappa,\theta}$, *uniformly bounded with respect to* $\varkappa$, *such that*

$$\begin{aligned}\mathcal{P}_0(\theta) &= \mathcal{P}_\varkappa(\theta)\\ &\quad + \varkappa\Big(\tilde{R}^{(m)}_{\varkappa,\theta}(w_{q,m}(\varkappa))V_\theta\mathcal{P}_\varkappa(\theta) + \mathcal{P}_\varkappa(\theta)V_\theta\tilde{R}^{(m)}_{\varkappa,\theta}(w_{q,m}(\varkappa))\Big) + \varkappa^2\mathcal{F}^{(m)}_{\varkappa,\theta}.\end{aligned} \tag{3.13}$$

P r o o f. By the resolvent identity, we have

$$\begin{aligned}R^{(m)}_{0,\theta}(\nu) &= R^{(m)}_{\varkappa,\theta}(\nu) + \varkappa\, R^{(m)}_{\varkappa,\theta}(\nu)V_\theta R^{(m)}_{\varkappa,\theta}(\nu)\\ &\qquad + \varkappa^2\, R^{(m)}_{\varkappa,\theta}(\nu)V_\theta R^{(m)}_{\varkappa,\theta}(\nu)V_\theta R^{(m)}_{0,\theta}(\nu).\end{aligned} \tag{3.14}$$

Moreover by definition of $\tilde{R}^{(m)}_{\varkappa,\theta}$ and of $\tilde{H}^{(m)} := H^{(m)}\,\mathcal{Q}_0$, we have:

$$R^{(m)}_{\varkappa,\theta}(\nu) = (w_{q,m}(\varkappa)-\nu)^{-1}\mathcal{P}_\varkappa(\theta) + \tilde{R}^{(m)}_{\varkappa,\theta}(\nu), \tag{3.15}$$

$$R^{(m)}_{0,\theta}(\nu) = (2bq+\lambda-\nu)^{-1}\mathcal{P}_0(\theta) + (\tilde{H}^{(m)}(\theta)-\nu)^{-1}\mathcal{Q}_0(\theta),$$

where $\nu \mapsto \tilde{R}^{(m)}_{\varkappa,\theta}(\nu)$ and $\nu \mapsto (\tilde{H}^{(m)}(\theta)-\nu)^{-1}\mathcal{Q}_0(\theta)$ are analytic near $2bq+\lambda$. Then, from the Cauchy formula, the integration of (3.14) on a small positively oriented circle centered at $2bq+\lambda$, yields (3.13) with $\mathcal{F}^{(m)}_{\varkappa,\theta}$ a linear combination of finite-rank operators of the form $P_1\, V_\theta\, P_2\, V_\theta\, P_3$, where $\{\mathcal{P}_0(\theta),\, \mathcal{P}_\varkappa(\theta)\} \cap \{P_j,\, j = 1,2,3\} \neq \emptyset$, and

$$P_j \in \{\mathcal{P}_0(\theta),\, \mathcal{P}_\varkappa(\theta),\, \tilde{R}^{(m)}_{\varkappa,\theta}(\nu),\, (\tilde{H}^{(m)}(\theta) - \nu)^{-1}\mathcal{Q}_0(\theta),$$
$$\text{with } \nu = w_{q,m}(\varkappa) \text{ or } \nu = 2bq + \lambda\}.$$

Since these operators are uniformly bounded in $\varkappa$ with $|\varkappa|$ small enough, $\mathcal{F}^{(m)}_{\varkappa,\theta}$ is a finite-rank operator which also is uniformly bounded in $\varkappa$ with $|\varkappa|$ small enough. □

P r o o f o f P r o p o s i t i o n 3.1. Pick at first any $g \in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $g = 1$ near $2bq + \lambda$. We have

$$\langle e^{-i(H^{(m)}+\varkappa V)t} g(H^{(m)} + \varkappa V)\Phi_{q,m}, \Phi_{q,m}\rangle = \mathrm{Tr}\,(e^{-i(H^{(m)}+\varkappa V)t} g(H^{(m)} + \varkappa V)\mathcal{P}_{q,m}).$$

By the Helffer-Sjöstrand formula,

$$\begin{aligned}(3.16)\quad & e^{-i(H^{(m)}+\varkappa V)t}\, g(H^{(m)} + \varkappa V)\, \mathcal{P}_{q,m} \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial \tilde{g}}{\partial \bar{z}}(z)\, e^{-izt}\, (H^{(m)} + \varkappa V - z)^{-1}\, \mathcal{P}_{q,m} dxdy\end{aligned}$$

where $z = x+iy$, $\bar{z} = x-iy$, $\tilde{g}$ is a compactly supported, quasi-analytic extension of g, and the convergence of the integral is understood in the operator-norm sense (see e.g. [15, Chapter 8]).

Consider the functions

$$\sigma_\pm(z) := \mathrm{Tr}\,((H^{(m)} + \varkappa V - z)^{-1}\, \mathcal{P}_{q,m}), \quad \pm\mathrm{Im} z > 0.$$

Following the arguments of the previous subsection, we find that

$$(3.17)\qquad \sigma_+(z) = \mathrm{Tr}\,(R^{(m)}_{\varkappa,\theta}(z)\, \mathcal{P}_{0,q,m}(\theta)), \quad \mathrm{Im}\, z > 0, \quad \theta_0 > \mathrm{Im}\,\theta > 0.$$

Inserting (3.13) into (3.17), and using the cyclicity of the trace, and the elementary identities

$$\mathcal{P}_\varkappa(\theta)\, R^{(m)}_{\varkappa,\theta}(z)\, \tilde{R}^{(m)}_{\varkappa,\theta}(w_{q,m}(\varkappa)) = 0 = \tilde{R}^{(m)}_{\varkappa,\theta}(w_{q,m}(\varkappa))\, \mathcal{P}_\varkappa(\theta)\, R^{(m)}_{\varkappa,\theta}(z),$$

we get

$$\sigma_+(z) = \mathrm{Tr}\,(R^{(m)}_{\varkappa,\theta}(z)\, \mathcal{P}_\varkappa(\theta)) + \varkappa^2 \mathrm{Tr}\,(R^{(m)}_{\varkappa,\theta}(z)\, \mathcal{F}^{(m)}_{\varkappa,\theta}).$$

Applying (3.15), we obtain

$$\sigma_+(z) = \Big(1 + \varkappa^2 r(\varkappa)\Big)(w_{q,m}(\varkappa) - z)^{-1} + \varkappa^2 G_+(\varkappa, z), \tag{3.18}$$

where $r(\varkappa) := \mathrm{Tr}\,(\mathcal{P}_\varkappa(\theta)\,\mathcal{F}^{(m)}_{\varkappa,\theta})$, and $G_+(\varkappa, z)$ is analytic near $2bq + \lambda$ and uniformly bounded with respect to $|\varkappa|$ small enough. Similarly,

$$\sigma_-(z) = \Big(1 + \varkappa^2 \overline{r(\varkappa)}\Big)\,(\overline{w_{q,m}(\varkappa)} - z)^{-1} + \varkappa^2 G_-(\varkappa, z), \tag{3.19}$$

where $G_-(\varkappa, z)$ is analytic near $2bq + \lambda$ and uniformly bounded with respect to $|\varkappa|$ small enough. Now, assume that the support of g is such that we can choose $\tilde{g}$ supported on a neighborhood of $2bq + \lambda$ where the functions $z \mapsto G_\pm(\varkappa, z)$ are holomorphic. Combining (3.16) with the Green formula, we get

$$\begin{aligned} &\mathrm{Tr}\,(e^{-i(H^{(m)}+\varkappa V)t}\, g(H^{(m)} + \varkappa V)\, \mathcal{P}_{q,m}) \\ &\qquad = \frac{1}{2i\pi}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\, (\sigma_+(\mu) - \sigma_-(\mu))d\mu. \end{aligned} \tag{3.20}$$

Making use of (3.18)–(3.19), we get

$$\begin{aligned} \frac{1}{2i\pi}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\, (\sigma_+(\mu) - \sigma_-(\mu))d\mu &= \frac{\varkappa^2}{2i\pi}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\, (G_+(\varkappa,\mu) - G_-(\varkappa,\mu))d\mu \\ &+ \frac{1 + \varkappa^2 r(\varkappa)}{2i\pi}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\, (w_{q,m}(\varkappa) - \mu)^{-1} d\mu \\ &- \frac{1 + \varkappa^2 \overline{r(\varkappa)}}{2i\pi}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\, (\overline{w_{q,m}(\varkappa)} - \mu)^{-1} d\mu. \end{aligned}$$

Pick $\varepsilon > 0$ so small that $g(\mu) = 1$ for $\mu \in [2bq + \lambda - 2\varepsilon, 2bq + \lambda + 2\varepsilon]$. Set

$$C_\varepsilon := (-\infty, 2bq + \lambda - \varepsilon] \cup \{2bq + \lambda + \varepsilon e^{it},\, t \in [-\pi, 0]\} \cup [2bq + \lambda + \varepsilon, +\infty),$$

$$g(\mu) := 1, \quad \mu \in C_\varepsilon \setminus \mathbb{R}.$$

Taking into account (3.20), bearing in mind that $\mathrm{Im}\, w_{q,m}(\varkappa) < 0$, and applying the Cauchy theorem, we easily find that

$$\begin{aligned} &\mathrm{Tr}\,(e^{-i(H^{(m)}+\varkappa V)t}\, g(H^{(m)} + \varkappa V)\, \mathcal{P}_{q,m}) \\ &\qquad = (1 + \varkappa^2 r(\varkappa))e^{-iw_{q,m}(\varkappa)t} + \varkappa^2 \sum_{j=1,2,3} I_j(t;\varkappa) \end{aligned} \tag{3.21}$$

where

$$I_1(t;\varkappa) := \frac{1}{2i\pi}\int_{\mathbb{R}} g(\mu)\, e^{-i\mu t}\, (G_+(\varkappa,\mu) - G_-(\varkappa,\mu))d\mu,$$

$$I_2(t;\varkappa) := \frac{1}{2i\pi}\int_{C_\varepsilon} g(\mu)\, e^{-i\mu t}(r(\varkappa)(w_{q,m}(\varkappa)-\mu)^{-1} - \overline{r(\varkappa)}(\overline{w_{q,m}(\varkappa)}-\mu)^{-1})d\mu,$$

$$I_3(t;\varkappa) := -\frac{\operatorname{Im} w_{q,m}(\varkappa)}{\varkappa^2\pi}\int_{C_\varepsilon} g(\mu)\, e^{-i\mu t}(w_{q,m}(\varkappa)-\mu)^{-1}(\overline{w_{q,m}(\varkappa)}-\mu)^{-1}\, d\mu.$$

Integrating by parts, we find that

$$|I_j(t;\varkappa)| = O((1+t)^{-n}), \quad t>0, \quad j=1,2,3, \quad \forall n\in\mathbb{Z}_+, \tag{3.22}$$

uniformly with respect to $\varkappa$, provided that $|\varkappa|$ is small enough; in the estimate of $I_3(t;\varkappa)$ we have taken into account that by Theorem 3.1 we have $|\operatorname{Im}(w_{q,m}(\varkappa))| = O(\varkappa^2)$. Putting together (3.21) and (3.22), we get (3.12).

4. Mourre estimates and dynamical resonances.

In this section we obtain Mourre estimates for the operator $H^{(m)}$ and apply them combined with a recent result of Cattaneo, Graf, and Hunziker (see [11]) in order to investigate the dynamics of the resonance states of the operator $H^{(m)}$ without analytic assumptions.

4.1. Let $v_0 : \mathbb{R}\to\mathbb{R}$. Set

$$v_j(x_3) := x_3^j v_0^{(j)}(x_3), \quad j\in\mathbb{Z}_+, \tag{4.1}$$

provided that the corresponding derivative $v_0^{(j)}$ of v_0 is well-defined.

Let

$$\mathcal{A} := -\frac{i}{2}\left(x_3\frac{d}{dx_3} + \frac{d}{dx_3}x_3\right)$$

be the self-adjoint operator defined initially on $C_0^\infty(\mathbb{R})$ and then closed in $L^2(\mathbb{R})$. Set $A := \tilde{I}_\perp\otimes\mathcal{A}$. Let T be an operator self-adjoint in $L^2(\mathbb{R}_+\times\mathbb{R};\varrho d\varrho dx_3)$ such that $e^{isA}D(T)\subseteq D(T)$, $s\in\mathbb{R}$. Define the commutator $[T,iA]$ in the sense of [21] and [11], and set $\operatorname{ad}_A^{(1)}(T) := -i[T,iA]$. Define recursively

$$\operatorname{ad}_A^{(k+1)}(T) = -i[\operatorname{ad}_A^{(k)}(T), iA], \quad k\geq 1,$$

provided that the higher order commutators are well-defined. Evidently, for each $m\in\mathbb{Z}$ we have

$$i^k\operatorname{ad}_A^{(k)}(H^{(m)}) = 2^k\tilde{I}_\perp\otimes H_{0,\parallel} + \sum_{j=1}^k c_{k,j}v_j, \quad k\in\mathbb{Z}_+, \tag{4.2}$$

with some constants $c_{k,j}$ independent of m; in particular, $c_{k,k} = (-1)^k$. Therefore the $H_{0,\|}$-boundedness of the multipliers v_j, $j = 1, \ldots, k$, guarantees the $H^{(m)}$-boundedness of all the operators $\mathrm{ad}_A^{(j)}(H^{(m)})$, $j = 1, \ldots, k$.

Let $J \subset \mathbb{R}$ be a Borel set, and T be a self-adjoint operator. Denote by $\mathbb{P}_J(T)$ the spectral projection of the operator T associated with J.

Proposition 4.1. *Fix $m \in \mathbb{Z}$. Let $\lambda \in (-2b, 0)$, $q \in \mathbb{Z}$, $q > m_-$. Put*

$$J = (2bq + \lambda - \delta, 2bq + \lambda + \delta) \tag{4.3}$$

where $\delta > 0$, $\delta < -\lambda/2$, and $\delta < (2b+\lambda)/2$. Assume that the operators $v_j(H_{0,\|} + 1)^{-1}$, $j = 0, 1$, are compact. Then there exist a positive constant $C > 0$ and a compact operator K such that

$$\mathbb{P}_J(H^{(m)})[H^{(m)}, iA]\mathbb{P}_J(H^{(m)}) \geq C\mathbb{P}_J(H^{(m)}) + K. \tag{4.4}$$

P r o o f. Let $\chi \in C_0^\infty(\mathbb{R};\mathbb{R})$ be such that $\mathrm{supp}\,\chi = [2bq + \lambda - 2\delta, 2bq + \lambda + 2\delta]$, $\chi(t) \in [0,1]$, $\forall t \in \mathbb{R}$, $\chi(t) = 1$, $\forall t \in J$. In order to prove (4.4), it suffices to show that

$$\chi(H^{(m)})[H^{(m)}, iA]\chi(H^{(m)}) \geq C\chi(H^{(m)})^2 + \tilde{K} \tag{4.5}$$

with a compact operator $\tilde{K}$. Indeed, if inequality (4.5) holds true, we can multiply it from the left and from the right by $\mathbb{P}_J(H^{(m)})$ obtaining thus (4.4) with $K = \mathbb{P}_J(H^{(m)})\tilde{K}\mathbb{P}_J(H^{(m)})$.

Next (4.2) yields

$$[H^{(m)}, iA] = 2\tilde{I}_\perp \otimes H_{0,\|} - v_1.$$

Therefore,

$$\chi(H^{(m)})[H^{(m)}, iA]\chi(H^{(m)}) = 2\chi(H^{(m)})\left(\tilde{I}_\perp \otimes H_{0,\|}\right)\chi(H^{(m)}) - \chi(H^{(m)})v_1\chi(H^{(m)})$$

$$= 2\chi(H_0^{(m)})\left(\tilde{I}_\perp \otimes H_{0,\|}\right)\chi(H_0^{(m)}) + 2K_1 - K_2 \tag{4.6}$$

where

$$K_1 := \chi(H^{(m)})\left(\tilde{I}_\perp \otimes H_{0,\|}\right)\chi(H^{(m)}) - \chi(H_0^{(m)})\left(\tilde{I}_\perp \otimes H_{0,\|}\right)\chi(H_0^{(m)}),$$

$$K_2 := \chi(H^{(m)}) v_1 \chi(H^{(m)}).$$

Since the operator $v_1(H_0^{(m)}+1)^{-1}$ is compact, the operators $v_1(H^{(m)}+1)^{-1}$, $v_1\chi(H^{(m)})$, and K_2 are compact as well. Let us show that K_1 is also compact. We have

$$K_1 = (\chi(H^{(m)}) - \chi(H_0^{(m)})) \left(\tilde{I}_\perp \otimes H_{0,\|}\right) \chi(H^{(m)}) +$$

$$\chi(H_0^{(m)}) \left(\tilde{I}_\perp \otimes H_{0,\|}\right) (\chi(H^{(m)}) - \chi(H_0^{(m)})). \tag{4.7}$$

By the Helffer-Sjöstrand formula and the resolvent identity,

$$\chi(H^{(m)}) - \chi(H_0^{(m)}) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)(H^{(m)}-z)^{-1} v_0 (H_0^{(m)}-z)^{-1} dxdy.$$

Since the support of $\tilde{\chi}$ is compact in $\mathbb{R}^2$, and the operator $\frac{\partial\tilde{\chi}}{\partial\bar{z}}(H^{(m)}-z)^{-1}v_0(H_0^{(m)} - z)^{-1}$ is compact for every $(x,y) \in \mathbb{R}^2$ with $y \neq 0$, and is uniformly norm-bounded for every $(x,y) \in \mathbb{R}^2$, we find that the operator $\chi(H^{(m)}) - \chi(H_0^{(m)})$ is compact. On the other hand, it is easy to see that the operators

$$\left(\tilde{I}_\perp \otimes H_{0,\|}\right) \chi(H^{(m)}) = \left(\tilde{I}_\perp \otimes H_{0,\|}\right) (H^{(m)}+1)^{-1}(H^{(m)}+1)\chi(H^{(m)}) =$$

$$\left(\tilde{I}_\perp \otimes H_{0,\|}\right) (H_0^{(m)}+1)^{-1}(I - v_0(H^{(m)}+1)^{-1})(H^{(m)}+1)\chi(H^{(m)})$$

and

$$\chi(H_0^{(m)}) \left(\tilde{I}_\perp \otimes H_{0,\|}\right) = \chi(H_0^{(m)})(H_0^{(m)}+1)(H_0^{(m)}+1)^{-1} \left(\tilde{I}_\perp \otimes H_{0,\|}\right)$$

are bounded. Taking into account (4.7), and bearing in mind the compactness of the operator $\chi(H^{(m)}) - \chi(H_0^{(m)})$, and the boundedness of the operators $\left(\tilde{I}_\perp \otimes H_{0,\|}\right) \chi(H^{(m)})$ and $\chi(H_0^{(m)}) \left(\tilde{I}_\perp \otimes H_{0,\|}\right)$, we conclude that the operator K_1 is compact.

Further, since $\delta < -\lambda/2$, and hence $2bj > 2bq + \lambda + 2\delta$ for all $j \geq q$, we have

$$\chi(H_{0,\|} + 2bj) = 0, \quad j \geq q.$$

Therefore,

$$\chi(H_0^{(m)}) = \sum_{j=m_-}^{\infty} \tilde{p}_{j,m} \otimes \chi(H_{0,\|} + 2bj) = \sum_{j=m_-}^{q-1} \tilde{p}_{j,m} \otimes \chi(H_{0,\|} + 2bj),$$

and

$$\chi(H_0^{(m)})\left(\tilde{I}_\perp \otimes H_{0,\|}\right)\chi(H_0^{(m)}) = \sum_{j=m_-}^{q-1} \tilde{p}_{j,m} \otimes \left(\chi(H_{0,\|}+2bj)^2 H_{0,\|}\right). \tag{4.8}$$

By the spectral theorem,

$$\sum_{j=m_-}^{q-1} \tilde{p}_{j,m} \otimes \left(\chi(H_{0,\|}+2bj)^2 H_{0,\|}\right)$$

$$\geq \sum_{j=m_-}^{q-1} (2b(q-j)+\lambda-2\delta)\tilde{p}_{j,m} \otimes \chi(H_{0,\|}+2bj)^2 \geq$$

$$(2b+\lambda-2\delta)\sum_{j=m_-}^{q-1} \tilde{p}_{j,m} \otimes \chi(H_{0,\|}+2bj)^2 = C_1\chi(H_0^{(m)})^2 \tag{4.9}$$

with $C_1 := 2b+\lambda-2\delta > 0$. Combining (4.6), (4.8), and (4.9), we get

$$\chi(H^{(m)})[H^{(m)}, iA]\chi(H^{(m)}) \geq 2C_1\chi(H^{(m)})^2 + 2C_1K_3 + 2K_1 - K_2 \tag{4.10}$$

where $K_3 := \chi(H_0^{(m)})^2 - \chi(H^{(m)})^2$ is a compact operator by the Helffer-Sjöstrand formula. Now we find that (4.10) is equivalent to (4.5) with $C = 2C_1$ and $\tilde{K} = 2C_1K_3 + 2K_1 - K_2$. $\square$

Remark. Mourre estimates for various magnetic quantum Hamiltonians can be found in [18, Chapter 3].

4.2. By analogy with (4.1) set

$$V_j(\varrho, x_3) = x_3^j \frac{\partial^j V(\varrho, x_3)}{\partial x_3^j}, \quad j \in \mathbb{Z}_+.$$

We have

$$i^k \mathrm{ad}_A^{(k)}(V) = \sum_{j=1}^{k} c_{k,j} V_j$$

with the same constants $c_{k,j}$ as in (4.2).

We will say that the condition $\mathcal{O}_\nu$, $\nu \in \mathbb{Z}_+$, holds true if the multipliers by v_j, $j = 0, 1$, are $H_{0,\|}$-compact, and the multipliers by v_j, $j \leq \nu$, are $H_{0,\|}$-bounded. Also, for a fixed $m \in \mathbb{Z}$ we will say that the condition $\mathcal{C}_{\nu,m}$, $\nu \in \mathbb{Z}_+$, holds true if the condition $\mathcal{O}_\nu$ is valid, the multiplier by V is $H_0^{(m)}$-bounded with zero relative bound, and the multipliers by V_j, $j = 1, \dots, \nu$, are $H_0^{(m)}$-bounded.

By Proposition 4.1 and [11, Lemma 3.1], the validity of condition $\mathcal{C}_{\nu,m}$ with $\nu \geq 5$ and a given $m \in \mathbb{Z}$ guarantees the existence of a finite limit $F_{q,m}(2bq+\lambda)$ with $q > m_-$ in (2.12), provided that (2.3) holds true, and λ is a discrete eigenvalue of $H_{0,\|} + v_0$.

Combining the results of Proposition 4.1 and [11, Theorem 1.2], we obtain the following

Theorem 4.1. *Fix $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$. Assume that:*

- *the condition $\mathcal{C}_{\nu,m}$ holds with $\nu \geq n + 5$;*
- *inequality (2.3) holds true, and λ is a discrete eigenvalue of $H_\|$;*
- *inequality (2.13) holds true, and hence the Fermi Golden Rule $\mathcal{F}_{q,m,\lambda}$ is valid.*

Then there exists a function $g \in C_0^\infty(\mathbb{R};\mathbb{R})$ such that $\operatorname{supp} g = \overline{J}$ (see (4.3)), $g = 1$ near $2bq + \lambda$, and

$$\begin{aligned} (4.11) \quad & \langle e^{-i(H^{(m)}+\varkappa V)t} g(H^{(m)} + \varkappa V)\Phi_{q,m}, \Phi_{q,m}\rangle \\ & \qquad = a(\varkappa)e^{-i\lambda_{q,m}(\varkappa)t} + b(\varkappa, t), \quad t \geq 0, \end{aligned}$$

where

$$\begin{aligned} (4.12) \quad \lambda_{q,m}(\varkappa) &= 2bq + \lambda + \varkappa\langle V\Phi_{q,m}, \Phi_{q,m}\rangle \\ & \quad - \varkappa^2 F_{q,m}(2bq + \lambda) + o_{q,m,V}(\varkappa^2), \quad \varkappa \to 0. \end{aligned}$$

In particular, we have $\operatorname{Im} \lambda_{q,m}(\varkappa) < 0$ for $|\varkappa|$ small enough. Moreover, a and b satisfy the asymptotic estimates

$$|a(\varkappa) - 1| = O(\varkappa^2),$$

$$|b(\varkappa, t)| = O(\varkappa^2 |\ln |\varkappa|| (1 + t)^{-n}),$$

$$|b(\varkappa, t)| = O(\varkappa^2 (1 + t)^{-(n+1)}),$$

as $\varkappa \to 0$ uniformly with respect to $t \geq 0$.

We will say that the condition $\mathcal{C}_\nu$, $\nu \in \mathbb{Z}_+$, holds true if the condition $\mathcal{O}_\nu$ is valid, the multiplier by V is H_0-bounded with zero relative bound, and the multipliers by V_j, $j = 1, \dots, \nu$, are H_0-bounded.

For $m \in \mathbb{Z}$ and $q \geq m_-$ denote by $\tilde{\Phi}_{q,m} : \mathbb{R}^3 \to \mathbb{C}$ the function written in cylindrical coordinates (ϱ, ϕ, x_3) as $\tilde{\Phi}_{q,m}(\varrho, \phi, x_3) = (2\pi)^{-\frac{1}{2}} e^{im\phi} \Phi_{q,m}(\varrho, x_3)$.

Corollary 4.1. *Fix $n \in \mathbb{Z}_+$. Assume that:*

- *the condition $\mathcal{C}_\nu$ holds with $\nu \geq n + 5$;*
- *inequality* (2.3) *is fulfilled, and λ is a discrete eigenvalue of $H_{0,\|} + v_0$;*
- *for each $m \in \mathbb{Z}$, $q > m_-$, inequality* (2.13) *holds true, and hence the Fermi Golden Rule $\mathcal{F}_{q,m,\lambda}$ is valid.*

Then for every fixed $q \in \mathbb{Z}_+$, and each $m \in \{-q+1, \ldots, 0\} \cup \mathbb{N}$ with $\mathbb{N} := \{1, 2, \ldots\}$, we have

$$\langle e^{-i(H+\varkappa V)t} g(H + \varkappa V) \tilde{\Phi}_{q,m}, \tilde{\Phi}_{q,m} \rangle_{L^2(\mathbb{R}^3)} = a(\varkappa) e^{-i\lambda_{q,m}(\varkappa)t} + b(\varkappa, t), \quad t \geq 0,$$

where g, $\lambda_{q,m}(\varkappa)$, a, and b are the same as in Theorem 4.1.

Remarks. (i) If $q \geq 1$, then Corollary 4.1 tells us that typically the eigenvalue $2bq + \lambda$ of the operator H, which has an infinite multiplicity, generates under the perturbation $\varkappa V$ infinitely many resonances with non-zero imaginary part. Note however that $2bq + \lambda$ is a discrete simple eigenvalue of the operator $H^{(-q)}$, and therefore the operator $H^{(-q)} + \varkappa V$ has a simple discrete eigenvalue provided that $|\varkappa|$ is small enough. Generically, this eigenvalue is an embedded eigenvalue for the operator $H + \varkappa V$.

(ii) If $q = 0$, then λ is an isolated eigenvalue of infinite multiplicity for H. By Theorem 6.1 below, in this case there exists an infinite series of discrete eigenvalues of the operator $H + V$ which accumulate at λ, provided that the perturbation V has a definite sign.

5. Sufficient conditions for the validity of the Fermi Golden Rule. In this section we describe certain classes of perturbations V compatible with the hypotheses of Theorems 3.1–4.1, for which the Fermi Golden rule $\mathcal{F}_{q,m,\lambda}$ is valid for every $m \in \mathbb{Z}$ and $q > m_-$. The results included are of two different types. Those of Subsection 5.1 are less general but they offer a constructive approximation of V by potentials for which the Fermi Golden Rule holds. On the other hand, the results of Subsection 5.2 are more general, but they are more abstract and less constructive.

5.1. Assume that $v_0 \in C^\infty(\mathbb{R})$ satisfies the estimates

$$|v_0^{(j)}(x)| = O_j\left(\langle x \rangle^{-m_0 - j}\right), \quad x \in \mathbb{R}, \quad j \in \mathbb{Z}_+, \quad m_0 > 1. \tag{5.1}$$

Then condition $\mathcal{O}_\nu$ is valid for every $\nu \in \mathbb{Z}_+$. Moreover, in this case the eigenfunction ψ (see (2.4)) is in the Schwartz class $\mathcal{S}(\mathbb{R})$, while the Jost solutions $y_j(\cdot; k)$, $j = 1, 2$, belong to $C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Suppose that (2.3) holds true, and the discrete spectrum of the operator $H_\|$ consists of a unique eigenvalue λ. Fix $m \in \mathbb{Z}$, and $q \in \mathbb{Z}_+$ such that $q > m_-$. Then it is easy to check that we have

$$\operatorname{Im} F_{q,m}(2bq + \lambda) =$$

(5.2)
$$\pi \sum_{l=1,2} \sum_{j=m_-}^{q-1} \left| \int_0^\infty \int_{\mathbb{R}} \varphi_{j,m}(\varrho)\varphi_{q,m}(\varrho)\psi(x_3)\Psi_l(x_3; 2b(q-j)+\lambda)V(\varrho, x_3)dx_3\varrho d\varrho \right|^2 .$$

In what follows we denote by $L^2_{\mathrm{Re}}(\mathbb{R}_+; \varrho d\varrho)$ the set of real functions $W \in L^2(\mathbb{R}_+; \varrho d\varrho)$.

Lemma 5.1. *The set of functions $W \in L^2_{\mathrm{Re}}(\mathbb{R}_+; \varrho d\varrho)$ for which*

$$\int_0^\infty \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)W(\varrho)\varrho d\varrho \neq 0 \tag{5.3}$$

for every $m \in \mathbb{Z}$, $q > m_-$, is dense in $L^2_{\mathrm{Re}}(\mathbb{R}_+; \varrho d\varrho)$.

Proof. Since the Laguerre polynomials $L_q^{(0)}$, $q \in \mathbb{Z}_+$, (see (2.2)) form an orthogonal basis in $L^2(\mathbb{R}_+; e^{-s}ds)$, the set of polynomials is dense in $L^2(\mathbb{R}_+; e^{-s}ds)$. Pick $W \in L^2_{\mathrm{Re}}(\mathbb{R}_+; \varrho d\varrho)$. Set $w(s) := W(\sqrt{2s/b})e^{s/2}$, $s > 0$. Evidently, $w = \overline{w} \in L^2(\mathbb{R}_+; e^{-s}ds)$. Pick $\varepsilon > 0$ and find a non-zero polynomial $\mathcal{P}$ with real coefficients such that

$$\int_0^\infty e^{-s} \left(\mathcal{P}(s) - w(s)\right)^2 ds < \frac{b\varepsilon^2}{4}.$$

Note that the coefficients of $\mathcal{P}$ could be chosen real since the coefficients of the Laguerre polynomials are real. Changing the variable $s = b\varrho^2/2$, we get

$$\int_0^\infty \left(\mathcal{P}(b\varrho^2/2)e^{-b\varrho^2/4} - W(\varrho)\right)^2 \varrho d\varrho < \frac{\varepsilon^2}{4}. \tag{5.4}$$

Now set $\mathcal{W}_\alpha(\varrho) := \mathcal{P}(b\varrho^2/2)e^{-\alpha b\varrho^2/2}$, $\varrho \in \mathbb{R}_+$, $\alpha \in (0, \infty)$, where the real polynomial $\mathcal{P}$ is fixed and satisfies (5.4). We will show that the set

(5.5)
$$\mathbb{A} := \left\{ \alpha \in (0,\infty) \mid \int_0^\infty \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)\mathcal{W}_\alpha(\varrho)\varrho d\varrho \neq 0, \ \forall m \in \mathbb{Z}, \ \forall q > m_- \right\}$$

is dense in $(0,\infty)$. Actually, for fixed $m \in \mathbb{Z}$ and $q > m_-$, we have

$$\int_0^\infty \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)\mathcal{W}_\alpha(\varrho)\varrho d\varrho = \Pi_{q,m}(\gamma(\alpha))$$

where $\Pi_{q,m}$ is a real polynomial of degree $2q+m+1+\deg\mathcal{P}$, and $\gamma(\alpha) := (1+\alpha)^{-1}$. Note that $\gamma : (0,\infty) \to (0,1)$ is a bijection. Denote by $\mathcal{N}_{q,m}$ the set of the zeros of $\Pi_{q,m}$ lying on the interval $(0,1)$. Set

$$\mathcal{N} := \bigcup_{m\in\mathbb{Z}} \bigcup_{q=m_-+1}^{\infty} \mathcal{N}_{q,m}.$$

Evidently, the sets $\mathcal{N}$ and $\gamma^{-1}(\mathcal{N})$ are countable, and $\mathbb{A} = (0,\infty) \setminus \gamma^{-1}(\mathcal{N})$. Therefore, $\mathbb{A}$ is dense in $(0,\infty)$. Now, pick $\alpha_0 \in \mathbb{A}$ so close to $1/2$ that

$$\int_0^\infty \mathcal{P}(b\varrho^2/2)^2 \left(e^{-b\varrho^2/4} - e^{-\alpha_0 b\varrho^2/2}\right)^2 \varrho d\varrho < \frac{\varepsilon^2}{4}. \tag{5.6}$$

Assembling (5.4) and (5.6), we obtain

$$\|W - \mathcal{W}_{\alpha_0}\|_{L^2(\mathbb{R}_+;\varrho d\varrho)} < \varepsilon. \tag{5.7}$$

□

Denote by $K(\mathbb{R})$ the class of real-valued continuous functions $u : [0,\infty) \to \mathbb{R}$ such that $\lim_{s\to\infty} u(s) = 0$. Set $\|u\|_{K(\mathbb{R})} := \max_{s\in[0,\infty)} |u(s)|$.

Lemma 5.2. *The set of functions $W \in K(\mathbb{R})$ for which* (5.3) *holds true for every $m \in \mathbb{Z}$, $q > m_-$, is dense in $K(\mathbb{R})$.*

P r o o f. By the Stone-Weierstrass theorem for locally compact spaces, we find that the set of functions $e^{-\alpha s}\mathcal{P}(s)$, $s > 0$ where $\alpha \in (0,\infty)$, and $\mathcal{P}$ is a polynomial, is dense in $K(\mathbb{R})$.

Let $W \in K(\mathbb{R})$; then we have $u \in K(\mathbb{R})$ where $u(s) := W(\sqrt{2s/b})$, $s > 0$. Pick $\varepsilon > 0$ and find $\alpha \in (0,\infty)$ and a polynomial $\mathcal{P}$ such that $\|W - \mathcal{W}_\alpha\|_{K(\mathbb{R})} < \varepsilon/2$ where, as in the proof of Lemma 5.1, $\mathcal{W}_\alpha(\varrho) = e^{-\alpha b\varrho^2/2}\mathcal{P}(b\varrho^2/2)$. Next pick $\alpha_0 \in \mathbb{A}$ (see (5.5)) such that $\|\mathcal{W}_\alpha - \mathcal{W}_{\alpha_0}\|_{K(\mathbb{R})} \leq \varepsilon/2$. Therefore, similarly to (5.7) we have $\|W - \mathcal{W}_{\alpha_0}\|_{K(\mathbb{R})} \leq \varepsilon$. □

Fix $\nu \in \mathbb{Z}_+$. We will write $V \in \mathcal{D}_\nu$ if

$$\|V\|^2_{\mathcal{D}_\nu} := \sum_{j=0}^{\nu} \int_0^\infty \int_{\mathbb{R}} \left(x_3^j \frac{\partial^j V(\varrho, x_3)}{\partial x_3^j}\right)^2 dx_3 \varrho d\varrho < \infty.$$

Note that if $\mathcal{O}_\nu$ holds, and $V \in \mathcal{D}_\nu$, then $\mathcal{C}_\nu$ is valid.

Theorem 5.1. *Assume that:*

- $v_0 \in C^\infty(\mathbb{R})$ *satisfies* (5.1)*;*
- *inequality* (2.3) *holds true;*
- *we have* $\sigma_{\mathrm{disc}}(H_{\|}) = \{\lambda\}$.

Fix $\nu \in \mathbb{Z}_+$. *Then the set of real perturbations* $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *for which the Fermi Golden Rule* $\mathcal{F}_{q,m,\lambda}$ *is valid for each* $m \in \mathbb{Z}$ *and* $q > m_-$, *is dense in* $\mathcal{D}_\nu$.

Proof. We will prove that the set of perturbations V for which the integral

$$I_{q,m,\lambda}(V) := \mathrm{Re} \int_0^\infty \int_{\mathbb{R}} \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)\psi(x_3)\Psi_1(x_3; 2b+\lambda)V(\varrho, x_3)dx_3\varrho d\varrho \tag{5.8}$$

does not vanish for each $m \in \mathbb{Z}$ and $q > m_-$, is dense in $\mathcal{D}_\nu$. By (5.2) this will imply the claim of the theorem. Set

$$\omega(x) := \psi(x)\mathrm{Re}\,\Psi_1(x; 2b+\lambda), \quad x \in \mathbb{R}.$$

Note that $0 \neq \omega = \overline{\omega} \in \mathcal{S}(\mathbb{R})$. Set

$$V_\perp(\varrho) = \int_{\mathbb{R}} \omega(x_3)V(\varrho, x_3)dx_3, \quad \varrho \in \mathbb{R}_+.$$

Evidently, $V_\perp \in L^2_{\mathrm{Re}}(\mathbb{R}_+; \varrho d\varrho)$. Fix $\varepsilon > 0$. Applying Lemma 5.1, we find $\tilde{V}_\perp \in L^2_{\mathrm{Re}}(\mathbb{R}_+; \varrho d\varrho)$ such that

$$\int_0^\infty \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)\tilde{V}_\perp(\varrho)\varrho d\varrho \neq 0 \tag{5.9}$$

for every $m \in \mathbb{Z}$, $q > m_-$, and

$$\|V_\perp - \tilde{V}_\perp\|_{L^2(\mathbb{R}_+;\varrho d\varrho)} < \varepsilon. \tag{5.10}$$

Set

$$\tilde{V}(\varrho, x_3) := \frac{\tilde{V}_\perp(\varrho)\omega(x_3)}{\|\omega\|^2_{L^2(\mathbb{R})}} + V(\varrho, x_3) - \frac{V_\perp(\varrho)\omega(x_3)}{\|\omega\|^2_{L^2(\mathbb{R})}}, \quad \varrho \in \mathbb{R}_+, \quad x_3 \in \mathbb{R}. \tag{5.11}$$

We have

$$I_{q,m,\lambda}(\tilde{V}) = \int_0^\infty \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)\tilde{V}_\perp(\varrho)\varrho d\varrho \neq 0$$

for every $m \in \mathbb{Z}$, $q > m_-$. On the other hand, (5.10) and (5.11) imply

$$\|V - \tilde{V}\|^2_{\mathcal{D}_\nu} \leq \varepsilon^2 \frac{\sum_{j=0}^{\nu} \int_{\mathbb{R}} x^{2j}\omega^{(j)}(x)^2 dx}{\|\omega\|^2_{L^2(\mathbb{R})}}. \qquad \square$$

Fix again $\nu \in \mathbb{Z}_+$. We will write $V \in \mathcal{E}_\nu$ if $V : \mathbb{R}_+ \to \mathbb{R}$ is continuous and tends to zero at infinity, and the functions $x_3^j \frac{\partial^j V(\varrho, x_3)}{\partial x_3^j}$, $j = 1, \ldots, \nu$, are bounded. If $\mathcal{O}_\nu$ holds, and $V \in \mathcal{E}_\nu$, then $\mathcal{C}_\nu$ holds true. For $V \in \mathcal{E}_\nu$ define the norm

$$\|V\|_{\mathcal{E}_\nu} := \sum_{j=0}^{\nu} \sup_{(\varrho, x_3) \in \mathbb{R}_+ \times \mathbb{R}} \left| x_3^j \frac{\partial^j V(\varrho, x_3)}{\partial x_3^j} \right|.$$

Arguing as in the proof of Theorem 5.1, from Lemma 5.2 we obtain the following

Theorem 5.2. *Assume that v_0 satisfies the hypotheses of Theorem 5.1. Fix $\nu \in \mathbb{Z}_+$. Then the set of perturbations $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ for which the Fermi Golden Rule $\mathcal{F}_{q,m,\lambda}$ is valid for each $m \in \mathbb{Z}$ and $q > m_-$, is dense in $\mathcal{E}_\nu$.*

5.2. For $\mathcal{H}$ a subspace of $L^r(\mathbb{R})$, let us introduce the space

$$\mathcal{H}^\dagger := \{\omega \in \mathcal{S}(\mathbb{R}) \mid \forall \chi \in \mathcal{H}, \int_{\mathbb{R}} \omega(x)\chi(x)dx = 0\}. \tag{12}$$

Clearly, if $C_b^\infty(\mathbb{R}) \subset \mathcal{H}$ then $\mathcal{H}^\dagger = \{0\}$. This property holds yet if $H^\infty(S_\theta) \subset \mathcal{H}$ where $H^\infty(S_\theta)$ is the set of smooth bounded functions on $\mathbb{R}$ admitting analytic extension on S_θ. It is enough to note that if $\omega(x_0) \neq 0$ then for C sufficiently large, the function $\chi(x_3) := e^{-C(x_3 - x_0)^2}$ is in $H^\infty(S_\theta)$ and satisfies $\int_{\mathbb{R}} \omega(x)\chi(x)dx \neq 0$.

Theorem 5.3. *Assume that v_0 satisfies the hypotheses of Theorem 5.1. Let $p \geq 1$, $r \geq 1$, $\delta \in \mathbb{R}$.*

Let $\mathcal{H}$ be a Banach space contained in $L^r(\mathbb{R})$ such that the injection $\mathcal{H} \hookrightarrow L^r(\mathbb{R})$ is continuous, and $\mathcal{H}^\dagger = \{0\}$.

Then the set of real perturbations $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, for which the Fermi Golden Rule $\mathcal{F}_{q,m,\lambda}$ is valid for each $m \in \mathbb{Z}$ and $q > m_-$, is dense in $L^p(\mathbb{R}_+, \langle\varrho\rangle^\delta \varrho d\varrho; \mathcal{H})$.

P r o o f. As in the case of Theorems 5.1–5.2, we will prove that the set of perturbations V for which the integral (5.8) does not vanish for each $m \in \mathbb{Z}$ and $q > m_-$, is dense in $L^p(\mathbb{R}_+, \langle\varrho\rangle^\delta \varrho d\varrho; \mathcal{H})$.

Let

$$\mathcal{M}_{q,m,\lambda} := \{V \in L^p(\mathbb{R}_+, \langle\varrho\rangle^\delta \varrho d\varrho; \mathcal{H});\ I_{q,m,\lambda}(V) \neq 0\}.$$

Since for $1/p' + 1/p = 1$ and $1/r' + 1/r = 1$ we have

$$|I_{q,m,\lambda}(V)| \leq \|\varphi_{q-1,m}\varphi_{q,m}\|_{L^{p'}(\mathbb{R}_+,\langle\varrho\rangle^{-\delta}\varrho d\varrho;\mathbb{R})}\ \|\omega\|_{L^{r'}(\mathbb{R})}\ \|V\|_{L^p(\mathbb{R}_+,\langle\varrho\rangle^\delta \varrho d\varrho; L^r(\mathbb{R}))},$$

the continuity of the injection $\mathcal{H} \hookrightarrow L^r(\mathbb{R})$ implies that $\mathcal{M}_{q,m,\lambda}$ is an open subset of the Banach space $L^p(\mathbb{R}_+, \langle\varrho\rangle^\delta \varrho d\varrho; \mathcal{H})$. Then according to the Baire lemma, we have only to check that each $\mathcal{M}_{q,m,\lambda}$ is dense in $L^p(\mathbb{R}_+, \langle\varrho\rangle^\delta \varrho d\varrho; \mathcal{H})$. Let $V \in L^p(\mathbb{R}_+, \langle\varrho\rangle^\delta \varrho d\varrho; \mathcal{H}) \setminus \mathcal{M}_{q,m,\lambda}$.

Since $0 \neq \omega = \overline{\omega} \in \mathcal{S}(\mathbb{R})$, the assumptions on $\mathcal{H}$ imply the existence of a $\Phi \in \mathcal{H}$ such that

$$\int_{\mathbb{R}} \omega(x_3)\Phi(x_3)dx_3 \neq 0.$$

Moreover, $\varrho \mapsto \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)\varrho$ is a product of polynomial and exponential functions. Then there exists $\varrho_0 \in \mathbb{R}_+$ such that $\varphi_{q-1,m}(\varrho_0)\varphi_{q,m}(\varrho_0)\varrho_0 \neq 0$, and for χ_0 supported near ϱ_0, we have

$$\int_0^\infty \varphi_{q-1,m}(\varrho)\varphi_{q,m}(\varrho)\chi_0(\varrho)\varrho d\varrho \neq 0.$$

Consequently, $\left\{V(\varrho, x_3) + \frac{1}{n}\chi_0(\varrho)\Phi(x_3)\right\}_{n\in\mathbb{N}}$ is a sequence of functions in $\mathcal{M}_{q,m,\lambda}$ tending to V in $L^p(\mathbb{R}_+, \langle\varrho\rangle^\delta \varrho d\varrho; \mathcal{H})$. $\square$

6. Singularities of the spectral shift function.

6.1. Suppose that v_0 satisfies (2.5). Assume moreover that the perturbation $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies (2.11) with $m_\perp > 2$ and $m_3 = m_0 > 1$. Then the multiplier by V is a relatively trace-class perturbation of H. Hence, the spectral shift function (SSF) $\xi(\cdot; H+V, H)$ satisfying the Lifshits-Krein trace formula

$$\mathrm{Tr}(f(H+V) - f(H)) = \int_{\mathbb{R}} f'(E)\xi(E; H+V, H)dE, \quad f \in C_0^\infty(\mathbb{R}),$$

and normalized by the condition $\xi(E; H+V, H) = 0$ for $E < \inf\sigma(H+V)$, is well-defined as an element of $L^1(\mathbb{R}; \langle E\rangle^{-2}dE)$ (see [24], [23]).

If $E < \inf \sigma(H)$, then the spectrum of $H + V$ below E could be at most discrete, and for almost every $E < \inf \sigma(H)$ we have

$$\xi(E; H + V, H) = -\mathrm{rank}\, \mathbb{P}_{(-\infty,E)}(H + V).$$

On the other hand, for almost every $E \in \sigma_{\mathrm{ac}}(H) = [0, \infty)$, the SSF $\xi(E; H + V, H)$ is related to the scattering determinant $\det S(E; H + V, H)$ for the pair $(H + V, H)$ by *the Birman-Krein* formula

$$\det S(E; H + V, H) = e^{-2\pi i \xi(E;H+V,H)}$$

(see [6]).

Set

$$\mathcal{Z} := \{E \in \mathbb{R} | E = 2bq + \mu,\ q \in \mathbb{Z}_+,\ \mu \in \sigma_{\mathrm{disc}}(H_{\|}) \text{ or } \mu = 0\}.$$

Arguing as in the proof of [9, Proposition 2.5], we can easily check the validity of the following

Proposition 6.1. *Let v_0 and V satisfy* (2.5), *and* (2.11) *with $m_\perp > 2$ and $m_0 = m_3 > 1$. Then the SSF $\xi(\cdot; H + V, H)$ is bounded on every compact subset of $\mathbb{R} \setminus \mathcal{Z}$, and is continuous on $\mathbb{R} \setminus (\mathcal{Z} \cup \sigma_p(H + V))$, where $\sigma_p(H + V)$ denotes the set of the eigenvalues of the operator $H + V$.*

In what follows we will assume in addition that

$$0 \le V(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{6.1}$$

and will consider the operators $H \pm V$ which are sign-definite perturbations of the operator H. The goal of this section is to investigate the asymptotic behaviour of the SSF $\xi(\cdot; H \pm V, H)$ near the energies which are eigenvalues of H of infinite multiplicity. More precisely, if (2.3) holds true, and $\lambda \in \sigma_{\mathrm{disc}}(H_{\|})$, we will study the asymptotics as $\eta \to 0$ of $\xi(2bq + \lambda + \eta; H \pm V, H)$, $q \in \mathbb{Z}_+$, being fixed.

Let T be a compact self-adjoint operator. For $s > 0$ denote

$$n_\pm(s; T) := \mathrm{rank}\, \mathbb{P}_{(s,\infty)}(\pm T), \quad n_*(s; T) := n_+(s; T) + n_-(s; T).$$

Put

$$U(X_\perp) := \int_{\mathbb{R}} V(X_\perp, x_3)\psi(x_3)^2 dx_3, \quad X_\perp \in \mathbb{R}^2,$$

the eigenfunction ψ being defined in (2.4).

Theorem 6.1. *Let v_0 and V satisfy* (2.5), (2.11) *with $m_\perp > 2$ and $m_0 = m_3 > 1$, and* (6.1). *Assume that* (2.3) *holds true, and $\lambda \in \sigma_{\rm disc}(H_{||})$. Fix $q \in \mathbb{Z}_+$. Then for each $\varepsilon \in (0,1)$ we have*

$$n_+((1+\varepsilon)\eta; p_q U p_q) + O(1) \leq \pm\xi(2bq + \lambda \pm \eta; H \pm V, H) \leq n_+((1-\varepsilon)\eta; p_q U p_q) + O(1), \tag{6.2}$$

$$\xi(2bq + \lambda \mp \eta; H \pm V, H) = O(1), \tag{6.3}$$

as $\eta \downarrow 0$.

Applying the well known results on the spectral asymptotics for compact Berezin-Toeplitz operators $p_q U p_q$ (see [29], [32]), we obtain the following

Corollary 6.1. *Assume the hypotheses of Theorem* 6.1.
(*i*) *Suppose that $U \in C^1(\mathbb{R}^2)$, and*

$$U(X_\perp) = u_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1+o(1)), \quad |X_\perp| \to \infty,$$

$$|\nabla U(X_\perp)| \leq C_1 \langle X_\perp \rangle^{-\alpha-1}, \quad X_\perp \in \mathbb{R}^2,$$

where $\alpha > 2$, and u_0 is a continuous function on $\mathbb{S}^1$ which is non-negative and does not vanish identically. Then we have

$$\xi(2bq + \lambda \pm \eta; H \pm V, H) = \pm\frac{b}{2\pi}\left|\{X_\perp \in \mathbb{R}^2 | U(X_\perp) > \eta\}\right| (1+o(1)) =$$

$$\pm\eta^{-2/\alpha}\frac{b}{4\pi}\int_{\mathbb{S}^1} u_0(s)^{2/\alpha} ds\, (1+o(1)), \quad \eta \downarrow 0,$$

where $|.|$ denotes the Lebesgue measure.

(*ii*) *Let $U \in L^\infty(\mathbb{R}^2)$. Assume that*

$$\ln U(X_\perp) = -\mu|X_\perp|^{2\beta}(1+o(1)), \quad |X_\perp| \to \infty,$$

for some $\beta \in (0,\infty)$, $\mu \in (0,\infty)$. Then we have

$$\xi(2bq + \lambda \pm \eta; H \pm V, H) = \pm\varphi_\beta(\eta)\, (1+o(1)), \quad \eta \downarrow 0, \quad \beta \in (0,\infty),$$

where

$$\varphi_\beta(\eta) := \begin{cases} \frac{b}{2\mu^{1/\beta}}|\ln\eta|^{1/\beta} & \text{if} \quad 0<\beta<1, \\ \frac{1}{\ln(1+2\mu/b)}|\ln\eta| & \text{if} \quad \beta=1, \\ \frac{\beta}{\beta-1}(\ln|\ln\eta|)^{-1}|\ln\eta| & \text{if} \quad 1<\beta<\infty, \end{cases} \qquad \eta\in(0,e^{-1}).$$

(iii) Let $U \in L^\infty(\mathbb{R}^2)$. Assume that the support of U is compact, and that there exists a constant $C>0$ such that $U \geq C$ on an open non-empty subset of $\mathbb{R}^2$. Then we have

$$\xi(2bq+\lambda\pm\eta; H\pm V, H) = \pm(\ln|\ln\eta|)^{-1}|\ln\eta|(1+o(1)), \quad \eta\downarrow 0.$$

Remarks. (i) The threshold behaviour of the SSF for various magnetic quantum Hamiltonians has been studied in [16] (see also [30], [31]), and recently in [8]. The singularities of the SSF described in Theorem 6.1 and Corollary 6.1 are of somewhat different nature since $2bq+\lambda$ is an infinite-multiplicity eigenvalue, and not a threshold in the continuous spectrum of the unperturbed operator.

(ii) By the strict mathematical version of the Breit-Wigner representation for the SSF (see [26], [27]), the resonances for various quantum Hamiltonians could be interpreted as the poles of the SSF. In [7] a Breit-Wigner approximation of the SSF near the Landau level was obtained for the 3D Schrödinger operator with constant magnetic field, perturbed by a scalar potential satisfying (2.11) with $m_\perp > 2$ and $m_3 > 1$. Moreover, it was shown in [7] that typically the resonances accumulate at the Landau levels. It is conjectured that the singularities of the SSF $\xi(\cdot; H\pm V, H)$ at the points $2bq+\lambda$, $q\in\mathbb{Z}_+$, are due to accumulation of resonances to these points. One simple motivation for this conjecture is the fact that if V is axisymmetric, then the eigenvalues of the operators p_qUp_q, $q\in\mathbb{Z}_+$, appearing in (6.2) are equal exactly to the quantities $\langle V\Phi_{q,m}, \Phi_{q,m}\rangle_{L^2(\mathbb{R}_+\times\mathbb{R};\varrho d\varrho dx_3)}$, $m\geq -q$, occurring in (3.1) and (4.12). We leave for a future work the detailed analysis of the relation between the singularities of the SSF at the points $2bq+\lambda$ and the eventual accumulation of resonances at these points. Hopefully, in this future work we will also extend our results of Sections 3 − 5 to the case of non-axisymmetric perturbations V.

(iii) As mentioned above, if $\lambda \in \sigma_{\rm disc}(H_\parallel)$, then λ is an isolated eigenvalue of H of infinite multiplicity. Set

$$\lambda_- := \begin{cases} \sup\{\mu \in \sigma(H), \quad \mu < \lambda\} & \text{if} \quad \lambda > \inf \sigma(H), \\ -\infty \quad \text{if} \quad \lambda = \inf \sigma(H), \end{cases}$$

$$\lambda_+ := \inf\{\mu \in \sigma(H), \quad \mu > \lambda\}.$$

By Pushnitski's representation of the SSF (see [28]), and the Birman-Schwinger principle for discrete eigenvalues in gaps of the essential spectrum, we have

$$\xi(\lambda - \eta; H - V; H) = -n_+(1; V^{1/2}(H - \lambda + \eta)^{-1}V^{1/2}) =$$
$$-\mathrm{rank}\,\mathbb{P}_{(\lambda_-, \lambda-\eta)}(H - V) + O(1), \quad \eta \downarrow 0,$$
$$\xi(\lambda + \eta; H + V; H) = n_-(1; V^{1/2}(H - \lambda - \eta)^{-1}V^{1/2}) =$$
$$\mathrm{rank}\,\mathbb{P}_{(\lambda+\eta, \lambda_+)}(H + V) + O(1), \quad \eta \downarrow 0.$$

Then Theorem 6.1 and Corollary 6.1 imply that the perturbed operator $H - V$ (resp., $H + V$) has an infinite sequence of discrete eigenvalues accumulating to λ from the left (resp., from the right).

6.2. This subsection contains some preliminary results needed for the proof of Theorem 6.1.

In what follows we denote by S_1 the trace class, and by S_2 the Hilbert-Schmidt class of compact operators, and by $\|\cdot\|_j$ the norm in S_j, $j = 1, 2$.

Suppose that $\eta \in \mathbb{R}$ satisfies

$$0 < |\eta| < \min\left\{2b + \lambda, \frac{1}{2}\mathrm{dist}\,\left(\lambda, \sigma(H_\parallel) \setminus \{\lambda\}\right)\right\}. \tag{6.4}$$

Note that inequalities (6.4) combined with (2.3), imply

$$\lambda + \eta \in (-2b, 0), \quad \lambda + \eta \notin \sigma(H_\parallel), \quad \mathrm{dist}\,(\lambda + \eta, \sigma(H_\parallel)) = |\eta|.$$

Set $P_j = p_j \otimes I_\parallel$, $j \in \mathbb{Z}_+$. For $z \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} | \mathrm{Im}\,\zeta > 0\}$, $j \in \mathbb{Z}_+$, and $W := V^{1/2}$, put

$$T_j(z) := W P_j (H - z)^{-1} W.$$

Proposition 6.2. *Assume the hypotheses of Theorem* 6.1. *Suppose that* (6.4) *holds true. Fix* $q \in \mathbb{Z}_+$. *Let* $j \in \mathbb{Z}_+$, $j \leq q$. *Then the operator-norm limit*

$$T_j(2bq + \lambda + \eta) = \lim_{\delta \downarrow 0} T_j(2bq + \lambda + \eta + i\delta) \tag{6.5}$$

exists in $\mathcal{L}(L^2(\mathbb{R}^3))$. *Moreover, if* $j < q$, *we have* $T_j(2bq+\lambda+\eta) \in S_1$, *and*

$$\|T_j(2bq+\lambda+\eta)\|_1 = O(1), \quad \eta \to 0. \tag{6.6}$$

P r o o f. We have

$$T_j(z) = M(t_{\perp,j} \otimes t_{\|}(z-2bj))M, \quad z \in \mathbb{C}_+, \tag{6.7}$$

where

$$M := W(X_\perp, x_3)\langle X_\perp\rangle^{m_\perp/2}\langle x_3\rangle^{m_3/2},$$

$$t_{\perp,l} := \langle X_\perp\rangle^{-m_\perp/2} p_l \langle X_\perp\rangle^{-m_\perp/2}, \quad l \in \mathbb{Z}_+,$$

$$t_{\|}(\zeta) := \langle x_3\rangle^{-m_3/2}(H_{\|}-\zeta)^{-1}\langle x_3\rangle^{-m_3/2}, \quad \zeta \in \mathbb{C}_+.$$

Since the operators M and $t_\perp$ are bounded, in order to prove that the limit (6.5) exists in $\mathcal{L}(L^2(\mathbb{R}^3))$, it suffices to show that the operator-norm limit

$$\lim_{\delta\downarrow 0} t_{\|}(2b(q-j)+\lambda+\eta+i\delta) \tag{6.8}$$

exists in $\mathcal{L}(L^2(\mathbb{R}))$. If $j<q$, the limit in (6.8) exists due to the existence of the limit in (2.6). If $j=q$, the limit in (6.8) exists just because $\lambda+\eta \notin \sigma(H_{\|})$.

Further, set

$$t_{\|,0}(\zeta) := \langle x_3\rangle^{-m_3/2}(H_{0,\|}-\zeta)^{-1}\langle x_3\rangle^{-m_3/2}, \quad \zeta \in \overline{\mathbb{C}}_+ \setminus \{0\}.$$

For $E = 2b(q-j)+\lambda+\eta$, from the resolvent equation we deduce

$$t_{\|}(E) = t_{\|,0}(E)(I_{\|} - \tilde{M} t_{\|}(E)) \tag{6.9}$$

where $\tilde{M} := v_0(x_3)\langle x_3\rangle^{m_3}$ is a bounded multiplier. By [9, Section 4.1], the operator $t_{\|,0}(E)$ with $E \in \mathbb{R}\setminus\{0\}$ is trace-class, and we have

$$\|t_{\|,0}(E)\|_1 \le \frac{c}{\sqrt{|E|}}(1+E_+^{1/4}) \tag{6.10}$$

with c independent of E.

Assume $j<q$. Then (6.9), (6.10), and (2.7) imply $t_{\|}(2b(q-j)+\lambda+\eta) \in S_1$, and

$$\|t_{\|}(2b(q-j)+\lambda+\eta)\|_1 = O(1), \quad \eta \to 0. \tag{6.11}$$

Finally, for any $l \in \mathbb{Z}_+$ we have $t_{\perp,l} \in S_1$, and

$$\|t_{\perp,l}\|_1 = \frac{b}{2\pi} \int_{\mathbb{R}^2} \langle X_\perp \rangle^{-m_\perp} dX_\perp \tag{6.12}$$

(see e.g. [9, Subsection 4.1]). Bearing in mind the structure of the operator T_j (see (6.7)) and the boundedness of the operator M, we find that $T_j(2b(q-j)+\lambda+\eta) \in S_1$, and due to (6.11) and (6.12), estimate (6.6) holds true. □

Now set $P_q^+ := \sum_{j=q+1}^{\infty} p_j$, $q \in \mathbb{Z}_+$, the convergence of the series being understood in the strong sense. For $z \in \mathbb{C}_+$ set

$$T_q^+(z) := W(H-z)^{-1} P_q^+ W.$$

Proposition 6.3. *Assume that v_0, V, and λ, satisfy the hypotheses of Theorem* 6.1, *and $\eta \in \mathbb{R}$ satisfies* (6.4). *Fix $q \in \mathbb{Z}_+$. Then the operator-norm limit*

$$T_q^+(2bq+\lambda+\eta) = \lim_{\delta \downarrow 0} T_q^+(2bq+\lambda+\eta+i\delta) \tag{6.13}$$

exists in $\mathcal{L}(L^2(\mathbb{R}^3))$. Moreover, $T_q^+(2bq+\lambda+\eta) \in S_2$, and

$$\|T_q^+(2bq+\lambda+\eta)\|_2 = O(1), \quad \eta \to 0. \tag{6.14}$$

P r o o f. Due to (2.3), the operator-valued function

$$\mathbb{C}_+ \ni z \mapsto (H-z)^{-1} P_q^+ \mapsto \mathcal{L}(L^2(\mathbb{R}^3))$$

admits an analytic continuation in $\{\zeta \in \mathbb{C} \,|\, \mathrm{Re}\,\zeta < 2bq\}$. Since $\lambda+\eta < 0$, and W is bounded, we immediately find that the limit in (6.13) exists. Evidently, the operator-valued function $\mathbb{C}_+ \ni z \mapsto (H_0-z)^{-1}P_q^+ \mapsto \mathcal{L}(L^2(\mathbb{R}^3))$ also admits an analytic continuation in $\{\zeta \in \mathbb{C} \,|\, \mathrm{Re}\,\zeta < 2bq\}$, and for $E = 2bq + \lambda + \eta$ we have

$$T_q^+(E) = W(H_0-E)^{-1}P_q^+(W - v_0(H-E)^{-1}P_q^+W). \tag{6.15}$$

Arguing as in the proof of [16, Proposition 4.2], we obtain

$$W(H_0 - 2bq - \lambda - \eta)^{-1} P_q^+ \in S_2,$$

and

$$\|W(H_0 - 2bq - \lambda - \eta)^{-1} P_q^+\|_2 = O(1), \quad \eta \to 0. \tag{6.16}$$

Since $\lambda < 0$, we have

$$\|W - v_0(H - 2bq - \lambda - \eta)^{-1} P_q^+ W\| = O(1), \quad \eta \to 0. \tag{6.17}$$

Putting together (6.15) and (6.16)-(6.17), we obtain (6.14). $\square$

6.3. In this subsection we prove Theorem 6.1.

Suppose that $\eta \in \mathbb{R}$ satisfies (6.4). Fix $q \in \mathbb{Z}_+$. Set

$$T(2bq + \lambda + \eta) := T_q^-(2bq + \lambda + \eta) + T_q(2bq + \lambda + \eta) + T_q^+(2bq + \lambda + \eta),$$

where

$$T_q^-(2bq + \lambda + \eta) = \sum_{j<q} T_j(2bq + \lambda + \eta).$$

Note that the operators $T_q(2bq+\lambda+\eta)$ and $T_q^+(2bq+\lambda+\eta)$ are self-adjoint.

By Pushnitski's representation of the SSF for sign-definite perturbations (see [28]), we have

$$\begin{aligned} &\xi(2bq + \lambda + \eta; H \pm V, H) \\ &\quad = \pm\frac{1}{\pi}\int_{\mathbb{R}} n_{\mp}(1, \operatorname{Re} T(2bq + \lambda + \eta) + s \operatorname{Im} T(2bq + \lambda + \eta))\frac{ds}{1 + s^2}. \end{aligned} \tag{6.18}$$

By (6.18) and the well-known Weyl inequalities, for each $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} n_{\mp}(1 + \varepsilon; T_q(2bq + \lambda + \eta)) - R_\varepsilon(\eta) \le \pm\xi(2bq + \lambda + \eta; H \pm V, H) \le \\ n_{\mp}(1 - \varepsilon; T_q(2bq + \lambda + \eta)) + R_\varepsilon(\eta) \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} &R_\varepsilon(\eta) := \\ &n_*(\varepsilon/3; \operatorname{Re} T_q^-(2bq+\lambda+\eta)) + n_*(\varepsilon/3; T_q^+(2bq+\lambda+\eta)) + \frac{3}{\varepsilon}\|T_q^-(2bq+\lambda+\eta)\|_1 \le \\ &\frac{6}{\varepsilon}\sum_{j<q}\|T_j(2bq + \lambda + \eta)\|_1 + \frac{9}{\varepsilon^2}\|T_q^+(2bq + \lambda + \eta)\|_2^2 = O(1), \quad \eta \to 0, \end{aligned} \tag{6.20}$$

due to Propositions 6.1–6.2.

Next, set

$$\tau_q = W(p_q \otimes p_{\|})W, \quad \tilde{T}_q(\lambda + \eta) := W(p_q \otimes (H_{\|} - \lambda - \eta)^{-1}(I_{\|} - p_{\|}))W,$$

provided that $\eta \in \mathbb{R}$ satisfies (6.4). Evidently,

$$T_q(2bq + \lambda + \eta) = -\eta^{-1}\tau_q + \tilde{T}_q(\lambda + \eta).$$

Applying again the Weyl inequalities, we get

$$n_\pm(s(1+\varepsilon)|\eta|; (\operatorname{sign}\eta)\tau_q) - n_*(s\varepsilon; \tilde{T}_q(\lambda + \eta)) \le n_\mp(s; T_q(2bq + \lambda + \eta)) \le$$

$$n_\pm(s(1-\varepsilon)|\eta|; (\operatorname{sign}\eta)\tau_q) + n_*(s\varepsilon; \tilde{T}_q(\lambda + \eta)) \tag{6.21}$$

for each $s > 0$, $\varepsilon \in (0,1)$, and η satisfying (6.4). Note that since $p_q U p_q \ge 0$ we have

$$n_\pm(s|\eta|; (\operatorname{sign}\eta)\,\tau_q) = \begin{cases} n_+(s|\eta|; p_q U p_q) & \text{if} \quad \pm\eta > 0, \\ 0 \quad \text{if} \quad \pm\eta < 0, \end{cases} \tag{6.22}$$

for every $s > 0$. Further,

$$\tilde{T}_q(\lambda + \eta) = M(t_{\perp,q} \otimes \tilde{t}_{\|}(\lambda + \eta))M \tag{6.23}$$

where

$$\tilde{t}_{\|}(\lambda + \eta) := \langle x_3 \rangle^{-m_3/2}(H_\| - \lambda - \eta)^{-1}(I_\| - p_\|)\langle x_3 \rangle^{-m_3/2}.$$

Obviously,

$$\|\tilde{t}_\|(\lambda + \eta)\| \le \|(H_\| - \lambda - \eta)^{-1}(I_\| - p_\|)\| = O(1), \quad \eta \to 0. \tag{6.24}$$

On the other hand, similarly to (6.9) we have

$$\tilde{t}_\|(\lambda+\eta) = t_{\|,0}(\lambda+\eta)(I_\| - \tilde{M}\tilde{t}_\|(\lambda+\eta)) - \langle x_3\rangle^{-m_3/2}(H_{0,\|} - \lambda - \eta)^{-1} p_\| \langle x_3\rangle^{-m_3/2}. \tag{6.25}$$

Since $p_\| \langle x_3 \rangle^{-m_3/2}$ is a rank-one operator, we have

$$\|\langle x_3\rangle^{-m_3/2}(H_{0,\|} - \lambda - \eta)^{-1} p_\| \langle x_3\rangle^{-m_3/2}\|_1$$
$$\le \|\langle x_3\rangle^{-m_3/2}(H_{0,\|} - \lambda - \eta)^{-1}\| \|p_\| \langle x_3\rangle^{-m_3/2}\|_2 \le$$

$$\int_{\mathbb{R}} \langle x \rangle^{-m_3} \psi(x)^2 dx\, |\lambda + \eta|^{-1} = O(1), \quad \eta \to 0. \tag{6.26}$$

Putting together (6.25), (6.10), (6.24), and (6.26), we get

$$\|\tilde{t}_{||}(\lambda+\eta)\|_1 = O(1), \quad \eta \to 0, \tag{6.27}$$

which combined with (6.23) and (6.12) yields

$$n_\pm(s; \tilde{T}_q(\lambda+\eta)) \leq s^{-1}\|\tilde{T}_q(\lambda+\eta)\| = O(1), \quad \eta \to 0. \tag{6.28}$$

Now (6.2) – (6.3) follow from estimates (6.19) – (6.22), and (6.28).

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